

# OPEN PROBLEMS IN TOPOLOGY

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## Introduction

This volume grew from a discussion by the editors on the difficulty of finding good thesis problems for graduate students in topology. Although at any given time we each had our own favorite problems, we acknowledged the need to offer students a wider selection from which to choose a topic peculiar to their interests. One of us remarked, “Wouldn’t it be nice to have a book of current unsolved problems always available to pull down from the shelf?” The other replied, “Why don’t we simply produce such a book?”

Two years later and not so simply, here is the resulting volume. The intent is to provide not only a source book for thesis-level problems but also a challenge to the best researchers in the field. Of course, the presented problems still reflect to some extent our own prejudices. However, as editors we have tried to represent as broad a perspective of topological research as possible. The topics range over algebraic topology, analytic set theory, continua theory, digital topology, dimension theory, domain theory, function spaces, generalized metric spaces, geometric topology, homogeneity, infinite-dimensional topology, knot theory, ordered spaces, set-theoretic topology, topological dynamics, and topological groups. Application areas include computer science, differential systems, functional analysis, and set theory. The authors are among the world leaders in their respective research areas.

A key component in our specification for the volume was to provide *current* problems. Problems become quickly outdated, and any list soon loses its value if the status of the individual problems is uncertain. We have addressed this issue by arranging a running update on such status in each volume of the journal *TOPOLOGY AND ITS APPLICATIONS*. This will be useful only if the reader takes the trouble of informing one of the editors about solutions of problems posed in this book. Of course, it will also be sufficient to inform the author(s) of the paper in which the solved problem is stated.

We plan a complete revision to the volume with the addition of new topics and authors within five years.

To keep bookkeeping simple, each problem has two different labels. First, the label that was originally assigned to it by the author of the paper in which it is listed. The second label, the one in the outer margin, is a global one and is added by the editors; its main purpose is to draw the reader’s attention to the problems.

A word on the indexes: there are two of them. The first index contains terms that are mentioned outside the problems, one may consult this index to find information on a particular subject. The second index contains terms that are mentioned in the problems, one may consult this index to locate problems concerning ones favorite subject. Although there is considerable overlap between the indexes, we think this is the best service we can offer the reader.

The editors would like to note that the volume has already been a success in the fact that its preparation has inspired the solution to several long-outstanding problems by the authors. We now look forward to reporting solutions by the readers. Good luck!

Finally, the editors would like to thank Klaas Pieter Hart for his valuable advice on T<sub>E</sub>X and METAFONT. They also express their gratitude to Eva Coplakova for composing the indexes, and to Eva Coplakova and Geertje van Mill for typing the manuscript.

*Jan van Mill*  
*George M. Reed*

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# PART I

## SET THEORETIC TOPOLOGY

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## **Toronto Problems**

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**Steve Watson<sup>1</sup>**  
**William Weiss<sup>1</sup>**

There are many set-theoretic topologists and set theorists in the Municipality of Metropolitan Toronto. We have had a seminar for 15 years or so and most of the regular participants have been there for around a decade. Thus the Editors thought it useful for us to compile a list of problems that interest us. This we have done in separate chapters arranged alphabetically by author below.

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## CHAPTER 1

### **Dow's Questions**

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**Question 1.** *Is there a ccc non-pseudocompact space which has no remote points?* 1. ?

This is probably the problem that I would most like to see answered. A remote point is a point of  $\beta X - X$  which is not in the closure of any nowhere dense subset of  $X$ . However there is a very appealing combinatorial translation of this in the case  $X$  is, for example, a topological sum of countably many compact spaces. It is consistent that there is a separable space with no remote points (DOW [1989]). If there is no such example then it is likely the case that  $V = L$  will imply that all such spaces do have remote points. I believe that **CH** implies this for spaces of weight less than  $\aleph_\omega$  (DOW [1988b]). Other references: for negative answers see VAN DOUWEN [1981], DOW [1983e], and DOW and PETERS [1987] and for positive answers see DOW [1982, 1989].

**Question 2.** *Find necessary and sufficient conditions on a compact space  $X$  so that  $\omega \times X$  has remote points.* 2. ?

Of course there may not be a reasonable answer to this question in **ZFC**, but it may be possible to obtain a nice characterization under such assumptions as **CH** or **PFA**. For example, I would conjecture that there is a model satisfying that if  $X$  is compact and  $\omega \times X$  has remote points then  $X$  has an open subset with countable cellularity. See DOW [1983d, 1987, 1988b].

**Question 3.** *Is there, for every compact space  $X$ , a cardinal  $\kappa$  such that  $\kappa \times X$  has remote points (where  $\kappa$  is given the discrete topology)?* 3. ?

It is shown in DOW and PETERS [1988] that this is true if there are arbitrarily large cardinals  $\kappa$  such that  $2^\kappa = \kappa^+$ .

**Question 4.** *If  $X$  is a non-pseudocompact space does there exist a point  $p \in \beta X$  which is not the limit of any countable discrete nowhere dense set?* 4. ?

It is shown in VAN MILL [1982] that the above follows from **MA**. It is known that **MA** can be weakened to  $\mathfrak{b} = \mathfrak{c}$ . However, if this is a theorem of **ZFC** it is likely the case that a new idea is needed. The main difficulty is in producing a point of  $\beta X - X$  which is not the limit of any countable discrete subset of  $X$  (an  $\omega$ -far point in VAN DOUWEN [1981]). The ideas in DOW [1982, 1989] may be useful in obtaining a negative answer.

**Question 5.** *Does  $U(\omega_1)$  have weak  $P_{\omega_2}$ -points?* 5. ?

A weak  $P_{\omega_2}$ -point is a point which is not the limit of any set of cardinality at most  $\omega_1$ . This question is the subject of DOW [1985]; it is known that  $U(\omega_3)$  has weak  $P_{\omega_2}$ -points.

? 6. **Question 6.** *Does every Parovichenko space have a  $\mathfrak{c} \times \mathfrak{c}$ -independent matrix?*

This is a technical question which probably has no applications but I find it interesting. A Parovichenko space is a compact F-space of weight  $\mathfrak{c}$  in which every non-empty  $G_\delta$  has infinite interior. The construction of a  $\mathfrak{c} \times \mathfrak{c}$ -independent matrix on  $\mathcal{P}(\omega)$  uses heavily the fact that  $\omega$  is strongly inaccessible, see KUNEN [1978]. In DOW [1985] it is shown that each Parovichenko space has a  $\mathfrak{c} \times \omega_1$ -independent matrix and this topic is also discussed in DOW [1984b, 1984a].

? 7. **Question 7.** *Is  $cf(\mathfrak{c}) = \omega_1$  equivalent to the statement that all Parovichenko spaces are co-absolute?*

It is shown in DOW [1983b] that the left to right implication holds.

? 8. **Question 8.** *Is there a clopen subset of the subuniform ultrafilters of  $\omega_1$  whose closure in  $\beta\omega_1$  is its one-point compactification?*

This is a desperate attempt to mention the notion and study of coherent sequences (DOW [1988c] and TODORČEVIĆ [1989]). These may be instrumental in proving that  $\omega^*$  is not homeomorphic to  $\omega_1^*$ .

? 9. **Question 9.** *What are the subspaces of the extremally disconnected spaces? More specifically, does every compact basically disconnected space embed into an extremally disconnected space?*

E. K. VAN DOUWEN and J. VAN MILL [1980] have shown that it is consistent that not every compact zero-dimensional F-space embeds and it is shown in DOW and VAN MILL [1982] that all P-spaces and their Stone-Čech compactifications do. It is independent of **ZFC** whether or not open subspaces of  $\beta\mathbb{N} \setminus \mathbb{N}$  are necessarily F-spaces (DOW [1983a]). There are other F-spaces with open subspaces which are not F-spaces. The references DOW [1982, 1983c] are relevant.

? 10. **Question 10.** *Find a characterization for when the product of a P-space and an F-space is again an F-space.*

A new necessary condition was found in DOW [1983c] and this had several easy applications. See also COMFORT, HINDMAN and NEGREPONTIS [1969] for most of what is known.

? 11. **Question 11.** *Is the space of minimal prime ideals of  $C(\beta\mathbb{N} \setminus \mathbb{N})$  basically disconnected?*

This problem is solved consistently in DOW, HENRIKSEN, KOPPERMAN and VERMEER [1988]. This problem sounds worse than it is. Enlarge the topology of  $\beta\mathbb{N} \setminus \mathbb{N}$  by declaring the closures of all cozero sets open. Now ask if this space is basically disconnected. If there are no large cardinals then it is not (Dow [1990]).

**Question 12.** *Consider the ideal of nowhere dense subsets of the rationals. Can this ideal be extended to a  $P$ -ideal in  $\mathcal{P}(\mathbb{Q})/\text{fin}$ ?* **12. ?**

This strikes me as a curiosity. A positive answer solves question 11.

**Question 13.** *Is every compact space of weight  $\omega_1$  homeomorphic to the remainder of a  $\psi$ -space?* **13. ?**

A  $\psi$ -space is the usual kind of space obtained by taking a maximal almost disjoint family of subsets of  $\omega$  and its remainder means with respect to its Stone-Čech compactification. Nyikos shows that the space  $2^{\omega_1}$  can be realized as such a remainder and the answer is yes under **CH** (this is shown in BAUMGARTNER and WEESE [1982]). This qualifies as an interesting question by virtue of the fact that it is an easily stated question (in **ZFC**) about  $\beta\mathbb{N}$ .

**Question 14.** *Is there a compact **ccc** space of weight  $\mathfrak{c}$  whose density is not less than  $\mathfrak{c}$ ?* **14. ?**

This is due to A. Błaszczyk. Todorčević showed that a yes answer follows from the assumption that  $\mathfrak{c}$  is regular. A reasonable place to look for a consistent no answer is the oft-called Bell-Kunen model (BELL and KUNEN [1981]); I had conjectured that all compact **ccc** spaces of weight at most  $\mathfrak{c}$  would have density  $\omega_1$  in this model but MERRILL [1986] shows this is not so. Todorčević is studying the consequences of the statement  $\Sigma_{\aleph_1}$ : “every **ccc** poset of size at most  $\mathfrak{c}$  is  $\aleph_1$ -centered”.

**Question 15.** *Is it consistent that countably compact subsets of countably tight spaces are always closed? Does it follow from **PFA**?* **15. ?**

This question is of course very similar to the Moore-Mrowka problem (BALOGH [1989]) and has been asked by Fleissner and Levy.

**Question 16.** *Does countable closed tightness imply countable tightness in compact spaces.* **16. ?**

This is due to Shapirovskii, I believe. Countable closed tightness means that if  $x \in \overline{A - \{x\}}$  then there should be a countable subset  $B \subset \overline{A}$  such that  $x \in \overline{B - \{x\}}$ .

? 17. **Question 17.** *Is every compact sequential space of character (or cardinality)  $\omega_1$  hereditarily  $\alpha$ -realcompact?*

This question is posed in Dow [1988a]. Nyikos defines a space to be  $\alpha$ -realcompact if every countably complete ultrafilter of closed sets is fixed.

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CHAPTER 2

**Steprāns’ Problems**

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## 1. The Toronto Problem

What has come to be known as the *Toronto problem* asks whether it is possible to have an uncountable, non-discrete, Hausdorff space which is homeomorphic to each of its uncountable subspaces. In order to convince the reader of the necessity of the various hypotheses in the question, define a *Toronto space* to be any space  $X$ , which is homeomorphic to all of its subspaces of the same cardinality as  $X$ . Hence the Toronto problem asks:

**Question 1.1.** *Are all Hausdorff, Toronto spaces of size  $\aleph_1$  discrete?* 18. ?

First note that the discrete space of size  $\aleph_1$  is a Toronto space and that, furthermore, so are the cofinite and cocountable topologies on  $\omega_1$ ; hence the requirement that the space be Hausdorff is a natural one. Moreover, it is easy to see that any infinite Hausdorff space contains an infinite discrete subspace and hence, any countable, Hausdorff Toronto space must be discrete. This is why the question is posed only for uncountable spaces.

Not much is known about the Toronto problem but the folklore does contain a few facts. First, any Hausdorff, Toronto space is scattered and the number of isolated points in any non-discrete, Hausdorff, Toronto space is countable. Consequently such a space must have derived length  $\omega_1$  and be hereditarily separable and, hence, must be an  $S$ -space. An even easier way to obtain a model where the answer to Question 1.1 is positive is to notice that this follows from the inequality  $2^{\aleph_0} \neq 2^{\aleph_1}$ . The reason is that hereditary separability implies that a space has only  $2^{\aleph_0}$  autohomeomorphisms while any Toronto space of size  $\lambda$  must have  $2^\lambda$  autohomeomorphisms.

While it has already been mentioned that the Toronto problem is easily answered for countable spaces, there is a version of the problem which remains open and which might have some significance for the original question. For any ordinal  $\alpha$  define an  $\alpha$ -Toronto space to be a scattered space of derived length  $\alpha$  which is homeomorphic to each subspace of derived length  $\alpha$ .

**Question 1.2.** *Is there an  $\omega$ -Toronto space?* 19. ?

Not even consistency results are known about this question and in fact answers are not available even if  $\omega$  is replaced by any  $\alpha \geq 2$ . For successor ordinals the question must be posed carefully though and it is more convenient to use the language of filters.

**1.1. DEFINITION.** If  $\mathcal{F}$  is a filter on  $X$  and  $\mathcal{G}$  a filter on  $Y$  then  $\mathcal{F}$  and  $\mathcal{G}$  are isomorphic if there is a bijection,  $\psi$ , from  $X$  to  $Y$  such that  $A \in \mathcal{F}$  if and only if  $\psi(A) \in \mathcal{G}$ .

**1.2. DEFINITION.** If  $\mathcal{F}$  is a filter on  $X$  then  $\mathcal{F}^2$  is the filter on  $X \times X$  defined by  $A \in \mathcal{F}^2$  if and only if  $\{a \in X; \{b \in X; (a, b) \in A\} \in \mathcal{F}\} \in \mathcal{F}$  and  $\mathcal{F}|A$  is the filter on  $X \setminus A$  defined by  $B \in \mathcal{F}|A$  if and only if  $B \cup A \in \mathcal{F}$ .

**1.3. DEFINITION.** A filter  $\mathcal{F}$  on  $\omega$  is *idempotent* if  $\mathcal{F}$  is isomorphic to  $\mathcal{F}^2$  and it is *homogeneous* if  $\mathcal{F}$  is isomorphic to  $\mathcal{F}|X$  for each  $X \notin \mathcal{F}$ .

By assuming that  $X$  is a counterexample to Question 1.1 and considering only the first two levels it can be shown that there is an idempotent homogeneous filter on  $\omega$ .

? **20. Question 1.3.** *Is there an idempotent, homogeneous filter on  $\omega$ ?*

As in the case of Question 1.2, not even a consistent solution to Question 1.3 is known. In fact only one example of an idempotent filter on  $\omega$  is known and it is not known whether this is homogeneous. Finally it should be mentioned that the questions concerning Toronto spaces of larger cardinalities and with stronger separation axioms also remain open.

? **21. Question 1.4.** *Is there some non-discrete, Hausdorff, Toronto space?*

? **22. Question 1.5.** *Are all regular (or normal) Toronto spaces of size  $\aleph_1$  discrete?*

## 2. Continuous colourings of closed graphs

Some attention has recently been focused on the question of obtaining analogs of finite combinatorial results, such as Ramsey or van der Waerden theorems, in topology. The question of graph colouring can be considered in the same spirit. Recall that a (directed) *graph*  $G$  on a set  $X$  is simply a subset of  $X^2$ . If  $Y$  is a set then a  *$Y$ -colouring* of  $G$  is a function  $\chi: X \mapsto Y$  such that  $(\chi^{-1}(i) \times \chi^{-1}(i)) \cap G = \emptyset$  for each  $i \in Y$ . By a *graph on a topological space* will be meant a closed subspace of the product space  $X^2$ . If  $Y$  is a topological space then a *topological  $Y$ -colouring* of a graph  $G$  on the topological space  $X$  is a continuous function  $\chi: X \mapsto Y$  such that  $\chi$  is a colouring of  $G$  when considered as an ordinary graph.

**2.1. DEFINITION.** If  $X$ ,  $Y$  and  $Z$  are topological spaces then define  $Y \leq_X Z$  if and only if for every graph  $G$  on  $X$ , if  $G$  has a topological  $Y$ -colouring then it has a topological  $Z$ -colouring.

Even for very simple examples of  $Y$  and  $Z$  the relation  $Y \leq_X Z$  provides unsolved questions. Let  $D(k)$  be the  $k$ -point discrete space and  $I(k)$  the  $k$ -point indiscrete space. The relation  $I(k) \leq_X D(n)$  says that every graph on  $X$  which can be coloured with  $k$  colours can be coloured with clopen sets and  $n$  colours. It is shown in KRAWCZYK and STEPRANS [19 $\infty$ ] that if  $X$  is compact and 0-dimensional and  $I(2) \leq_X D(k)$  holds for any  $k \in \omega$  then  $X$  must be scattered. Moreover,  $I(k) \leq_{\omega+1} D(k)$  is true for each  $k$  and  $I(2) \leq_X D(3)$  if  $X$  is a compact scattered space whose third derived set is empty. This is the reason for the following question.



**Question 2.1.** *If  $X$  is compact and scattered does  $I(2) \leq_X D(3)$  hold?* 23. ?

**Question 2.2.** *If the answer to Question 2.1 is negative then what is the least ordinal for which there is a compact scattered space of that ordinal height,  $X$ , such that  $I(2) \leq_X D(3)$  fails?* 24. ?

**Question 2.3.** *More generally, what is the least ordinal for which there is a compact scattered space of that ordinal height,  $X$ , such that  $I(n) \leq_X D(m)$  fails?* 25. ?

The preceding discussion has been about zero-dimensional spaces but the notation  $Y \leq_X Z$  was introduced in order to pose questions about other spaces as well. Let  $A(2)$  be the Alexandrov two point space with precisely one isolated point.

**Question 2.4.** *Does  $I(2) \leq_{\mathbb{R}} A(2)$  hold? What about  $I(2) \leq_{\mathbb{I}} A(2)$  where  $\mathbb{I}$  is the unit interval?* 26. ?

**Question 2.5.** *Characterize the triples of spaces  $X$ ,  $Y$  and  $Z$  such that  $X \leq_Z Y$  holds.* 27. ?

### 3. Autohomeomorphisms of the Čech-Stone Compactification on the Integers

The autohomeomorphism group of  $\beta\mathbb{N} \setminus \mathbb{N}$ , which will be denoted by  $\mathbb{A}$ , is the subject of countless unsolved questions so this section will not even attempt to be comprehensive but, instead will concentrate on a particular category of problems. W. RUDIN [1956] was the first to construct autohomeomorphisms of  $\beta\mathbb{N} \setminus \mathbb{N}$  which were non-trivial in the sense that they were not simply induced by a permutation of the integers. It was then shown by Shelah that it is consistent that every autohomeomorphism of  $\beta\mathbb{N} \setminus \mathbb{N}$  is induced by an *almost permutation*—that is a one-to-one function whose domain and range are both cofinite. This was later shown to follow from **PFA** by SHELAH and STEPRĀNS in [1988] while Veličković has shown that this does not follow from **MA**.

Let  $\mathbb{T}$  denote the subgroup of  $\mathbb{A}$  consisting of the trivial autohomeomorphisms—in other words, those which are induced by almost permutations of the integers. In every model known, the number of cosets of  $\mathbb{T}$  in  $\mathbb{A}$  is either 1 or  $2^{2^{\aleph_0}}$ .

**Question 3.1.** *Is it consistent that the number of cosets of  $\mathbb{T}$  in  $\mathbb{A}$  is strictly between 1 and  $2^{2^{\aleph_0}}$ ?* 28. ?

In his proof that  $\mathbb{T} = \mathbb{A}$  Shelah introduced the ideal of sets on which an autohomeomorphism is trivial.

**3.1. DEFINITION.** If  $\Phi \in \mathbb{A}$  define  $\mathcal{J}(\Phi) = \{X \subset \omega : (\exists f: X \mapsto \omega) f \text{ is one-to-one and } \Phi|_{\mathcal{P}(X)} \text{ is induced by } f\}$

Hence  $\Phi$  is trivial precisely if  $\mathcal{J}(\Phi)$  is improper—that is, contains  $\omega$ . It was shown in Shelah's argument that, under certain circumstances, if  $\mathcal{J}(\Phi)$  is merely sufficiently large then  $\Phi$  is trivial. This is of course not true in general because if there is a  $P$ -point of character  $\aleph_1$  then there is an autohomeomorphism of  $\beta\mathbb{N} \setminus \mathbb{N}$  which is trivial on precisely this  $P$ -point. It might be tempting to conjecture however, that if  $\mathcal{J}(\Phi)$  is either, improper or a prime ideal for every autohomeomorphism  $\Phi$ , then this implies that all such autohomeomorphisms are trivial. This is true but only for the reason that the hypothesis is far too strong—after all if  $\Phi_i: \mathcal{P}(A_i) \mapsto \mathcal{P}(A_i)$  is an autohomeomorphism for  $i \in k$  and the sets  $A_i$  are pairwise disjoint, then it is easy to see how to define

$$\oplus\{\Phi_i; i \in k\}: \cup\{\mathcal{P}(A_i); i \in k\} \mapsto \cup\{\mathcal{P}(A_i); i \in k\}$$

in such a way that  $\mathcal{J}(\oplus\{\Phi_i; i \in k\}) = \oplus\{\mathcal{J}(\Phi_i); i \in k\}$  Notice that this implies that  $\{\mathcal{J}(\Phi); \Phi \in \mathbb{A}\}$  is closed under finite direct sums; but not much else is known. In particular, it is not known what restrictions on  $\{\mathcal{J}(\Phi); \Phi \in \mathbb{A}\}$  imply that every member of  $\mathbb{A}$  is trivial.

? **29. Question 3.2.** Suppose that for every  $\Phi$ ,  $\mathcal{J}(\Phi)$  is either improper or the direct sum of prime ideals. Does this imply that every automorphism is trivial?

Even the much weaker hypothesis has not yet been ruled out.

? **30. Question 3.3.** If  $\mathcal{J}(\Phi) \neq \emptyset$  for each  $\Phi \in \mathbb{A}$  does this imply that each  $\Phi \in \mathbb{A}$  is trivial?

Rudin's proof of the existence of non-trivial autohomeomorphisms shows even more than has been stated. He showed in fact that, assuming **CH**, for any two  $P$ -points there is an autohomeomorphism of  $\beta\mathbb{N} \setminus \mathbb{N}$  which takes one to the other.

**3.2. DEFINITION.**  $R_H(\kappa)$  is defined to be the statement that, given two sets of  $P$ -points,  $A$  and  $B$ , both of size  $\kappa$ , there is  $\Phi \in \mathbb{A}$  such that  $\Phi(A) = \Phi(B)$ . Define  $R_T(\kappa)$  to mean that, given two sequences of  $P$ -points of length  $\kappa$ ,  $a$  and  $b$ , there is  $\Phi \in \mathbb{A}$  such that  $\Phi(a(\alpha)) = \Phi(b(\alpha))$  for each  $\alpha \in \kappa$ .

In this notation, Rudin's result says that **CH** implies that  $R_T(1)$  holds. It is easy to see that, in general,  $R_T(1)$  implies  $R_T(n)$  for each integer  $n$ . Observe

also, that  $R_T(\kappa)$  implies  $R_H(\kappa)$  and  $\kappa \leq \lambda$  and  $R_T(\lambda)$  implies  $R_T(\kappa)$ . However the answers to the following questions are not known.

**Question 3.4.** *If  $R_H(\kappa)$  holds and  $\lambda \leq \kappa$  must it be true that  $R_H(\lambda)$  also holds?* **31. ?**

**Question 3.5.** *Does  $R_T(1)$  imply  $R_T(\omega)$ ?* **32. ?**

**Question 3.6.** *Does  $R_H(\kappa)$  imply  $R_T(\kappa)$ ?* **33. ?**

It should be observed that  $R_T(1)$  is quite a rare property since an easy way to get it to fail is to have a  $P_\kappa$ -point and a  $P_\lambda$ -point which is not a  $P_\kappa$ -point and both  $\kappa$  and  $\lambda$  are uncountable. Indeed, the only models known to satisfy even  $R_T(1)$  are:

- models of **CH** (RUDIN [1956]),
- models obtained by adding  $\aleph_2$  Cohen reals to a model of **CH** (STEPRĀNS [1987]),
- models where there are no  $P$ -points (SHELAH [1982]), and
- models where every  $P$ -point has character  $\aleph_1$  (BLASS [1989]).

Hence the first question which should be answered is the following.

**Question 3.7.** *Is  $R_T(\omega_1)$  false?* **34. ?**

In order to make the property a bit easier to satisfy, the following definitions can be formulated.

**3.3. DEFINITION.** An *exact- $P_\kappa$ -point* is a  $P_\kappa$ -point which is not a  $P_\lambda$ -point for any  $\lambda$  such that  $\kappa < \lambda$ .

**3.4. DEFINITION.** Define  $R_H^\lambda(\kappa)$  to be the statement that, given two sets of exact- $P_\lambda$ -points,  $A$  and  $B$ , both of size  $\kappa$ , there is  $\Phi \in \mathbb{A}$  such that  $\Phi(A) = \Phi(B)$ . Define  $R_T^\lambda(\kappa)$  to mean that, given two sequences of exact- $P_\lambda$ -points of length  $\kappa$ ,  $a$  and  $b$ , there is  $\Phi \in \mathbb{A}$  such that  $\Phi(a(\alpha)) = \Phi(b(\alpha))$  for each  $\alpha \in \kappa$ .

The Questions 3.4 to 3.7 can all be asked in this context as well.

The main reason for asking the questions in this section has been to provoke some thought on how to construct autohomeomorphisms of  $\beta\mathbb{N} \setminus \mathbb{N}$ . At the moment, all non-trivial such constructions fall into two categories: Inductive constructions and approximations by trivial autohomeomorphisms along a prime ideal. The final question might be considered as a proposal for a new way of constructing autohomeomorphisms of  $\beta\mathbb{N} \setminus \mathbb{N}$ . Notice that if it has a positive answer then so does Question 3.5.

**Question 3.8.** If  $\Phi_i: \mathcal{P}(A_i) \mapsto \mathcal{P}(A_i)$  is an autohomeomorphism for  $i \in \omega$  and the sets  $A_i$  are pairwise disjoint, is there  $\Phi \in \mathbb{A}$  such that  $\Phi|_{\mathcal{P}(A_i)} = \Phi_i$  for each  $i \in \omega$ ? **35. ?**

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## CHAPTER 3

### **Tall's Problems**

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Here are some problems that interest me. Most I have worked on; some I have not. I have in general avoided listing well-known problems that I am not particularly associated with, since they surely will be covered elsewhere in this volume.

## A. Normal Moore Space Problems

TALL [1984, 1979] and FLEISSNER [1984] are good references for normal Moore spaces.

**Question A1.** *Is it consistent with  $2^{\aleph_0} = \aleph_2$  that every normal Moore space is metrizable?* **36. ?**

It is known to be consistent with  $2^{\aleph_0}$  being weakly inaccessible (NYIKOS [1982], DOW, TALL and WEISS [1984]). If—as once conjectured by Steve Watson— $2^{\aleph_0}$  not weakly inaccessible implies the existence of a normal non-metrizable Moore space, there would be a simple proof of the necessity of large cardinals to prove the consistency of the Normal Moore Space Conjecture. The game plan would be to work with Fleissner’s **CH** example of a normal non-metrizable Moore space (FLEISSNER [1982]) and weaken the hypothesis. However, Fleissner and I conjecture the other way—namely that the Conjecture *is* consistent with  $2^{\aleph_0} = \aleph_2$ . In particular I conjecture that the Conjecture holds in the model obtained by Mitchell-collapsing a supercompact cardinal. (For Mitchell collapse, see MITCHELL [1972] and ABRAHAM [1983].) There are enough Cohen reals in this model so that normal Moore spaces of cardinality  $\aleph_1$  are metrizable (DOW, TALL and WEISS [1984]), so this conjecture is a “reflection problem”—see below.

**Question A2.** *Is it consistent with **GCH** that normal Moore spaces are para-Lindelöf?* **37. ?**

A space is *para-Lindelöf* if every open cover has a locally countable open refinement. This is an attempt to get as much of the Normal Moore Space Conjecture as possible consistent with **GCH**. It is done for spaces of cardinality  $\leq \aleph_1$  in TALL [1988]. Any consistency proof would likely establish the consistency with **GCH** of every first countable countably paracompact submetacompact space being para-Lindelöf. Again, this is done for spaces of cardinality  $\leq \aleph_1$  in TALL [1988]; indeed, first countability is weakened to character  $\leq \aleph_1$ . It’s consistent with **GCH** that there’s a normal Moore space that’s not collectionwise Hausdorff, hence *not* para-Lindelöf (DEVLIN and SHELAH [1979]).

**Question A3.** *Does the existence of a normal non-metrizable Moore space imply the existence of one which is in addition is metacompact?* **38. ?**

This is probably due to D. Traylor. It has been popular among Moore space aficionados. If there is a normal first countable non-collectionwise Hausdorff space or if there is a normal locally metrizable non-metrizable Moore space, there is a metacompact normal non-metrizable Moore space. The former result is in TALL [1974c]; the latter in TALL [1984] (due to Watson). The interest is whether metacompactness makes normal Moore spaces that much closer to being metrizable. Fleissner's examples (FLEISSNER [1982]) are metacompact, so all the usual discussion about consistency results and the Normal Moore Space Conjecture apply to the question of whether metacompact normal Moore spaces are metrizable.

- ? 39. **Question A4.** *Does the consistency of para-Lindelöf normal Moore spaces being metrizable require large cardinals?*

Probably it does—although this has not been proved; the question is whether one can get by with say a measurable instead of a strong compact. This idea is due to Watson. Although Fleissner's **CH** example is para-Lindelöf, his singular cardinal one is not, which is why the question is open.

- ? 40. **Question A5.** *Does the consistency of normal Moore spaces of cardinality  $2^{\aleph_0}$  being metrizable require large cardinals?*

A weakly compact cardinal will do (NYIKOS [1983], DOW, TALL and WEISS [19∞a]) but I don't know whether it's necessary. I suspect some (small) large cardinal is necessary; it would be very interesting if that were not the case.

- ? 41. **Question A6.** *Is it consistent that every  $\aleph_1$ -collectionwise normal Moore space is metrizable?*

This is discussed in §F (Reflection Problems) below.

## B. Locally Compact Normal Non-collectionwise Normal Problems

- ? 42. **Question B1.** *Does the consistency of locally compact normal spaces being collectionwise normal require large cardinals?*

Presumably the answer is “yes”, by methods like those Fleissner used to show them necessary for first countable spaces (FLEISSNER [1982]). In [19∞] BALOGH used a supercompact cardinal to obtain consistency. The result one would hope to generalize as Fleissner generalized his **CH** example is the Daniels-Gruenhage example from  $\diamond^*$  of a locally compact non-collectionwise normal space (DANIELS and GRUENHAGE [1985]).

**Question B2.** *Is there a consistent example of a locally compact normal metacompact space that's not paracompact?* **43. ?**

Under  $V = L$  (or indeed in any model in which normal spaces of character  $\leq \aleph_1$  are collectionwise Hausdorff) there is no such example (WATSON [1982]). In **ZFC** there is none such that for each open cover  $\mathcal{U}$  there is an  $n \in \omega$  such that  $\mathcal{U}$  has a point  $n$ -refinement (DANIELS [1983]).

**Question B3.** *Is there a consistent example of a locally compact locally connected normal space that's not collectionwise normal?* **44. ?**

This problem is due to Nyikos. The only connection I know between local connectivity and collectionwise normality is that locally compact locally connected perfectly normal spaces are collectionwise normal with respect to submetacompact closed sets (ALSTER and ZENOR [1976], or see TALL [1984]).

**Question B4.** *Is it consistent that normal  $k$ -spaces are collectionwise normal?* **45. ?**

$k$ -spaces are precisely the quotients of locally compact spaces. Partial results have been achieved by DANIELS [19 $\infty$ ].

**Question B5.** *Is it consistent without large cardinals that normal manifolds are collectionwise normal?* **46. ?**

Nyikos noted that a weakly compact cardinal suffices (NYIKOS [1983]), or see TALL [1982]. Rudin obtained a counterexample from  $\diamond^+$  (RUDIN [19 $\infty$ ]). This problem is related to **A5** above, since the components have size  $\leq 2^{\aleph_0}$ .

## C. Collectionwise Hausdorff Problems

**Question C1.** *Is it consistent (assuming large cardinals) that every first countable  $\aleph_1$ -collectionwise Hausdorff space is collectionwise Hausdorff?* **47. ?**

This is discussed in §F below.

**Question C2.** *Suppose  $\kappa$  is a singular strong limit and  $X$  is a normal space of character less than  $\kappa$ . Suppose  $X$  is  $\lambda$ -collectionwise Hausdorff for all  $\lambda < \kappa$ . Is  $X$   $\kappa$ -collectionwise Hausdorff?* **48. ?**

There is a consistent first countable counterexample if normality is dropped (FLEISSNER and SHELAH [19 $\infty$ ]). There is a counterexample if there are no inner models with large cardinals, again without normality (FLEISSNER and SHELAH [19 $\infty$ ]). The reason the conjecture implicitly stated is plausible is



that it is true for singular  $\kappa$  such that **GCH** holds on a tail of the cardinals below  $\kappa$  (FLEISSNER [1974]). The conjecture has applications in TALL [1988] (there Fleissner's result is misstated).

The next problem is more technical. In general, anything one can prove about collectionwise Hausdorffness in  $L$ , one can prove in the (reverse) Easton model via forcing, and vice versa. The one remaining exception follows:

- ? 49. **Question C3.** *Prove via forcing (in a natural way, not by forcing  $\diamond$  for stationary systems) in the (reverse) Easton model that  $\aleph_1$ -para-Lindelöf regular spaces of character  $\leq \aleph_2$  are collectionwise Hausdorff.*

A space is  $\aleph_1$ -para-Lindelöf if every open cover of size  $\leq \aleph_1$  has a locally countable open refinement. See FLEISSNER [1983] and TALL [1988].

#### D. Weak Separation Problems

In TALL [1976c] I defined a space to be *weakly* ( $\lambda$ -) *collectionwise Hausdorff* if each closed discrete subspace (of size  $\lambda$ ) included one of the same cardinality which was separated by disjoint open sets. **GCH** (actually, for every  $\kappa$ ,  $2^\kappa < (2^\kappa)^+$ ) implies normal spaces of character  $\leq 2^{\aleph_0}$  are weakly collectionwise Hausdorff (TALL [1976c], or see TALL [1984]), but, as mentioned previously, is consistent with the existence of a normal Moore space which is not collectionwise Hausdorff (DEVLIN and SHELAH [1979]). Analogously define *weak collectionwise normality* as the possibility of separating  $\lambda$  members of a discrete collection of size  $\lambda$ , for any  $\lambda$ .

- ? 50. **Question D1.** *Is it consistent (assuming large cardinals) that every first countable weakly  $\aleph_1$ -collectionwise Hausdorff space is weakly collectionwise Hausdorff?*

This is discussed in §F below.

- ? 51. **Question D2.** *Is it consistent that normal first countable spaces are all weakly collectionwise normal but that there is one that's not collectionwise normal?*

The next three problems are from TALL [1981]. Given a normal space  $X$  and a closed discrete subspace  $Y$ , a *basic function* is a function which assigns to each point in  $Y$  an open set about it which contains no other point of  $Y$ . Given such a basic function  $f$  and a  $Z \subseteq Y$ , the open set

$$f(Z) = \bigcup \{f(y) : y \in Z\}$$

may or may not be disjoint from  $f(Y - Z)$ . Since  $X$  is normal, there will be some  $f$  for which these sets are disjoint. In general, one would expect  $2^{|Y|}$  basic functions would be needed to *witness the normality* of (the  $2^{|Y|}$  subsets of)  $Y$ . In TALL [1981] I proved for  $Y$ 's of cardinality  $\aleph_1$ :

**D.1. THEOREM.**

- (a) If  $\leq \aleph_1$  functions witness the normality of  $Y$ , then  $Y$  is separated.
- (b) Assuming a generalized Martin's Axiom (e.g. **BACH**), if  $< 2^{\aleph_1}$  functions witness the normality of  $Y$ , then  $Y$  is separated.
- (c) If  $2^{\aleph_0} < 2^{\aleph_1}$  and  $< 2^{\aleph_1}$  functions witness the normality of  $Y$ , then there is an uncountable separated subset of  $Y$ . □

**Question D3.** Is it consistent that there is a space  $X$  and a closed discrete **52.** ?  
 $Y$  such that  $< 2^{\aleph_1}$  (better,  $< 2^{\aleph_0}$ ) functions witness the normality of  $Y$ , but (every uncountable  $Z \subseteq Y$  is not separated)?

**Question D4.** Is **CH** equivalent to the assertion that whenever  $< 2^{\aleph_0}$  **53.** ?  
functions witness the normality of  $Y$ ,  $Y$  is separated?

**Question D5.** Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply that assertion? **54.** ?

(See WATSON [1985]) STEPRĀNS and WATSON proved that  $2^{\aleph_0} < 2^{\aleph_1} \leq \aleph_{\omega_1}$  implies countably paracompact separable spaces are collectionwise normal. (Note weakly collectionwise Hausdorff implies collectionwise normal for separable spaces.) Of course  $2^{\aleph_0} < 2^{\aleph_1}$  suffices if we replace countable paracompactness by normality.

**Question D6.** Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply countably paracompact separable (first **55.** ?  
countable?) spaces are collectionwise normal?

Both this and Problem D5 are related to the following long-open hard problem, which I believe is due to Laver.

**Question D7.** Is it consistent that there is an  $\mathcal{F} \subseteq {}^{\omega_1}\omega$ , with  $|\mathcal{F}| < 2^{\aleph_1}$ , **56.** ?  
such that  $\mathcal{F}$  dominates all functions from  $\omega_1$  to  $\omega$ ?

There is no such family if  $\text{cf}(2^{\aleph_0}) < \min(2^{\aleph_1}, \aleph_{\omega_1})$ , while the existence of such a family implies the existence of a measurable cardinal in an inner model (STEPRĀNS [1982], JECH and PRIKRY [1984]).

## E. Screenable and Para-Lindelöf Problems

*Screenability* (every open cover has a  $\sigma$ -disjoint open refinement) and para-Lindelöfness are not as well-behaved (or well-understood) as more familiar covering properties. Among the many open problems, I have chosen the ones that particularly interest me. I have already mentioned one: A4 above.

- ? 57. **Question E1.** *Is there a real example of a screenable normal space that's not collectionwise normal?*

There's an example under  $\diamond^{++}$  (RUDIN [1983]). Such an example would be a Dowker space since screenable normal spaces are collectionwise normal with respect to countably metacompact closed sets (TALL [1982], or see TALL [1984]).

- ? 58. **Question E2.** *Is there a consistent example of a normal space with a  $\sigma$ -disjoint base that's not collectionwise normal?*

Such spaces are of course screenable. Since they are first countable, there are no absolute examples unless strongly compact cardinals are inconsistent.

- ? 59. **Question E3.** *Is there a real example of a para-Lindelöf first countable space which in addition is*  
 (a) *regular but not paracompact,*  
 (b) *countably paracompact (and/or normal) but not paracompact.*

Under **MA** plus not **CH**, there is even a para-Lindelöf normal Moore space that's not metrizable (NAVY [19 $\infty$ ]). Similarly under **CH** (FLEISSNER [1982]). Without first countability, there exists a real example of a para-Lindelöf normal space which is not collectionwise normal (NAVY [19 $\infty$ ], or see FLEISSNER [1984]).

## F. Reflection Problems

The problems in this section all ask whether, if a proposition holds at  $\aleph_1$  or other small cardinal, it holds for all larger cardinals.

- ? 60. **Question F1.** *Is it consistent (assuming large cardinals) that if a first countable space has all its subspaces of size  $\leq \aleph_1$  metrizable, then it's metrizable?*

This is due to P. Hamburger. A non-reflecting stationary set of  $\omega$ -cofinal ordinals in  $\omega_2$  is a counterexample (HAJNAL and JUHÁSZ [1976]), so large cardinals are needed. By Lévy-collapsing a supercompact cardinal, DOW [19 $\infty$ ]

establishes consistency for spaces that in addition are locally of cardinality  $\leq \aleph_1$ .

The following problems were raised earlier.

**Question A6.** *Is it consistent that every  $\aleph_1$ -collectionwise normal Moore space is metrizable?*

A space is  $\aleph_1$ -collectionwise normal if any discrete collection of size  $\aleph_1$  can be separated. In TALL [19 $\infty$ b], assuming the consistency of a huge cardinal, I proved it consistent that  $\aleph_1$ -collectionwise normal Moore spaces of size  $\leq \aleph_2$  are metrizable. Assuming a not unreasonable axiom the consistency of which is, however, not currently known to follow from the usual large cardinal axioms, the cardinality restriction can be removed.

**Question C1.** *Is it consistent (assuming large cardinals) that every first countable  $\aleph_1$ -collectionwise Hausdorff space is collectionwise Hausdorff?*

This question is due to Fleissner. Again, in the Lévy model, the proposition holds for spaces of local cardinality  $\leq \aleph_1$  (SHELAH [1977]). The question here is whether countably closed forcing can separate an unseparated discrete collection in a first countable space.

**Question D1.** *Is it consistent (assuming large cardinals) that every first countable weakly  $\aleph_1$ -collectionwise Hausdorff space is weakly collectionwise Hausdorff?*

In TALL [19 $\infty$ b], from the consistency of a huge cardinal I proved the consistency of first countable weakly  $\aleph_1$ -collectionwise Hausdorff spaces being weakly  $\aleph_2$ -collectionwise Hausdorff. Using an axiom the consistency of which is not known to follow from the usual large cardinality axioms—but which is considerably weaker than one previously alluded to—and a result of WATSON [19 $\infty$ ], I can indeed get from  $\aleph_1$  to all larger cardinals.

DANIELS [1988] obtained a first countable weakly  $\aleph_1$ -collectionwise Hausdorff space that is not weakly  $\aleph_2$ -collectionwise Hausdorff, assuming **MA** plus  $2^{\aleph_0} = \aleph_2$ .

**Question F2.** *Is there a (real) example of a first countable space  $X$  such that  $X \times (\omega_1 + 1)$  is normal, but  $X$  is not paracompact?* **61. ?**

The hypothesis that  $X \times (\omega_1 + 1)$  is normal is equivalent to  $X$  being normal and  $\aleph_1$ -paracompact (Kunen, see PRZYMUSIŃSKI [1984]). (A space is  $\kappa$ -paracompact if every open cover of size  $\leq \kappa$  has a locally finite open refinement.) A non-reflecting stationary set of  $\omega$ -cofinal ordinals in  $\omega_2$  is again

a counterexample (TALL [19 $\infty$ b]). Assuming a huge cardinal, it's consistent that there's no first countable normal *hereditarily*  $\aleph_1$ -paracompact non-paracompact space of cardinality  $\leq \aleph_2$  (TALL [19 $\infty$ b]). It is also consistent from a huge that first countable  $T_2$   $\aleph_2$ -paracompact spaces of size  $\leq \aleph_3$  are paracompact (TALL [19 $\infty$ a]); thus the example called for must likely depend essentially on  $\omega_1$ . In view of the situation at  $\aleph_2$ , we can also ask without normality.

- ? 62. **Question F3.** *Is there a (real example of a) first countable  $\aleph_1$ -paracompact space that's not paracompact?*

### G. Countable Chain Condition Problems

- ? 63. **Question G1.** *Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply there is an  $S$ -space (or an  $L$ -space)?*

An  $S$ -space is a hereditarily separable regular space that's not hereditarily Lindelöf. An  $L$ -space is a hereditarily Lindelöf space that's not hereditarily separable. See ROITMAN [1984] for a survey on the subject. The question is whether **CH** can be weakened to  $2^{\aleph_0} < 2^{\aleph_1}$ .

- ? 64. **Question G2.** *Is every first countable  $T_2$  space satisfying the Šanin condition separable?*

A space satisfies the *Šanin condition* if it has caliber  $\kappa$  for every regular uncountable  $\kappa$ . If the density of a space with countable tightness satisfying the Šanin condition is less than  $\aleph_\omega$ , it is countable (TALL [1974a]). Thus  $2^{\aleph_0} < \aleph_\omega$  implies first countable  $T_2$  spaces satisfying the Šanin condition are separable. Compare this with the facts that **CH** implies first countable  $T_2$  spaces with caliber  $\aleph_1$  are separable (EFIMOV [1969], or see TALL [1974a]), and that **MA** plus not **CH** implies there is a first countable  $T_2$  space with caliber  $\aleph_1$  that is not separable (TALL [1977a]).

- ? 65. **Question G3.** *Find interesting necessary and sufficient conditions for the inclusion ordering on a topology to include a Souslin tree.*

See KUREPA [1967], RUDIN [1952], and TALL [1976b]. In the latter, it is shown for example that it's sufficient for the space to satisfy the countable chain condition, be  $T_1$  and locally connected, and to have every first category set nowhere dense.

- ? 66. **Question G4.** *Does there exist a real example of a first countable hereditarily normal countable chain condition space which is not hereditarily separable?*

Under **CH**, there is a first countable  $L$ -space (VAN DOUWEN, TALL and WEISS [1977]) and hence an example. If  $2^{\aleph_0} < 2^{\aleph_1}$ , such an example would yield a first countable  $L$ -space (TALL [1974b]). A model in which there were no such space would both have to have no first countable  $L$ -space and yet have every first countable normal space be collectionwise Hausdorff (TALL [1974b])—a very curious combination indeed!

**Question G5.** *Is it consistent with **GCH** that precaliber  $\aleph_1$  implies precal-* **67. ?**  
*iber  $\aleph_{\omega+1}$ ?*

Spaces with precaliber  $\aleph_1$  do have precaliber  $\aleph_{\omega+1}$  if one assumes the axiom alluded to in §F (TALL [1980b]).

## H. Real Line Problems

The rational sequence topology (see STEEN and SEEBACH [1978]), the Pixley-Roy topology (see e.g., VAN DOUWEN [1977]), and the density topology (see e.g., TALL [1976a]) are all strengthenings of the usual topology on the real line. For the first two, there is a characterization of normal subspaces in terms of their properties as sets of reals. By the same proof (BING [1951], TALL [1977b]) as for the tangent disk space, a set  $X$  of reals is normal in the rational sequence topology iff it's a  $Q$ -set in the usual topology, while  $X$  is normal in the Pixley-Roy topology iff it's a strong  $Q$ -set (RUDIN [1983]).

**Question H1.** *Characterize the normal subspaces of the density topology.* **68. ?**

In TALL [1978] I obtained the following partial result:

**H.1. THEOREM.** *If  $Y$  is a normal subspace of the density topology, then  $Y = S \cup T$ , where  $S$  is generalized Sierpiński,  $T$  is a nullset such that  $\bar{Z} \cap T = \emptyset$  for every nullset  $Z \subseteq S$ , every subset of  $Z$  is the intersection of  $Z$  with a Euclidean  $F_{\sigma\delta}$ .*  $\square$

(A set  $S$  of reals is *generalized Sierpiński* if its intersection with every nullset has cardinality less than continuum.) The closure referred to is in the density topology, so that even if the converse were proved, the resulting characterization would not be quite satisfactory. On the other hand, under **MA** plus not **CH**, one can construct a generalized Sierpiński  $S$  (namely one of outer measure 1) and a nullset  $T$  disjoint from  $S$  such that  $S \cup T$  is not normal (since  $|T| = 2^{\aleph_0}$ ) and yet every null  $Z \subseteq S$  is a  $Q$ -set.

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## CHAPTER 4

### Problems I wish I could Solve

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## 1. Introduction

This assortment is a list of problems that I worked on between 1979 and 1989 which I failed to solve. Some of the problems are due to other topologists and set theorists and I have attributed them when references are available in the literature. An earlier list appeared in 1984 in the Italian journal, *Rend. Circ. Mat. Palermo*. This list was entitled “Sixty questions on regular non-paracompact spaces”. Thirteen of these questions have since been answered (we shall give the numbering in that earlier paper in each case).

- CARLSON, in [1984], used Nyikos’ solution from NYIKOS [1980] of the normal Moore space problem to show that if it is consistent that there is a weakly compact cardinal then it is consistent that normal Moore spaces of cardinality at most  $2^{\aleph_0}$  are metrizable. This solved Palermo #11.
- In [1980a] BALOGH showed at the STACY conference at York University that assuming the consistency of the existence of a supercompact cardinal, it is consistent that normal locally compact spaces are collectionwise normal thus solving Palermo #16. Tall had earlier established this result for spaces of cardinality less than  $\beth_\omega$ . See Tall’s B1.
- In [1985] DANIELS and GRUENHAGE constructed a perfectly normal locally compact collectionwise Hausdorff space under  $\diamond^*$  which is not collectionwise normal, thus answering Palermo #22, Palermo #23 and Palermo #24 in one blow.
- Balogh showed that under  $V = L$  countably paracompact locally compact spaces are collectionwise Hausdorff and that under  $V = L$  countably paracompact locally compact metacompact spaces are paracompact thus answering Palermo #26 and giving a partial solution to Palermo #28.
- Burke showed that under **PMEA**, countably paracompact Moore spaces are metrizable thus solving a famous old problem and incidentally answering Palermo #30, Palermo #33 and Palermo #34.
- It turned out that Palermo #37 and Palermo #47 were somewhat ill-posed since Fleissner’s **CH** space already in existence at that time answered both in its **ZFC** version by being a para-Lindelöf metacompact normal space of character  $2^{\aleph_0}$  which is not collectionwise normal.
- Daniels solved Palermo #56 by showing that in **ZFC** the Pixley-Roy space of the co-countable topology on  $\omega_1$  is collectionwise Hausdorff (it was her question to begin with).

## 2. Normal not Collectionwise Hausdorff Spaces

- ? **69. Problem 1.** (Palermo #2) Does **CH** imply that there is a normal not collectionwise Hausdorff space of character  $\aleph_2$ ?
- ? **70. Problem 2.** (Palermo #4) Does  $\neg$ **CH** imply that there is a normal not collectionwise Hausdorff space of character  $2^{\aleph_0}$ ?

A natural question which has never been asked explicitly but has occupied a huge amount of thought is “What is the least character of a normal space which is not collectionwise Hausdorff?”. There are many independence results here so we ask only that this number be calculated under each possible cardinal arithmetic. The above are the two simplest cases.

These questions point out the two kinds of consistent theorems which we have for getting normal spaces of small character to be collectionwise Hausdorff. One is the  $V = L$  argument which requires character  $\leq \aleph_1$ . The other is the Cohen real or **PMEA** argument which requires character less than  $2^{\aleph_0}$ . To answer either of the above questions negatively would thus require, I think, a new kind of consistency proof and that would certainly be interesting—and challenging. To answer either of the above questions positively would be simply astounding.

- ? **71. Problem 3.** Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply that there is no family of size less than  $2^{\aleph_1}$  which generates the power set of  $\omega_1$  under countable unions?
- ? **72. Problem 4.** Is  $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$  consistent with the existence of a family of size  $\aleph_2$  which generates the power set of  $\omega_1$  under countable unions?

These questions arose in a paper (WATSON [1988a]) which studied possible methods of lowering the character of Bing’s space directly. That paper showed that the existence of such families of cardinality less than  $2^{\aleph_1}$  implies the existence of a normal space of character less than  $2^{\aleph_1}$  which is not collectionwise Hausdorff. I am most interested in these questions however purely as problems in combinatorial set theory.

JURIS STEPRĀNS in [1982], and independently JECH and PRIKRY in [1984] showed that the answer to the first question is yes so long as either  $2^{\aleph_0} < \aleph_{\omega_1}$  holds or else  $2^{\aleph_0} = \kappa, (\kappa^+)^L < 2^{\aleph_1}$  and the covering lemma over  $L$  is true (note the connection with problem 26). Thus a solution in either direction would be quite startling.

The other case of  $2^{\aleph_0} = 2^{\aleph_1}$  brings to mind **MA** $_{\aleph_1}$  which, however, implies that the minimal size of such a family is  $2^{\aleph_1}$ . The model where  $\aleph_\omega$  Cohen reals are added to a model of **GCH** has such a subfamily of cardinality  $\aleph_\omega < 2^{\aleph_0}$ . This is a rather unsatisfying result and the second question is designed

to exploit this. This question probably has something to do with Kurepa's hypothesis.

**Problem 5.** (Palermo #7) *Is there an axiom which implies that first countable normal spaces are  $\aleph_1$ -collectionwise Hausdorff, which follows from the product measure extension axiom, from  $\diamond^*$ , from  $\diamond$  for stationary systems and which holds in the reverse Easton model?* **73. ?**

**Problem 6.** (Palermo #5) *If  $\diamond_S$  holds for each stationary  $S \subset \omega_1$ , then are normal first countable spaces  $\aleph_1$ -collectionwise Hausdorff?* **74. ?**

The first problem was proposed to the author by Frank Tall. He wanted to unify the various proofs (TALL [1977], FLEISSNER [1974], SHELAH [1979], NYIKOS[1980]) of the consistency of the statement that first countable normal spaces are collectionwise Hausdorff. I was unable to solve this question but wrote a paper (WATSON [1984]) in which I introduced an axiom  $\Phi$  which implies that first countable normal spaces are  $\aleph_1$ -collectionwise Hausdorff, which follows from  $\diamond^*$  and from  $\diamond$  for stationary systems. It thus provided a unified (and simple) proof for both Fleissner's result and Shelah's result. The referee noted that  $\Phi$  held in a forcing model similar to the reverse Easton model but I never checked whether it held in that reverse Easton model used by Tall in 1968 to show the consistency of normal Moore spaces of cardinality less than  $\aleph_{\omega_1}$  being metrizable. I was also never able to determine whether the product measure extension axiom used by Nyikos implies  $\Phi$ . I am still interested in knowing whether this (or another) axiom can unify these four proofs. An observer from outside the normal Moore space fraternity might feel that this is a somewhat esoteric question but the fact of the matter is that the consistency of normal first countable spaces being  $\aleph_1$ -collectionwise Hausdorff will remain of interest in the decades to come and a single proof would enhance our understanding of the set theoretic nature of this property.

The second problem is an attempt to ascertain why  $\diamond$  is not enough. Shelah's model in DEVLIN and SHELAH [1979] in which there is a non-metrizable normal Moore space satisfies  $\diamond$  but exploits a stationary set on which  $\diamond$  does not hold. It is that result together with the two consistent theorems of FLEISSNER [1974] and SHELAH [1979] which give rise to this desperate attempt to figure out what is going on with  $\diamond$ . After all, it is Fleissner who created  $\diamond$  for stationary systems so this is a question about the nature of  $\diamond$ -principles, not really a question about general topology at all. The Easton model can also be included in problem 5 (see TALL [1988]).

**Problem 7.** (Fleissner; Palermo #57; Tall's C1) *Does ZFC imply that there is a first countable  $\aleph_1$ -collectionwise Hausdorff space which fails to be collectionwise Hausdorff?* **75. ?**

This is a central question on reflection. It has been much worked on. In [1977] SHELAH showed that the answer is yes for locally countable spaces if a supercompact cardinal is Lévy-collapsed to  $\aleph_2$ . On the other hand  $E(\omega_2)$  is enough to construct a counterexample. Thus large cardinals are needed to establish a consistency result if indeed it is consistent. In [1977b] Fleissner has conjectured that Lévy-collapsing a compact cardinal to  $\aleph_2$  will yield a model in which first countable  $\aleph_1$ -collectionwise Hausdorff spaces are collectionwise Hausdorff. As in the pursuit of problem 11, far more effort has gone into obtaining a consistency result than has gone into trying to construct a counterexample. The conventional wisdom is that the set-theoretic technology is simply not ready yet and that we just have to wait. I conjecture that there is a **ZFC** example and that we really have to look somewhere other than ordinals for large first countable spaces which are not paracompact.

- ? 76. **Problem 8.** (Palermo #1) Does **GCH** imply that normal first countable  $\aleph_1$ -collectionwise Hausdorff spaces are collectionwise Hausdorff?

In [1977], SHELAH constructed two consistent examples of a normal Moore space which is  $\aleph_1$ -collectionwise Hausdorff but which fails to be  $\aleph_2$ -collectionwise Hausdorff. The first satisfied  $2^{\aleph_0} = \aleph_1$  and  $2^{\aleph_1} = \aleph_3$  and the second satisfied  $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ . These cardinal arithmetics make the question quite natural. FLEISSNER asked this question in [1977b].

- ? 77. **Problem 9.** (Palermo #6; Tall's # C2) If  $\kappa$  is a strong limit cardinal, then are normal first countable spaces which are  $< \kappa$ -collectionwise Hausdorff,  $\kappa$ -collectionwise Hausdorff?

In WATSON [198a], we showed that the answer is yes for strong limit cardinals of countable cofinality even without normality (answering a question from FLEISSNER [1977c]; in that paper Fleissner had proved the same result assuming **GCH**). Aside from that nothing seems to be known about getting *normal* first countable spaces where collectionwise Hausdorff first fails at a limit cardinal. In SHELAH [1977], a consistent counterexample which is not normal was obtained.

- ? 78. **Problem 10.** (Palermo #3) Does **CH** imply that normal first countable spaces are weakly  $\aleph_2$ -collectionwise Hausdorff?

The motive for this question is that the only examples of spaces which fail to be weakly collectionwise Hausdorff are those which do not use pressing-down arguments but rather simply  $\Delta$ -system arguments. Such examples always seem to use an identification of the unseparated set with a subset of the real line. Of course, there is no reason why it should be this way but I spent a lot of time trying to construct an example some other way and got nowhere.

On the other hand, a theorem would be quite surprising and would provide convincing evidence that the Moore plane is canonical.

### 3. Non-metrizable Normal Moore Spaces

**Problem 11.** (Palermo #8; Tall's A1) *Does  $2^{\aleph_0} = \aleph_2$  imply the existence of a non-metrizable normal Moore space?* **79. ?**

I only add that it is dangerous to spend 95% of the effort on a question trying to prove it in one direction. Very little effort has gone into trying to modify FLEISSNER'S [1982b, 1982a] construction of a non-metrizable Moore space from the continuum hypothesis. In fact, there's probably only about two or three people who really understand his construction (I am not one of them). If the conventional wisdom that collapsing a large cardinal should get the consistency of the normal Moore space conjecture with  $2^{\aleph_0} = \aleph_2$ , then why hasn't it been done?

**Problem 12.** (Palermo #9; Tall's A3) *Does the existence of a non-metrizable normal Moore space imply the existence of a metacompact non-metrizable normal Moore space?* **80. ?**

**Problem 13.** (Palermo #12) *Does the existence of a non-metrizable normal Moore space imply the existence of a normal Moore space which is not collectionwise normal with respect to metrizable sets?* **81. ?**

The reason this question remains of interest is that a tremendous amount of effort has gone into obtaining partial positive results. RUDIN and STARBIRD [1977] and NYIKOS [1981] both obtained some technical results of great interest. Nyikos showed, in particular, that if there is a non-metrizable normal Moore space, then there is a metacompact Moore space with a family of closed sets which is normalized but not separated. In WATSON [1988a], it is shown that if there is a non-metrizable normal Moore space which is non-metrizable because it has a nonseparated discrete family of closed metrizable sets then there is a metacompact non-metrizable normal Moore space. This means that if you believe in a counterexample you had better solve problem 32 first! If you believe in a theorem as Rudin and Starbird and Nyikos did, you have a lot of reading to do. I think there is a counterexample.

**Problem 14.** (Palermo #10; see Tall's A2 and A4) *Does the existence of a non-metrizable normal Moore space imply the existence of a para-Lindelöf non-metrizable normal Moore space?* **82. ?**

The normal Moore space problem enjoyed lots of consistent counterexamples long before para-Lindelöf raised its head. However in [1981] CARYN



NAVY managed to show how to use **MA** +  $\neg$ **CH** to get a para-Lindelöf example. FLEISSNER's **CH** example of a non-metrizable normal Moore space from [1982b] turns out to be para-Lindelöf (after all, he was modifying Navy's example). The only part that is not clear is whether the singular cardinals hypothesis can get you a para-Lindelöf counterexample to the normal Moore space conjecture. Probably the best positive result that can be hoped for is to show that Fleissner's **SCH** counterexample can be modified to be para-Lindelöf. That would be a good result since a deep understanding of Fleissner's space would be required and that space does have to be digested. A negative result is more likely and would really illustrate the difference between FLEISSNER's **CH** example and his **SCH** example from [1982b] and [1982a] and that would be valuable.

- ? 83. **Problem 15.** (Palermo #14) *Are Čech-complete locally connected normal Moore spaces metrizable?*
- ? 84. **Problem 16.** (Palermo #13) *Are normal Moore spaces sub-metrizable?*

#### 4. Locally Compact Normal Spaces

- ? 85. **Problem 17.** (Palermo #15; Tall's B2) *Are locally compact normal metacompact spaces paracompact?*

I have spent a lot of time on this question. It was originally stated by TALL [1974] although it would be difficult to appreciate a result proved by ARHANGEL'SKIĬ in [1971] without thinking of it. In WATSON [1982], it was shown that, under  $V = L$ , locally compact normal metacompact spaces are paracompact. However, the techniques for constructing examples under **MA** +  $\neg$ **CH** (the usual place to look) did not seem to provide any way of getting locally compact spaces and metacompact spaces at the same time. In [1983] PEG DANIELS showed in **ZFC** that locally compact normal boundedly metacompact spaces are paracompact. This surprising result raised the hopes of everyone (except Frank Tall) that the statement in question might actually be a theorem in **ZFC**. It hasn't worked out that way so far. Boundedly metacompact really is different from metacompact although most examples don't show this. If there is a theorem here in **ZFC** it would be an astounding result. If there is an example in some model (as I think there is), it would require a deeper understanding of the Pixley-Roy space than has so far existed. Either way this is a central question. Another related question with a (much?) higher probability of a consistent counterexample is problem 18.

**Problem 18.** (Palermo #28) *Are countably paracompact locally compact metacompact spaces paracompact?* **86. ?**

**Problem 19.** *Does  $\mathbf{MA}_{\aleph_1}$  imply that normal locally compact meta-Lindelöf spaces are paracompact?* **87. ?**

This problem is related to problem 17. We mentioned that no consistent example of a locally compact normal metacompact space which is not paracompact is known. Actually, even constructing a consistent example of a locally compact normal meta-Lindelöf space which is not paracompact is non-trivial. We constructed such a space in WATSON [1986] but the proof makes essential use of a compact hereditarily Lindelöf space which is not hereditarily separable. These spaces do not exist under  $\mathbf{MA} + \neg\mathbf{CH}$ . Thus what this question does is up the ante. Anyone except Frank Tall working on a counterexample to problem 17 is probably using Martin's axiom. So what we are saying is "you'll never do it—bet you can't even get meta-Lindelöf". Of course, maybe there is an example but that would require a completely different approach to getting meta-Lindelöf together with locally compact and normal. That would be just as interesting to me because I tried for a long time to get the results of WATSON [1986] using Martin's axiom. Of course, such an example cannot be done in  $\mathbf{ZFC}$  because of BALOGH's result from [1980b] that, under  $V = L$ , locally compact normal meta-Lindelöf spaces are paracompact.

**Problem 20.** (Palermo #17) *Does  $\mathbf{ZFC}$  imply that there is a perfectly normal locally compact space which is not paracompact?* **88. ?**

This is my favorite question. If there is an example then what a strange creature it must be. A series of results running from MARY ELLEN RUDIN's result from [1979] that under  $\mathbf{MA} + \neg\mathbf{CH}$  perfectly normal manifolds are metrizable runs through results of LANE [1980] and GRUENHAGE [1980] to culminate in a result of Balogh and Junnila that, under  $\mathbf{MA} + \neg\mathbf{CH}$ , perfectly normal locally compact collectionwise Hausdorff spaces are paracompact. On the other hand, under  $V = L$ , normal locally compact spaces are collectionwise Hausdorff. This means that, if there is in  $\mathbf{ZFC}$  a perfectly normal locally compact space which is not paracompact, then under  $\mathbf{MA} + \neg\mathbf{CH}$  it is not collectionwise Hausdorff but that under  $V = L$  it *is* collectionwise Hausdorff. Now there are two ways this can be done. First, by stating a set-theoretic condition, using it to construct one space and then using its negation to construct another space.  $2^{\aleph_0} = 2^{\aleph_1}$  is the only worthwhile axiom I know whose negation is worth something (although see WEISS [1975] and [1977]). Second, by constructing a space whose collectionwise Hausdorffness happens to be independent. This is fine but the fragment of  $V = L$  is small enough to force with countably closed forcing so the definition of such a space better depend pretty strongly on what the subsets of *both*  $\omega$  and  $\omega_1$  are. There

are however examples in most models. Under **CH**, the Kunen line (JUHÁSZ, KUNEN and RUDIN [1976]) is an example of a perfectly normal locally compact  $S$ -space which is not paracompact. Under **MA** +  $\neg\mathbf{CH}$ , the Cantor tree with  $\aleph_1$  branches is an example.

On the other hand, a consistent theorem would be amazing. To get it by putting together the two consistency proofs would be quite hard. The **MA** +  $\neg\mathbf{CH}$  result uses both  $\mathfrak{p} = \mathfrak{c}$  and the non-existence of a Suslin line so that's a lot of set theory. The  $V = L$  result can be done without **CH** but then you have to add weakly compact many Cohen reals or something like that (TALL [1984]). I haven't tried this direction at all, although set-theorists have. It looks impossible to me.

- ? 89. **Problem 21.** (Palermo #19) *Does the existence of a locally compact normal space which is not collectionwise Hausdorff imply the existence of a first countable normal space which is not collectionwise Hausdorff?*

I think the answer is yes. This belief stems from WATSON [1982] where an intimate relation between the two existence problems was shown. The only thing that is missing is the possibility that there might be a model in which normal first countable spaces are  $\aleph_1$ -collectionwise Hausdorff and yet that in that model there is a normal first countable space which fails to be  $\aleph_2$ -collectionwise Hausdorff. This seems unlikely, though it is open, and yet the question might be answered positively in any case (see SHELAH [1977]). A counterexample would have been more interesting before Balogh showed the consistency of locally compact normal spaces being collectionwise normal. However it may yet provide a clue to an answer to problem 17.

- ? 90. **Problem 22.** (Palermo #18; Tall's B5) *Are large cardinals needed to show that normal manifolds are collectionwise normal?*

In 1986, Mary Ellen Rudin built a normal manifold which is not collectionwise normal under the axiom  $\diamond^+$ . Meanwhile, PETER NYIKOS [1989] has shown that if the existence of a weakly compact cardinal is consistent then it is consistent that normal manifolds are collectionwise normal. The most likely solution to this problem is doing it on the successor of a singular cardinal where  $\diamond$ -like principles tend to hold unless there are large cardinals (see FLEISSNER [1982a]). A consistent theorem that normal manifolds are collectionwise normal probably means starting from scratch, where so many have started before.

- ? 91. **Problem 23.** (Palermo #21) *Does ZFC imply that normal manifolds are collectionwise Hausdorff?*

Mary Ellen Rudin's example in problem 22 is collectionwise Hausdorff. This can be deduced from the fact that under  $V = L$  normal first countable spaces

are collectionwise Hausdorff (FLEISSNER [1974]). Thus no example of any kind has yet been demonstrated to exist and all we have are a few consistent theorems.

**Problem 24.** (Palermo #20; Tall's B3) *Are normal locally compact locally connected spaces collectionwise normal?* **92. ?**

REED and ZENOR showed in [1976] that locally connected locally compact normal Moore spaces are metrizable in **ZFC**. ZOLTAN BALOGH showed in [19 $\infty$ d] that connected locally compact normal submeta-Lindelöf spaces are paracompact under  $2^\omega < 2^{\omega_1}$ . BALOGH showed in [19 $\infty$ c] that locally connected locally compact normal submeta-Lindelöf spaces are paracompact in **ZFC**. GRUENHAGE constructed in [1984] a connected locally compact non-metrizable normal Moore space under **MA** +  $\neg$ **CH**. Problem 24 attempts to determine whether covering properties have anything central to do with these phenomena.

**Problem 25.** (Palermo #25) *Does **ZFC** imply that there is a normal extremally disconnected locally compact space which is not paracompact?* **93. ?**

In [1978] KUNEN and PARSONS showed that if there is a weakly compact cardinal then there is a normal extremally disconnected locally compact space which is not paracompact. In [1977] KUNEN showed that there is an normal extremally disconnected space which is not paracompact. This is a great question. I suspect that useful ideas may be found in WATSON [19 $\infty$ e] where normal spaces which are not collectionwise normal with respect to extremally disconnected spaces are constructed.

## 5. Countably Paracompact Spaces

**Problem 26.** (Palermo #27; Tall's D6) *Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply that separable first countable countably paracompact spaces are collectionwise Hausdorff?* **94. ?**

In [1937] JONES showed that, under  $2^{\aleph_0} < 2^{\aleph_1}$  normal separable spaces have no uncountable closed discrete sets (and thus that separable normal Moore spaces are metrizable). In [1964] HEATH showed that, in fact, the existence of a normal separable space with an uncountable closed discrete set is equivalent to  $2^{\aleph_0} = 2^{\aleph_1}$ .

These results blend well with the ongoing problem of determining the relation between normality and countable paracompactness. The normal separable space with an uncountable closed discrete set is, in fact, countably paracompact and so FLEISSNER [1978], PRZYMUSIŃSKI [1977] and REED [1980] asked whether the existence of a countably paracompact separable space with

an uncountable closed discrete set is also equivalent to  $2^{\aleph_0} = 2^{\aleph_1}$ . FLEISSNER [1978] fueled this suspicion with a proof that the continuum hypothesis implies that countably paracompact separable spaces have no uncountable closed discrete set. In WATSON [1985], we showed that the existence of such a space is equivalent to the existence of a dominating family in  ${}^{\omega_1}\omega$  of cardinality  $2^{\aleph_0}$ . This was somewhat satisfying since the equivalence of the existence of such a family with  $2^{\aleph_0} < 2^{\aleph_1}$  was known as an open problem in set theory. In 1983, Steprāns and Jech and Prikry independently showed that if the continuum is a regular cardinal and there are no measurable cardinals in an inner model, then the latter equivalence holds. The general set-theoretic problem remains open.

Back in general topology, what about first countable spaces? The examples that are used in all these results have character equal to the continuum. One expects first countability to be a big help but so far it seems useless in this context. The drawback to this question is that if the answer is no, one first has to solve the set-theoretic question and then figure out how to lower the character from the continuum to  $\aleph_0$ . Getting the character down is always interesting. On the other hand, if there is a theorem, that might involve a hard look at the weak version of  $\diamond$  invented by Keith Devlin (DEVLIN and SHELAH [1978]) and lots of people would be interested in an essential use of that axiom.

? 95. **Problem 27.** (Palermo #31) *Does  $2^{\aleph_0} < 2^{\aleph_1}$  imply that special Aronszajn trees are not countably paracompact?*

This question is quite attractive to some precisely because it is not a topological question. It is however a natural question about the structure of Aronszajn trees.

In [1980] FLEISSNER noted that the proof in FLEISSNER [1975] that under  $\mathbf{MA} + \neg\mathbf{CH}$ , special Aronszajn trees are normal could be modified to show that under  $\mathbf{MA} + \neg\mathbf{CH}$  special Aronszajn trees are countably paracompact. We showed in WATSON [1985], that under  $(\forall \text{ stationary } S \subset \omega_1) \diamond_S$ , special Aronszajn trees are not countably paracompact. This proof, however, was implicit in FLEISSNER [1975, 1980]. Fleissner had shown that under  $V = L$ , special Aronszajn trees are not normal but the key result that gave rise to the present question is the proof by SHELAH and DEVLIN [1979], that  $2^{\aleph_0} < 2^{\aleph_1}$  implies that special Aronszajn trees are not normal. FLEISSNER [1980] cites a result of Nyikos that normal implies countably paracompact in trees. This means that the present statement in question is weaker than the Devlin-Shelah result.

? 96. **Problem 28.** (Palermo #32) *If the continuum function is one-to-one and  $X$  is a countably paracompact first countable space, then is  $e(X) \leq c(X)$ ?*

This is just an attempt to conjecture a form of Shapirovskii's improvement of Jones' lemma at each cardinal. Recall that JONES proved in [1937] that separable normal spaces have no uncountable closed discrete set under  $2^{\aleph_0} < 2^{\aleph_1}$ . That proof was sharpened by Shapirovskii to show that in normal first countable spaces  $2^{\aleph_0} < 2^{\aleph_1}$  implies that closed discrete sets of cardinality  $\aleph_1$  have a subset of cardinality  $\aleph_1$  which can be separated by disjoint open sets. Thus if there are no disjoint families of more than  $\aleph_0$  many open sets then there are no closed discrete sets of cardinality  $\aleph_1$ . That proof was observed to extend to higher cardinals by FRANK TALL [1976] and neatly summarized in the form: if the continuum function is one-to-one and  $X$  is a normal first countable space then  $e(X) \leq c(X)$ . The question is just asking whether this nice statement about cardinal functions applies equally to countably paracompact first countable spaces. I believe that it does.

**Problem 29.** (Palermo #29) Does  $\diamond^*$  imply that countably paracompact first countable spaces are  $\aleph_1$ -collectionwise Hausdorff? **97. ?**

This question is just something that I expected would have a positive answer but couldn't make any headway on. SHELAH [1979] showed that  $\diamond^*$  implies that normal first countable spaces are  $\aleph_1$ -collectionwise Hausdorff and every other separation theorem which used normality eventually was extended to countable paracompactness (see WATSON [1985] and Burke's use of **PMEA**). The real question here is vague: "is there a distinction between the separation properties of normality and countable paracompactness". A negative answer to the specific question would answer the vague question quite clearly. A positive answer would get a little closer to the combinatorial essence underlying separation and that would be a worthy accomplishment.

**Problem 30.** Does the existence of a countably paracompact non-normal Moore space imply the existence of a normal non-metrizable Moore space? **98. ?**

This question was first asked by WAGE [1976] since, in that paper, he showed the converse to be true. All the available evidence indicates that the answer is yes. PETER NYIKOS [1980] showed that **PMEA** implies that normal Moore spaces are metrizable. This was later extended in a non-trivial way, by Dennis Burke, who showed that, under **PMEA**, countably paracompact Moore spaces are metrizable.

**Problem 31.** (Palermo #59) Does **ZFC** imply that collectionwise Hausdorff  $\omega_1$ -trees are countably paracompact? **99. ?**

## 6. Collectionwise Hausdorff Spaces

- ? 100. **Problem 32.** (Palermo #38) *Is there a normal not collectionwise normal space which is collectionwise normal with respect to collectionwise normal sets?*

This apparently frivolous main question is intended to be a specific version of a more serious question (first asked in WATSON [1988b]): Characterize those spaces  $Y$  and categories  $\mathcal{C}$  for which there exists a normal space which is collectionwise normal with respect to discrete families of sets in  $\mathcal{C}$  but not collectionwise normal with respect to copies of  $Y$ . This tries to get at the heart of many constructions like those in WATSON [19 $\infty$ e]. A less extreme question is: Characterize those spaces  $Y$  for which there exists a normal collectionwise Hausdorff space which is not collectionwise normal with respect to copies of  $Y$ . The main question is worth solving. If the answer is no, then I would be amazed and it would solve problem 12. If the answer is yes, then I think we would be getting at the heart of a topic on which I have spent a great deal of time (WATSON [19 $\infty$ e] is devoted to establishing partial results).

- ? 101. **Problem 33.** (Palermo #60) *Does **ZFC** imply that there is a collectionwise Hausdorff Moore space which is not collectionwise normal with respect to compact sets?*
- ? 102. **Problem 34.** (Palermo #39) *Is there a normal collectionwise Hausdorff space which is not collectionwise normal with respect to  $\aleph_1$  many compact sets?*
- ? 103. **Problem 35.** (Palermo #40) *Is it consistent that there is a normal first countable collectionwise Hausdorff space which is not collectionwise normal with respect to compact sets?*

In an unpublished result from 1980, Fleissner and Reed constructed, by using a measurable cardinal, a regular collectionwise Hausdorff space which is not collectionwise normal with respect to compact sets. In [1983] MIKE REED constructed in **ZFC** a collectionwise Hausdorff first countable regular space which is not collectionwise normal with respect to compact metric sets. He also obtained a collectionwise Hausdorff Moore space which is not collectionwise normal with respect to compact metric sets under the continuum hypothesis or Martin's axiom and asked the first question. The answer could well turn out to be yes since normality is not required and of course that would be preferable to Reed's results. If the answer is no, that is more interesting because the proof would penetrate into the manner in which the closed unit interval can be embedded in a Moore space and that would be quite exciting. In WATSON [19 $\infty$ e], an example was constructed of a normal collectionwise

Hausdorff space which is not collectionwise normal with respect to copies of  $[0, 1]$ . In that example, the proof of not collectionwise normality is not a  $\Delta$ -system argument but rather a measure-theoretic argument. As a result, we have no hope of using that method to get an example in which  $\aleph_1$  many copies of  $[0, 1]$  cannot be separated unless there is a subset of the reals of cardinality  $\aleph_1$  which has positive measure. Thus an example answering the second question would have to be essentially different from that of WATSON [1900e] and I do not believe such an example exists. On the other hand, a theorem, under **MA** +  $\neg$ **CH** for example, would be quite interesting. The example of WATSON [1900e] is badly not first countable. Anyway, the only normal first countable collectionwise Hausdorff spaces which are not collectionwise normal are Fleissner's space of FLEISSNER [1976] and the ones based on Navy's space of NAVY [1981] and FLEISSNER [1982b, 1982a]. The first of these requires the unseparated sets to be badly non-compact. The second requires the unseparated sets to be non-separable metric sets. Neither of these constructions is going to be easily modified to a positive solution to the third question. I don't think such a consistent example exists—it's asking too much. On the other hand, a negative answer means that rare thing: a **ZFC** result!

**Problem 36.** (Palermo #41) *Does **ZFC** imply that normal first countable collectionwise Hausdorff spaces are collectionwise normal with respect to scattered sets?* **104. ?**

**Problem 37.** (Palermo #35) *Does  $V = L$  imply that normal first countable spaces are collectionwise normal with respect to separable sets?* **105. ?**

**Problem 38.** (Palermo #36) *Does  $V = L$  imply that normal first countable spaces are collectionwise normal with respect to copies of  $\omega_1$ ?* **106. ?**

The prototypes of normal collectionwise Hausdorff spaces which are not collectionwise normal are Fleissner's space of FLEISSNER [1976] and Navy's space of NAVY [1981]. The latter space has an unseparated discrete family of Baire spaces of weight  $\aleph_1$ . These Baire spaces are very non-scattered. On the other hand Fleissner's space has a unseparated discrete family of copies of the ordinal space  $\omega_1$ . These sets are scattered. The latter type has been successfully modified to be first countable: that is Fleissner's solution to the normal Moore space conjecture (FLEISSNER [1982b, 1982a]). The first question is trying to ask whether the former type can be modified to be first countable. This is a question which has great intrinsic interest. A consistent method which succeeds in lowering the character of the former prototype would undoubtedly be quite useful in many other contexts. A positive answer would be unthinkable. This first question is however mostly a question about my own inability to follow a proof in the literature. A paper (FLEISSNER [1982c]) has appeared which gives a negative answer to this question. The idea of this paper is very



clever and introduces an axiom which has since been used (RUDIN [1983]) to construct what is possibly the most clever example in general topology. However I have spent a great deal of effort trying to understand the proof in FLEISSNER [1982c]. I have conversed with the author who has suggested several changes. For various reasons I have been unable to locate anyone who has checked all of the details. It is undoubtedly the case that the proof is simply over my head but I just cannot follow it. In my stubborn fashion, I still want to know the answer to this first question. If these comments succeed in provoking someone to read FLEISSNER [1982c] and then to explain it to me, then they will have done both of us a favor, for they will have read an inspired paper and, in addition, set my mind at ease (in August 1989 Bill Fleissner circulated a corrigendum to that paper).

The second and third questions are follow-ups in my tribute to Fleissner's George (FLEISSNER [1976]), a normal collectionwise Hausdorff space which fails to be collectionwise normal with respect to copies of  $\omega_1$ . This space has been modified (WATSON [19 $\infty$ a]) to fail to be collectionwise normal with respect to separable sets ( $S$ -spaces actually), under suitable set-theoretic assumptions. The example of FLEISSNER [1982c] fails to be collectionwise normal with respect to copies of a space something like  $\omega_1$ . The second question asks: "Can a  $S$ -space (like Ostaszewski's space [1976]) be used?". The third question asks "Was it really necessary to use something different from  $\omega_1$ ?". FLEISSNER showed in [1977a] that it is consistent that normal first countable spaces are collectionwise normal with respect to copies of  $\omega_1$  by collapsing an inaccessible in a model of the constructible universe (see also DOW, TALL and WEISS [19 $\infty$ ]). I think that the techniques that one would have to develop in order to solve these questions would be useful in many areas of general topology, and thus worth the effort.

## 7. Para-Lindelöf Spaces

? 107. **Problem 39.** (Palermo #43) *Are para-Lindelöf regular spaces countably paracompact?*

This is the main open problem on para-Lindelöf spaces. The original question was whether para-Lindelöf was equivalent to paracompact— one more feather in the cap of equivalences of paracompactness established by Stone and Michael in the 1950s (see Burke's article in the handbook of Set-Theoretic Topology BURKE [1984]). This question was finally solved by Caryn Navy, a student of Mary Ellen Rudin, in NAVY [1981]. Her construction was a rather general one that permitted quite a lot of latitude; she obtained first countable ones under **MA** +  $\neg$ **CH** using the Moore plane, she obtained a **ZFC** example using Bing's space. FLEISSNER [1982b, 1982a] later modified this example to be a Moore space under the continuum hypothesis, thus solving

the normal Moore space conjecture. Certain properties seemed hard to get however. These difficulties each gave rise to questions which were listed in Navy's thesis. The main open problem listed above is due to the fact that all the constructions are intrinsically countably paracompact. I tried for a long time to build in the failure of countable paracompactness but each time para-Lindelöf failed as well. It may be useful to note that the whole idea of Navy's construction was to take Fleissner's space of FLEISSNER [1979] which was  $\sigma$ -para-Lindelöf but not paracompact and build in a way to "separate" the countably-many locally countable families so that one locally countable refinement is obtained. This way was normality. No other way of getting para-Lindelöf is known. I don't think another way of getting para-Lindelöf is even possible— Navy's method looks quite canonical to me (although see WATSON [1900e]). I think the easiest way of getting a para-Lindelöf space which is not countably paracompact (at least consistently) is to iterate a normal para-Lindelöf space which is not collectionwise normal in an  $\omega$ -sequence (see WATSON [1900c]) and solve problem 40. I tried to do this but got bowled over by the details:

**Problem 40.** (Palermo #44) *Is there a para-Lindelöf Dowker space?* 108. ?

Another question which has not really been looked at but which I think is extremely important is:

**Problem 41.** (Palermo #42) *Are para-Lindelöf collectionwise normal spaces paracompact?* 109. ?

This was first asked by FLEISSNER and REED [1977]. So far, there are no ideas at all on how to approach this. Even the much weaker property of meta-Lindelöf creates big problems here:

**Problem 42.** (Palermo #58) *Is it consistent that meta-Lindelöf collectionwise normal spaces are paracompact?* 110. ?

In [1983] RUDIN showed that under  $V = L$ , there is a screenable normal space which is not paracompact. This space is collectionwise normal and meta-Lindelöf reducing our search to a **ZFC** example (although to use such a difficult space to solve this question consistently seems overkill— but I don't know of a simpler one).

**Problem 43.** (Palermo #46) *Are para-Lindelöf screenable normal spaces paracompact?* 111. ?

This question just throws in all the hardest properties and asks whether a theorem pops out. I predict a **ZFC** example will not be seen in this century (at

least not from me). If para-Lindelöf does indeed imply countably paracompact then such an example does not exist in any case, since normal screenable countably paracompact spaces are paracompact (NAGAMI [1955]).

? 112. **Problem 44.** (Palermo #45) *Are para-Lindelöf screenable spaces normal?*

There is an example of a screenable space which is not normal in BING [1951] but a lot of work has to be done to make it para-Lindelöf. Maybe that is the place to start. Keep in mind that para-Lindelöf spaces are strongly collectiowise Hausdorff (FLEISSNER and REED [1977]).

## 8. Dowker Spaces

The next few questions are **ZFC** questions about Dowker spaces. It's fairly easy to come up with a question about Dowker spaces. Just find a property that Mary Ellen Rudin's Dowker example in **ZFC** (RUDIN [1971]) does not have and ask if there is a Dowker space with that property. A lot can be done in particular models of **ZFC** to obtain very nice, well-behaved examples of Dowker spaces (see RUDIN [1955], JUHÁSZ, KUNEN and RUDIN [1976], DE CAUX [1976], WEISS [1981], BELL [1981] and RUDIN [1984, 1983]) but, in **ZFC**, there is only that one example around. I tried to construct another one in 1982 but only succeeded in getting one from a compact cardinal (WATSON [19 $\infty$ c]). On the one hand, this is worse than using **CH** or **MA** +  $\neg$ **CH** but on the other hand, postulating the existence of a compact cardinal has a different flavour than the other axioms. Anyway that example was scattered of height  $\omega$  and hereditarily normal thus giving rise to the next three questions:

? 113. **Problem 45.** (Palermo #48) *Does **ZFC** imply that there is a hereditarily normal Dowker space?*

? 114. **Problem 46.** (Palermo #55) *Does **ZFC** imply that there is a  $\sigma$ -discrete Dowker space?*

? 115. **Problem 47.** (Palermo #54) *Does **ZFC** imply that there is a scattered Dowker space?*

The next two questions have been around for a while and rest on the following pathological properties of Mary Ellen Rudin's example (RUDIN [1971]): It has cardinality and character  $(\aleph_\omega)^\omega$ .

? 116. **Problem 48.** (Palermo #50) *Does **ZFC** imply that there is a Dowker space of cardinality less than  $\aleph_\omega$ ?*

**Problem 49.** (Palermo #51; Rudin [1971]) *Does **ZFC** imply that there is a first countable Dowker space?* 117. ?

**Problem 50.** (RUDIN [1971]) *Is there a separable Dowker space?* 118. ?

In [1983], RUDIN showed that under  $V = L$ , there is a screenable normal space which is not paracompact. This space was quite difficult to construct. A **ZFC** example seems a long, long way off (although  $\diamond$  has been known to hold at large enough cardinals. On the other hand a consistent theorem would finish off this nearly forty year old question implicit in BING [1951]:

**Problem 51.** (Palermo #49) *Does **ZFC** imply that there is a screenable normal space which is not paracompact?* 119. ?

An even stronger property than screenable is that of having a  $\sigma$ -disjoint base. It remains completely open whether a normal space with a  $\sigma$ -disjoint base must be paracompact. The next question is conjectured to have a positive answer. This would start to clear up the mystery surrounding screenability and having a  $\sigma$ -disjoint base. A negative answer would require a good hard study of Rudin's space RUDIN [1983] and that is worthwhile anyway.

**Problem 52.** (Palermo #52) *Does **ZFC** imply that normal spaces with a  $\sigma$ -disjoint base are collectionwise normal (or paracompact)?* 120. ?

In reply to a question of Frank Tall, RUDIN [1983] showed that the existence of a screenable normal non-paracompact space implies the existence of a screenable normal non-collectionwise normal space. The next question asks whether collectionwise normality really is quite irrelevant.

**Problem 53.** (Palermo #53) *Does the existence of a screenable normal space which is not paracompact imply the existence of a screenable collectionwise normal space which is not paracompact?* 121. ?

## 9. Extending Ideals

If  $I$  is an ideal on  $X$  then  $I$  measures  $A$  if and only if  $A$  is a subset of  $X$  and either  $A \in I$  or  $X - A \in I$ . If an ideal  $I$  on  $X$  has the property that whenever  $\mathcal{A}$  is a family of  $\kappa$  many subsets of  $X$  there is a countably complete ideal which extends  $I$  and which measures each of the elements of  $\mathcal{A}$ , then we say that the ideal  $I$  is  $\kappa$ -extendible. We say that an ideal  $I$  is  $\kappa$ -completable if there is a proper ideal  $J$  which is  $\kappa$ -complete and which contains  $I$ . If an ideal  $I$  on  $X$  has the property that whenever  $\mathcal{A}$  is a family of  $\kappa$  many subsets of  $X$  there is a countably complete ideal which extends  $I$  and which measures at least  $\lambda$  many elements of  $\mathcal{A}$ , then we say that the ideal  $I$  is  $(\kappa, \lambda)$ -extendible.

The idea of investigating these questions is due to Frank Tall whose interest is responsible for all the questions in this section. In a paper with Steprāns (STEPRĀNS and WATSON [1986]), we investigated many problems on the  $\kappa$ -extendibility and the  $(\kappa, \lambda)$ -extendibility of ideals. Many of these questions have remained open.

We showed in STEPRĀNS and WATSON [1986] that if an ideal  $I$  is  $(\kappa^\omega)^+$ -completable then  $I$  is  $\kappa$ -extendible. We showed that the converse is true unless  $\kappa$  is greater or equal to either a weakly compact cardinal or something called a  $\Xi$ -cardinal (in particular an ideal  $I$  is  $\omega$ -extendible if and only if  $I$  is  $(2^{\aleph_0})^+$ -completable). We also showed that, if there are no measurable cardinals in an inner model and  $\kappa$  is not a  $\Xi$ -cardinal, then the  $\kappa$ -extendibility of an ideal is directly dependent on the completeness of the ideal. However, if  $\kappa$  is a  $\Xi$ -cardinal and there are measurable cardinals in an inner model, then the best we can say is that  $\kappa^+$ -completable implies  $\kappa$ -extendible which implies  $\kappa$ -completable. We were able to show that adding ineffably-many Cohen reals produced a model in which there is a  $\kappa$ -extendible ideal which is not  $\kappa^+$ -completable. Problem 54 tries to establish whether we can get a cardinal (in a model which uses a large cardinal consistent with  $V = L$ ) which is not weakly compact but which acts like one with respect to extendibility. Problem 55 asks whether we need an ineffable cardinal or could get away with a weakly compact cardinal (which would be more satisfying).

- ? 122. **Problem 54.** *Does the consistency of the existence of an ineffable cardinal imply the consistency of the existence of a cardinal  $\kappa$  which is not weakly compact such that each  $\kappa$ -completable ideal is  $\kappa$ -extendible?*
- ? 123. **Problem 55.** *Does the consistency of the existence of a weakly compact cardinal imply the consistency of the existence of a cardinal  $\kappa$  which is not weakly compact and a  $\kappa$ -extendible ideal which is not  $\kappa^+$ -completable?*

The case of measurable cardinals is a bit different. If  $\kappa$  is a measurable cardinal then there is a  $\kappa^+$ -extendible ideal which is not  $\kappa^+$ -completable. If  $\kappa$  is a compact cardinal then any  $\kappa$ -completable ideal is  $\kappa^+$ -extendible. On the other hand, if it is consistent that there is a supercompact cardinal, then it is consistent that there is a cardinal  $\kappa$  which is not measurable and a  $\kappa$ -completable ideal on  $\kappa$  which is  $\kappa^+$ -extendible. Problem 56 asks whether a supercompact cardinal is needed for the simplest  $\kappa$ -completable ideal.

- ? 124. **Problem 56.** *Does the consistency of the existence of a measurable cardinal imply the consistency of the existence of a cardinal  $\kappa$  which is not measurable and yet so that  $[\kappa]^{<\kappa}$  is  $\kappa^+$ -extendible?*

The set theory involved in  $(\kappa, \lambda)$ -extendibility is even more interesting.

In [1978] LAVER showed that if it is consistent that there is a huge cardinal then it is consistent that **GCH** holds and  $[\omega_1]^{<\omega_1}$  is  $(\omega_2, \omega_2)$ -extendible. On the other hand if  $\aleph_3$  Cohen subsets of  $\omega_1$  are added to a model of **GCH** then  $[\omega_1]^{<\omega_1}$  is  $(\omega_3, \omega_3)$ -extendible. This latter argument really needs  $\aleph_3$  thus provoking problem 57.

**Problem 57.** *Does the  $(\omega_2, \omega_2)$ -extendibility of  $[\omega_1]^{<\omega_1}$  imply the consistency of large cardinals?* **125. ?**

Todorčević has shown that  $[\omega_1]^{<\omega_1}$  is not  $(\omega_1, \omega_1)$ -extendible in **ZFC**. This takes away any idea that a right-hand coordinate  $\aleph_2$  is any stronger than a right-hand coordinate of  $\aleph_1$  thus giving rise to problem 58.

**Problem 58.** *Does the fact that  $[\omega_1]^{<\omega_1}$  is  $(\omega_2, \omega_1)$ -extendible imply that  $[\omega_1]^{<\omega_1}$  is  $(\omega_2, \omega_2)$ -extendible?* **126. ?**

We know that  $[2^{\omega_1}]^{<2^{\omega_1}}$  is  $(\mu, \mu)$ -extendible whenever  $\mu < 2^{\omega_1}$ . On the other hand,  $[2^{\omega_1}]^{<2^{\omega_1}}$  is not  $(2^{\omega_1}, \omega)$ -extendible whenever  $2^\omega = 2^{\omega_1}$ . This is not the strongest conceivable negative consistency result thus raising problem 59.

**Problem 59.** *Does **ZFC** imply that  $[2^{\omega_1}]^{<2^{\omega_1}}$  is  $(2^{2^{\omega_1}}, \omega)$  extendible?* **127. ?**

We showed that it is consistent with  $2^\omega = \omega_1$  and  $2^{\omega_1} = \omega_2$  that  $[2^{\omega_1}]^{<2^{\omega_1}}$  is not  $(2^{\omega_1}, 2^{\omega_1})$ -extendible. There is no reason to think that this is dependent on a particular cardinal arithmetic which brings up problem 60.

**Problem 60.** *Is it consistent with every cardinal arithmetic that  $[2^{\omega_1}]^{<2^{\omega_1}}$  is not  $(2^{\omega_1}, 2^{\omega_1})$ -extendible?* **128. ?**

Problem 61 is an even broader question on which we have made no progress at all. On the other hand, if it is consistent that there is a weakly compact cardinal, then it is consistent that **CH** holds and that  $[2^{\omega_1}]^{<2^{\omega_1}}$  is  $(2^{\omega_1}, 2^{\omega_1})$ -extendible. It is not at all clear that a large cardinal is needed for this result and that is problem 62.

**Problem 61.** *Does **ZFC** imply that  $[\kappa]^{<\kappa}$  is  $(2^\kappa, 2^\kappa)$ -extendible whenever  $2^\kappa > 2^{\omega_1}$ ?* **129. ?**

**Problem 62.** *Does the consistency of the  $(2^{\omega_1}, 2^{\omega_1})$ -extendibility of  $[2^{\omega_1}]^{<2^{\omega_1}}$  imply the consistency of the existence of an inaccessible cardinal?* **130. ?**

## 10. Homeomorphisms

In the 1895 volume of *Mathematische Annalen*, Georg Cantor wrote an article CANTOR [1895] in which a “back-and-forth” argument, which has become standard (see, for example EBERHART [1977]), was used to show, among other things, that, for any countable dense subsets  $A$  and  $B$  of the real line  $\mathbb{R}$ , there is a homeomorphism  $h: \mathbb{R} \rightarrow \mathbb{R}$  which takes  $A$  onto  $B$  in a monotonic manner. The study of the existence of homeomorphisms taking one countable dense set to another was extended to more general spaces by FORT in [1962]. He showed that the product of countably many manifolds with boundary (for example, the Hilbert cube) admits such homeomorphisms. In 1972, the abstract study was begun by BENNETT [1972] who called such spaces “countable dense homogeneous” or “CDH”.

Bennett asked a series of questions including: “Are CDH continua locally connected?”. This question was answered by FITZPATRICK [1972] in the same year who showed that CDH connected locally compact metric spaces are locally connected. Fitzpatrick has now asked the natural question:

? **131. Problem 63.** *Are CDH connected complete metric spaces locally connected?*

Ungar raised a new question in 1978: Is each open subspace of a CDH space also a CDH space? He showed that the answer is yes for dense open subsets of locally compact separable metric spaces with no finite cut-set. In [1954] FORD defined a space  $X$  to be strongly locally homogeneous if there is a base of open sets  $U$  such that, for each  $x, y \in U$  there is a homeomorphism of  $X$  which takes  $x$  to  $y$  but is the identity outside  $U$ . In 1982, van Mill also raised a question: Are connected CDH spaces strongly locally homogeneous? In [19∞a] FITZPATRICK and ZHOU answered these two questions negatively for the class of Hausdorff spaces. Unfortunately, their spaces were not regular.

In SIMON and WATSON [19∞], Petr Simon and the author exhibit a completely regular CDH space which fails to be strongly locally homogeneous and which contains an open subset which is not CDH. The construction uses the classical tradition (see Stackel’s 1895 article STACKEL [1895]) of constructing smooth functions on  $\mathbb{R}^2$  which take one countable dense set to another as developed by DOBROWOLSKI in [1976].

These examples still leave some natural questions:

? **132. Problem 64.** *Is there a connected metric CDH space which has an open subspace which is not CDH? Is there a connected normal (compact) CDH space which has an open subspace which is not CDH?*

FITZPATRICK and ZHOU [19∞b] have raised an interesting question which nicely turns a strong property of our example against itself. The open subset of that space is *not* homogeneous and so:

**Problem 65.** *Is there a connected CDH space with an open subset which is connected and homogeneous but not CDH?* **133. ?**

Jan van Mill probably had in mind metric spaces (he does open the paper with the comment that all spaces are separable metric) and so the next problem is his:

**Problem 66.** *Are connected CDH metrizable (or compact Hausdorff) spaces strongly locally homogeneous?* **134. ?**

In [1972] BENNETT showed that strongly locally homogeneous locally compact separable metric spaces are CDH. Bennett's theorem proved to be influential. In [1969] DE GROOT weakened locally compact to complete (see proof on p.317 of ANDERSON, CURTIS and VAN MILL [1982]). In [1974] RAVDIN replaced strongly locally homogeneous by "locally homogeneous of variable type" (which implies "representable" which is apparently weaker than strongly locally homogeneous) for complete separable metric spaces. In [1974] FLETCHER and MCCOY showed that "representable" complete separable metric spaces are CDH while in [1976] BALES showed that "representable" is equivalent to strongly locally homogeneous in any case.

In [1982] VAN MILL put an end to the sequence of weakenings of Bennett's theorem by constructing a subset of  $\mathbb{R}^2$  which is connected locally connected and strongly locally homogeneous but not CDH. In [19 $\infty$ ] STEPRĀNS and ZHOU contributed another lower bound by constructing a separable manifold (thus strongly locally homogeneous and locally compact) which is not CDH. Their manifold had weight  $2^\omega$  and they asked whether it is consistent that there is a separable manifold of weight less than the continuum which is not CDH. In WATSON [19 $\infty$ d], we constructed a consistent example of a separable manifold of weight less than the continuum which fails to be CDH. All of these results leave:

**Problem 67.** *Which cardinal invariant describes the minimum weight of a separable manifold which fails to be CDH?* **135. ?**

The cardinal invariant which is the least cardinality of a dominating family in  ${}^\omega\omega$  was shown by Steprāns and Zhou to be a lower bound and I believe that there is a nice answer to this question. I was unable to find it myself but think that a connection between these cardinal invariants and manifolds would be quite interesting.

The next question uses the concept of  $n$ -homogeneity. A space  $X$  is said to be  $n$ -homogeneous if for any two subsets  $\{a_i : i \leq n\}$  and  $\{b_i : i \leq n\}$  of  $X$  there is a homeomorphism of  $X$  onto itself which takes each  $a_i$  to  $b_i$ . Thus homogeneity is the same as 1-homogeneity.



? 136. **Problem 68.** *Are CDH connected spaces 2-homogeneous?*

In [1972] BENNETT showed that CDH connected first countable spaces are homogeneous and asked “Are CDH continua  $n$ -homogeneous?”. UNGAR [1978] answered this question in 1978 by showing that, for compact metric spaces, CDH is equivalent to  $n$ -homogeneous. Meanwhile, in [19 $\infty$ ] FITZPATRICK and LAUER showed that the assumption of first countability in Bennett’s theorem was unnecessary.

In a paper with Steprāns (STEPRĀNS and WATSON [1987]), we investigated the problem of when there is an autohomeomorphism of Euclidean space which takes one uncountable dense set to another. A little thought makes it clear that to have a chance of finding such an autohomeomorphism we need to try to send one  $\kappa$ -dense set to another. A set  $A \subset X$  is  $\kappa$ -dense if  $|A \cap V| = \kappa$  for every non-empty open set  $V$  in  $X$ . We say that  $BA(X, \kappa)$  holds if, for every two  $\kappa$ -dense subsets of  $X$  there is an autohomeomorphism  $H$  of  $X$  such that  $H(A) = B$ . The axiom  $BA(\mathbb{R}, \aleph_1)$  was shown to be consistent with  $\mathbf{MA} + \neg\mathbf{CH}$  by BAUMGARTNER in [1973] and then shown to be independent of  $\mathbf{MA} + \neg\mathbf{CH}$  by ABRAHAM and SHELAH in [1981]. We showed in STEPRĀNS and WATSON [1987] that  $\mathbf{MA} + \neg\mathbf{CH}$  implies that  $BA(\mathbb{R}^n, \kappa)$  for each  $n > 1$  and  $\kappa < 2^\omega$ . This showed that  $\mathbb{R}$  and  $\mathbb{R}^n$  are different insofar as these autohomeomorphisms are concerned when  $n > 1$  but leaves the question of whether  $\mathbb{R}^2$  is different from  $\mathbb{R}^3$  in this context.

? 137. **Problem 69.** *Does  $BA(\mathbb{R}^m, \aleph_1)$  imply  $BA(\mathbb{R}^n, \aleph_1)$  when  $n$  and  $m$  are greater than 1 and unequal?*

We do think however that there is a positive answer for:

? 138. **Problem 70.** *Does  $BA(\mathbb{R}, \aleph_1)$  imply  $BA(\mathbb{R}^n, \aleph_1)$ ?*

There seems to be no apparent monotonicity on the second coordinate either:

? 139. **Problem 71.** *Does  $BA(\mathbb{R}^n, \kappa)$  imply  $BA(\mathbb{R}^n, \lambda)$  when  $\kappa \neq \lambda$ ?*

... and the best problem of all is Baumgartner’s:

? 140. **Problem 72.** *Is  $BA(\mathbb{R}, \aleph_2)$  consistent?*

We actually obtained under  $\mathbf{MA} + \neg\mathbf{CH}$  a stronger result which we denote by  $BA^+(\mathbb{R}^n, \kappa)$  which states that if  $\{A_\alpha : \alpha \in \kappa\}$  and  $\{B_\alpha : \alpha \in \kappa\}$  are families of disjoint countable dense subsets of  $\mathbb{R}^n$ , then there is an autohomeomorphism  $H$  of  $\mathbb{R}^n$  such that, for each  $\alpha \in \kappa$ ,  $H(A_\alpha) = B_\alpha$ . The advantage of this axiom is that at least it is monotonic in the second coordinate. However

it is not clear that it is the same as the original question:

**Problem 73.** *Does  $BA(\mathbb{R}^n, \kappa)$  imply  $BA^+(\mathbb{R}^n, \kappa)$  when  $n \geq 2$ ?* **141. ?**

## 11. Absoluteness

In the summer of 1986, Alan Dow, Bill Weiss, Juris Steprāns and I met for a few weeks to discuss problems of absoluteness in topology. We compiled a list of what we knew and what we did not know:

If one adds a Cohen subset of  $\omega_1$ , then the collectionwise normal space  $2^{\omega_1}$  gets a closed copy of the Tychonoff plank and so becomes not normal.

If one forces with a Suslin tree, then one can embed the square of the Alexandroff compactification of the discrete space of cardinality  $\aleph_1$  minus the “corner point” in  $2^{\omega_1}$ , thus making that collectionwise normal space not normal.

If adds a new subset of  $\omega_1$  with finite conditions (that is, adds  $\aleph_1$  many Cohen reals), then Bing’s space will cease to be normal.

**Problem 74.** *Can Cohen forcing make a collectionwise normal space not collectionwise normal? not normal?* **142. ?**

**Problem 75.** *Can one Cohen real kill normality?* **143. ?**

**Problem 76.** *Is there, in **ZFC**, a countable chain condition partial order which kills collectionwise normality?* **144. ?**

If you take  $(\omega_2 + 1) \times (\omega_1 + 1) - \{\omega_2, \omega_1\}$  and then collapse  $\omega_2$  into an ordinal of cardinality  $\omega_1$  and uncountable cofinality, it becomes collectionwise normal.

If you take a  $(\omega, \mathfrak{b}^*)$  gap and fill it, you make a non-normal space into one which is collectionwise normal. Cohen forcing does preserve non-normality and non-collectionwise normality as demonstrated by DOW, TALL and WEISS in [19∞].

**Problem 77.** *Can countably-closed cardinal-preserving forcing make a non-normal space normal?* **145. ?**

If you take the Alexandroff double of the reals but only use some of the isolated points, then this space is metrizable if and only if the isolated points are a relative  $F_\sigma$ . Thus countable chain condition forcing can make a non-metrizable space into a metrizable space. If you take a non-normal ladder system on a stationary costationary set and make it into a nonstationary set by forcing a club, then you have taken a non-normal space and made it into a metrizable space by a cardinal-preserving forcing.

**Problem 78.** *Can countable chain condition forcing make a non-normal space metrizable?* **146. ?**

? **147. Problem 79.** *Is there, in **ZFC**, a cardinal-preserving forcing which makes a non-normal space metrizable?*

? **148. Problem 80.** *Can countably-closed forcing make a non-metrizable space metrizable?*

? **149. Problem 81.** *Does countably-closed forcing preserve hereditary normality?*

We also looked at problems involving cardinal functions: If you take the Alexandroff double with a Bernstein set of isolated points and make that set into a relative  $F_\sigma$  by means of countable chain condition forcing then the Lindelöf number will have increased from  $\omega$  up to  $2^\omega$ .

On the other hand, an elementary submodel argument shows that the Lindelöf number cannot increase up to  $(2^\omega)^+$  under countable chain condition forcing.

The Lindelöf number can however increase arbitrarily under countably closed forcing—just add a Cohen subset of  $\kappa$  and look at  $2^\kappa$ .

Density (or Lindelöf number) cannot be decreased under the covering lemma but the topology on  $\kappa$  in which the bounded sets are the only proper closed sets can be made separable by forcing  $\kappa$  to have cofinality  $\omega$  and that forcing is cardinal-preserving but denies the covering lemma.

Tightness can be increased by countable chain condition forcing from  $\omega$  to  $2^\omega$  but not past the continuum by using the quotient of  $\omega_1$  many convergent sequences.

Finally tightness can also be increased by countably-closed forcing by using the binary  $\omega_1$ -tree and defining a neighborhood to be the complement of finitely-many branches.

? **150. Problem 82.** *Can character be lowered by cofinality-preserving forcing?*

? **151. Problem 83.** *Does **ZFC** imply that cardinal-preserving forcing cannot decrease the density of Hausdorff spaces?*

? **152. Problem 84.** *Can countably-closed forcing lower density?*

? **153. Problem 85.** *Can cardinal-preserving forcing make a first countable non-Lindelöf space Lindelöf?*

**Problem 86.** Does **ZFC** imply that countably-closed forcing preserves Lindelöf for first countable spaces? **154. ?**

**Problem 87.** Does **ZFC** imply that countably-closed forcing preserves compactness (or the Lindelöf property) in sequential spaces? **155. ?**

Under **PFA**, the answer is yes for compactness.

## 12. Complementation

In 1936, Birkhoff published “On the Combination of Topologies” in *Fundamenta Mathematicae* (BIRKHOFF [1936]). In this paper, he ordered the family of all topologies on a set by letting  $\tau_1 < \tau_2$  if and only if  $\tau_1 \subset \tau_2$ . He noted that the family of all topologies on a set is a lattice. That is to say, for any two topologies  $\tau$  and  $\sigma$  on a set, there is a topology  $\tau \wedge \sigma$  which is the greatest topology contained in both  $\tau$  and  $\sigma$  (actually  $\tau \wedge \sigma = \tau \cap \sigma$ ) and there is a topology  $\tau \vee \sigma$  which is the least topology which contains both  $\tau$  and  $\sigma$ . This lattice has a greatest element, the discrete topology and a smallest element, the indiscrete topology whose open sets are just the null set and the whole set. In fact, the lattice of all topologies on a set is a complete lattice; that is to say there is a greatest topology contained in each element of a family of topologies and there is a least topology which contains each element of a family of topologies.

The study of this lattice ought to be a basic pursuit both in combinatorial set theory and in general topology.

This section is concerned with the nature of complementation in this lattice. We say that topologies  $\tau$  and  $\sigma$  are complementary if and only if  $\tau \wedge \sigma = 0$  and  $\tau \vee \sigma = 1$ . For simplicity, we call any topology other than the discrete and the indiscrete a proper topology (both the discrete topology and the indiscrete topology are uniquely complemented). As a result in finite combinatorics, JURIS HARTMANIS showed, in [1958], that the lattice of all topologies on a finite set is complemented. He also asked whether the lattice of all topologies on an infinite set is complemented. He showed that, in fact, there are at least two complements for any proper topology on a set of size at least 3.

The next series of results were obtained by MANUEL BERRI [1966], Haim Gaifman and Anne Steiner. GAIFMAN [1961] brought some startling new methods to play that foreshadowed some of the arguments of Hajnal and Juhasz in their work on  $L$ -spaces and  $S$ -spaces (HAJNAL and JUHÁSZ [1968, 1969]) and showed in 1961 that the lattice of all topologies on a countable set is complemented. In fact, Gaifman showed that any proper topology on a countable set has at least two complements. In [1966] STEINER used a careful analysis of Gaifman’s argument to show that the lattice of all topologies on any set is complemented. A slightly modified proof of Steiner’s result was given by

VAN ROOIJ in [1968]. The question of the number of distinct complements a topology on a set must possess was first raised by BERRI [1966] before Steiner's theorem was obtained. He asked if every complemented proper topology on an infinite set must have at least two complements. SCHNARE [1968] showed that any proper topology (even not  $T_0$ ) on a infinite set has indeed infinitely many complements (see also DACIC [1969]).

The last paper on this subject appeared in 1969 and was written by Paul Schnare as well (SCHNARE [1969]). In this paper, Schnare showed that any proper topology on an infinite set of cardinality  $\kappa$  has at least  $\kappa$  distinct complements. He also pointed out that there are at most  $2^{2^\kappa}$  many complements on a set of cardinality  $\kappa$ . By exhibiting examples of topologies on a set of cardinality  $\kappa$  which possess exactly  $\kappa$  complements, exactly  $2^\kappa$  complements and exactly  $2^{2^\kappa}$  complements, Schnare showed under the generalized continuum hypothesis that three values are possible for the number of complements of a topology on an infinite set and that these three values are attained.

In 1989, I obtained some results in WATSON [1989] which solved the problem of establishing the exact number of complements of a topology on a fixed set of cardinality  $\aleph_n$  by showing that there are exactly  $2n + 4$  possible values (although, depending on the cardinal arithmetic, some of these may coincide). This removed the assumption of the generalized continuum hypothesis in Schnare's theorem in the countable case and showed that some assumption of cardinal arithmetic is needed in all other cases. To be exact, the number of distinct complements of any topology on a set of cardinality  $\aleph_n$  is either 1 or  $\aleph_n^\kappa$  where  $0 < \kappa \leq \aleph_n$  or  $2^{\aleph_n + 2^{\omega_i}}$  where  $0 \leq i \leq n$ . In particular, on a countable set, exactly four values are possible: 1 or  $\aleph_0$  or  $2^{\aleph_0}$  or  $2^{2^\omega}$ .

However, this still leaves open the original 1966 question of Berri:

- ? **156. Problem 88.** (Berri, rephrased in light of new results) *Let  $\kappa$  be a fixed cardinal. What is the set of possible numbers of complements of topologies on a set of cardinality  $\kappa$ ?*

To make a specific conjecture:

- ? **157. Problem 89.** *Is the set of possible numbers of complements of topologies on a set of cardinality  $\kappa$  precisely:*

$$\{1\} \cup \{(\sup\{2^{2^\alpha} : \alpha < \lambda\})^\kappa : \lambda \leq \kappa\} \cup \{\kappa^\lambda : \lambda \leq \kappa\}?$$

A special case of this question is:

- ? **158. Problem 90.** *Can the number of complements of a topology on  $\aleph_\omega$  be at least  $(2^{\aleph_\omega})^+ + \sup\{2^{2^\kappa} : \kappa < \aleph_\omega\}$  and yet equal to neither  $2^{2^{\omega_\omega}}$  nor  $(\sup\{2^{2^\kappa} : \kappa < \aleph_\omega\})^\omega$ ?*

Some insight into this question may be gained by pointing out that all topologies on a set of cardinality  $\kappa$ , except for some simple and easy to describe ones, have at least  $2^\kappa$  many complements. The interested reader may find a proof of this non-trivial fact in WATSON [1989].

An additional piece of the puzzle may be provided by the fact that if  $\kappa$  is a regular cardinal and  $X$  is a topological space of cardinality at least  $\kappa$  which does not have  $2^{2^\kappa}$  many complements, then, in every complement, either each point has a neighborhood of cardinality less than  $\kappa$  or the number of complements is exactly  $\lambda^\mu$  for some cardinals  $\lambda$  and  $\mu$ .

I think that an answer to this question is going to involve a mixture of set-theoretic topology and finite combinatorics. I worked on this question for quite a while and just ran out of steam when I got to the singular cardinals. Although questions 89 and 90 may seem a little technical, all they really ask is “Is the answer to question 88 a *definable* set?”.

Other interesting questions on the number of complements deal with the concept of a  $T_1$ -complement. This is the appropriate notion for  $T_1$  spaces where the 0 in the lattice is just the cofinite topology:

**Problem 91.** *How many  $T_1$ -complements can a  $T_1$  topology on a set of cardinality  $\kappa$  have?* **159. ?**

In [1967], STEINER and STEINER showed that of any pair of  $T_1$ -complements on a countable set, at least one is not Hausdorff. In [1969], ANDERSON and STEWART showed that of any pair of  $T_1$ -complements, at least one is not first countable Hausdorff. Anderson and Stewart also asked: Can a Hausdorff topology on an uncountable set have a Hausdorff  $T_1$ -complement? We showed in WATSON [1989b] that there is a completely regular topology on a set of cardinality  $(2^\omega)^+$  which is its own complement.

**Problem 92.** *It is consistent that any Hausdorff topology which is its own  $T_1$  complement must lie on a set of cardinality at least  $(2^\omega)^+$ ?* **160. ?**

A few months after we lectured on this result in Srní in January 1989, Aniszczyk constructed two  $T_1$ -complementary compact Hausdorff spaces.

**Problem 93.** *Can two homeomorphic compact Hausdorff spaces be  $T_1$ -complementary?* **161. ?**

**Problem 94.** *Does every Hausdorff topology have a  $T_1$ -complement? What about every completely regular topology?* **162. ?**

Other information on this topic can be found in STEINER and STEINER [1968] and ANDERSON [1970].

In the notation of BIRKHOFF [1967], the maximum number of mutually  $T_1$ -complementary topologies on a set of cardinality  $\kappa$  is the *complementary width* of the lattice of  $T_1$  topological spaces on a set. In [1971] ANDERSON showed by a beautiful construction that there are at least  $\kappa$  mutually  $(T_1)$  complementary topologies on a set of cardinality  $\kappa$ . In STEPRĀNS and WATSON [19 $\infty$ ], we showed that there do not exist uncountably many mutually  $T_1$ -complementary topologies on  $\omega$ . It was also shown that it is consistent that there do not exist  $\aleph_2$  many mutually  $T_1$ -complementary topologies on  $\omega_1$ . On the other hand, it was shown that, under **CH**, there are  $2^{\aleph_1}$ -many mutually  $T_1$ -complementary topologies on  $\omega_1$ .

- ? 163. **Problem 95.** *Does there exist, in **ZFC**, a cardinal  $\kappa$  so that there are  $2^\kappa$  (or  $\kappa^+$ ) many mutually  $T_1$ -complementary topologies on  $\kappa$ ? What about if  $\kappa = 2^\omega$ ?*

The general problem remains:

- ? 164. **Problem 96.** *How many mutually  $T_1$ -complementary topologies are there on a set of cardinality  $\kappa$ ?*

I think problems 95 and 96 are extremely interesting and believe that a solution will require a potent mixture of finite combinatorics and set-theoretic virtuosity.

- ? 165. **Problem 97.** *Is there a triple of mutually complementary topologies which does not admit a fourth topology complementary to each of them? What are the sizes of maximal families of mutually complementary topologies?*
- ? 166. **Problem 98.** *Is there a set of infinitely many but fewer than  $\kappa$  many mutually complementary topologies on a set of cardinality  $\kappa$  which does not admit another mutually complementary topology?*

In forthcoming papers with Jason Brown (BROWN and WATSON [19 $\infty$ , 1989b, 1989a]) we study topologies on a finite set. These are identical with preorders on a finite set ( $T_0$  topologies are identical with partial orders on a finite set) and are thus of substantial interest to finite combinatorists. My interest in the subject originates in the fact that the somewhat difficult construction of WATSON [19 $\infty$ b] can be viewed as a preimage of a non-trivial topology on 18 elements. Many questions in this area have remained immune to our efforts. I mention only a few:

- ? 167. **Problem 99.** *Which topology on a set of size  $n$  has the largest number of complements?*

We know which topology (other than the discrete or indiscrete) has the least number of complements but we do not know which  $T_0$  topology (i.e., partial order) has the least number of complements.

**Problem 100.** *What is the maximum number of pairwise complementary  $T_0$  topologies on a set of size  $n$  ?* **168. ?**

Specifically, we know that the answer to this question lies asymptotically between  $\frac{n}{100 \log n}$  and  $0.486n$  but do not know whether there is a linear lower bound. See also Anderson's beautiful paper ANDERSON [1973].

**Problem 101.** *If  $G$  is the graph on the set of topologies on the integers formed by putting an edge between two topologies if and only if they are complementary, then does  $G$  contain a copy of each countable graph ?* **169. ?**

**Problem 102.** *What is the diameter of the graph  $G$ ?* **170. ?**

We know that the answer is either 6 or 7.

We say that a topology is “self-complementary” if some complement of that topology is homeomorphic to the original topology.

**Problem 103.** (Jason Brown) *Characterize the self-complementary finite topologies.* **171. ?**

We have established a characterization of the self-complementary finite  $T_0$  topologies (i.e., the self-complementary finite partial orders) and the finite equivalence relations (viewed as preorders).

**Problem 104.** *Can every lattice with 1 and 0 be homomorphically embedded in the lattice of topologies on some set?* **172. ?**

Problem 104 is the most important question in this section. I guess that the answer is yes but I have no idea how to prove this. Note that the image of the meet of two elements must be the topological meet of the images of the two elements, the image of the join of two elements must be the topological join of the images of the two elements and that, furthermore, the image of 0 must be the indiscrete topology and the image of 1 must be the discrete topology. It is this last requirement which is so difficult. Embedding the infinite lattice all of whose elements except 0 and 1 are incomparable means producing an infinite mutually complementary family of topologies and there are only a few ways of doing that—the intricate construction of BRUCE ANDERSON [1971] and the methods of STEPRĀNS and WATSON [19∞]. Modifying those arguments will not be easy.



### 13. Other Problems

? **173. Problem 105.** *Give a topological proof of the fact that any connected metrizable manifold is the countable increasing union of compact connected manifolds.*

A few years ago, Raj Prasad asked me whether there was an elementary proof of this fact, since he had needed it and had to resort to quoting fairly deep results in low-dimensional topology to prove it. I came up with a proof which had a hole in it and then another . . . . I still don't know why this fact is true. By the way, the manifold may have boundary but I can't see why this makes the problem any harder.

? **174. Problem 106.** *Are there, under **CH** or otherwise, sets **LARGE**, **SMALL**  $\subset [\omega_1]^\omega$  such that*

- (i) *If  $A$  is large and  $B$  is small, then there is a small infinite  $C$  such that  $C \subset A - B$ ,*
- (ii) *If each  $A_n$  is small then there are finite sets  $F_n$  such that  $\cup\{A_n - F_n : n \in \omega\}$  is small,*
- (iii) *If  $A$  is uncountable then there is a large  $B \subset A$ ,*
- (iv) *If  $A$  is small and  $B \subset A$  then  $B$  is small.*

I like this problem a lot. Under  $\diamond$ , the answer is yes (see OSTASZEWSKI [1976]). Deeper is the fact that under the existence of a Suslin line, the answer is yes (see RUDIN [1972], although it's hidden a little). I managed to get a few set theorists interested in this question, two of whom promptly announced that the answer is no under **PFA**. They both later withdrew this claim. The reason I was looking at this in the first place is slightly less interesting than the sheer combinatorics. Kunen's line (JUHÁSZ, KUNEN and RUDIN [1976]) had one advantage over Ostaszewski's line: it could be done under **CH**. The disadvantage is that it wasn't countably compact and worse yet it didn't have that beautiful property that closed sets are either countable or cocountable. If the answer to the above question is yes under **CH** then Ostaszewski's construction can be done under **CH**. To get a feel for the question take a P-point ultrafilter on each element of a  $\clubsuit$ -sequence.

? **175. Problem 107.** *Is there, in **ZFC**, a linear ordering in which every disjoint family of open sets is the union of countably many discrete subfamilies and yet in which there is no dense set which is the union of countably many closed discrete sets? Is there such a linear ordering if and only if there is a Suslin line?*

The Urysohn metrization theorem is to the Nagata-Smirnov-Stone metrization theorem as the Suslin problem is to this problem. It is incredible that

such a basic question about linear orderings is unsolved (and yet well-known in various disguises).

**Problem 108.** *Is there a topological space (or a completely regular space) in which the connected sets (with more than one point) are precisely the cofinite sets?* **176. ?**

This question was created while looking at an interesting paper of TSVID [1978]. He was asking simply whether (in a countable connected Hausdorff space) the connected sets could be a filter. That also remains unknown. I hawked this question at the 1989 Spring Topology Conference at Tennessee State University, asking for either a topological example (not necessarily even  $T_0$ ) or on the other hand a proof that one couldn't find such a subset of the plane. John Kulesza at George Mason University sent a proof to me a few weeks later that there are no examples which are hereditarily normal Fréchet spaces. Later B. D. Garrett discovered a proof in ERDŐS [1944] (which Erdős attributes to Arthur Stone) of the fact that there are no metrizable examples (Kulesza rediscovered the same proof). I conjecture that there is an example (probably even completely regular) and that the existence of such an example depends on some hard finite combinatorics.

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## CHAPTER 5

### **Weiss's Questions**

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#### *Contents*

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What constitutes a good mathematical problem? This is a difficult question for mathematicians—we are more capable of solving problems than evaluating them. However for me, there is a sense in which every mathematical problem is a good one. If you would ever hear me say “I don’t want to know the answer to that question”, you would hear me wishing to be ignorant—and likely succeeding!

However, this is not to say that I find some problems better than others, and certainly I don’t have time to work on all problems; so I have given some thought to what constitutes a “better” problem, and I have obtained a list of criteria, admittedly subjective and perhaps incomplete. It is just a list of four attributes, such that if any mathematical problem has one of them, then I really like it. First, I like a problem in which the solution will give information about our basic objects of study—the unit interval,  $[0, 1]$ , powers of the two-point discrete space, the ordinals,  $\beta\mathbb{N}$ , the irrationals and a few other natural variations. Just as I wish to know all the topological properties about some topological spaces, I want to know all the topological spaces which possess some basic topological properties. So, second, I like a problem which develops the theory by shedding light among cardinal invariants and other basic topological properties. Third, I like a problem for which the solution would give theorems analogous to existing important theorems. The hope is that these results would someday be almost as useful as the originals. Fourth, and most subjective of all, I like a problem which is simply nice—elementary and easy to state, and intriguing.

With this in mind, here is my list of some problems that I would like to solve. And I admit it—I have worked unsuccessfully on each of them.

## A. Problems about Basic Spaces

A topological space  $X$  is said to have the *Blumberg property* whenever for each function  $f: X \rightarrow \mathbb{R}$  there is a dense set  $D_f$  such that  $f|D_f$  is continuous. H. BLUMBERG [1922] proved that every compact metrizable space has the Blumberg property; however, not every compact Hausdorff space has it (WEISS [1977]).  $\beta\mathbb{N}$  has it, but it is independent whether  $\beta\mathbb{N} \setminus \mathbb{N}$  has it. The following are still wide open.

**Question A1.** For  $\kappa \geq \aleph_1$  does  $[0, 1]^\kappa$  have the Blumberg property? 177. ?

**Question A2.** For  $\kappa \geq \aleph_1$  does  $\{0, 1\}^\kappa$  have the Blumberg property? 178. ?

A topological space  $X$  is said to have the **CIP**, *complete invariance property*, whenever for each closed subset  $Y$  of  $X$  there is a continuous  $f_Y: X \rightarrow X$  such that  $Y = \{x \in X : f_Y(x) = x\}$ . Not every compact metrizable space has the **CIP**, but  $[0, 1]^\kappa$  does, for  $\kappa < \aleph_1$  (WARD [1973]). For other  $\kappa$ , I do not know.

? 179. **Question A3.** *For which  $\kappa$  does  $[0, 1]^\kappa$  have the **CIP**?*

Now  $\{0, 1\}^\kappa$  does have the **CIP** for each  $\kappa$ , as does  $\mathbb{R}^\kappa$ . However the **CIP** behaves strangely, for example  $(\beta\mathbb{N})^c$  has it, whereas  $\beta\mathbb{N}$  does not (MARTIN and WEISS [1984]).

## B. Problems about Cardinal Invariants

The next problem was originally formulated by E. K. van Douwen. It is obvious that a compact space is globally small if it is locally small. However,

? 180. **Question B1.** *Is there a bound on the size of countably compact, locally countable regular spaces?*

In fact there are countably compact, locally countable regular spaces of size  $\aleph_n$  for each positive integer  $n$ , JUHÁSZ, NAGY and WEISS [1979], and assuming  $V = L$  there are such spaces of arbitrarily large size. Not much more is known; however it still may be reasonable to extend the problem.

? 181. **Question B2.** *Let  $X$  be a regular space and let  $\lambda$  be the least cardinal such that some open cover of size  $\lambda$  has no finite subcover and let  $\kappa$  be the least cardinal such that every point has a neighbourhood of size  $< \kappa$ . How are  $\lambda, \kappa$  and  $|X|$  related?*

For a space  $X$ , let us consider the least cardinal  $\kappa$  such that any collection  $\mathcal{U}$  of fewer than  $\kappa$  open subsets of  $X$  has a nowhere dense choice function, i.e.,  $F: \mathcal{U} \rightarrow X$  such that  $F(U) \in U$  and the set  $\{F(U) : U \in \mathcal{U}\}$  is nowhere dense. This  $\kappa$  is actually  $\pi_d(X)$ , which is defined as the minimum of:  $\{|\mathcal{B}| : \mathcal{B} \text{ is a family of non-empty open sets and every dense open set contains an element of } \mathcal{B}\}$ . Also,  $\pi_0(X)$  is defined as the minimum of  $\{\pi(U) : U \text{ is a non-empty open subset of } X\}$ . We always have

$$\pi_d(X) \leq \pi_0(X) \leq 2^{\pi_d(X)}.$$

The question of M. van de Vel and E. K. van Douwen can now be stated.

? 182. **Question B3.** *Is there a regular space  $X$  such that  $\pi_d(X) < \pi_0(X)$ ?*

The answer is consistently yes. However (JUHÁSZ and WEISS [1986]), any such  $X$  cannot be compact and  $\pi_d(X)$  must be uncountable. If **MA** is assumed,  $X$  cannot be separable.

? 183. **Question B4.** *What is the relationship between the height and the width of a compact Hausdorff scattered space?*

Each scattered space  $X$  decomposes into a Cantor-Bendixson hierarchy  $\{I_\alpha : \alpha < \beta\}$  in which  $I_\alpha$  is the non-empty set of isolated points of  $X - \bigcup\{I_\gamma : \gamma < \alpha\}$  and  $X = \bigcup\{I_\alpha : \alpha < \beta\}$ . The ordinal  $\beta$  is called the height of  $X$  and  $\sup\{|I_\alpha| : \alpha < \beta\}$  is called the width of  $X$ . R. Telgársky originally asked about compact scattered spaces of countable width. There are such spaces of height  $\beta$  for each  $\beta < \omega_2$ , but whether or not there exists one of height larger than or equal to  $\omega_2$  is independent (JUHÁSZ and WEISS [1978], BAUMGARTNER and SHELAH [1987], JUST [1985]). However, the following problem is not completely solved.

**Question B5.** *Does each compact Hausdorff space of countable width have height less than  $\omega_3$ ?* **184. ?**

There are many omitting cardinals problems and no doubt they will be discussed somewhere in this volume. A related problem, due to A. Hajnal and I. Juhász (RUDIN [1975]) follows.

**Question B6.** *Does each Lindelöf regular space of cardinality  $\aleph_2$  have a Lindelöf subspace of cardinality  $\aleph_1$ ?* **185. ?**

Now, one might think that  $\aleph_1$  and  $\aleph_2$  are artificial here and should really be replaced by  $2^{\aleph_0}$  and  $2^{2^{\aleph_0}}$ . However, if this is done, then the answer is no—at least consistently, but as far as I know, not consistently with **GCH**, see JUHÁSZ and WEISS [1989]. I think the problem is best left as stated.

## C. Problems about Partitions

Two of the earliest and nicest theorems of general topology are due to F. Bernstein and R. Baire. Any separable metric space  $X$  can be partitioned into two pieces such that no piece contains an uncountable closed subset of  $X$ . If any complete metric space  $X$  is partitioned into countably many pieces, then one piece contains a subset dense in some open set of  $X$ . I would like to consider a class of problems analogous to these theorems, in which the sets to be avoided or captured by the partition are not defined relative to the whole space  $X$ , but are those subsets of  $X$  which are homeomorphic to some prearranged space  $Y$ .

The basic relation to be studied can be written as  $X \rightarrow (Y)_\omega^1$  which means that whenever  $X$  is partitioned in countably many pieces, there is one piece which contains a subset homeomorphic to  $Y$ . An oblique line through the arrow will denote the negation of the relation.

**Question C1.** *Does each regular space  $X$  have the property that  $X \not\rightarrow (\{0, 1\}^\omega)_\omega^1$ ?* **186. ?**

Here  $\{0, 1\}^\omega$  is the usual Cantor set. Such partitions have been found for all  $X$  of cardinality  $\kappa$  for which  $\{\lambda : \lambda \text{ is a cardinal and } \mathfrak{c} < \lambda < \kappa\}$  is finite (WEISS [1990]) and consistently for any regular space. Also, not much more than this is known about the following restricted version.

? **187. Question C2.** *Does every metrizable space  $X$  have the property  $X \not\rightarrow (\{0, 1\}^\omega)_\omega^1$ ?*

On the other hand, there is the following general problem.

? **188. Question C3.** *Which regular  $Y$  have the property that for some class of regular spaces  $\mathcal{C}$ ,  $X \rightarrow (Y)_\omega^1$  for all  $X \in \mathcal{C}$ ?*

It is known that if the metrizable space  $X$  is not  $\sigma$ -discrete, and  $\text{top } \alpha$  denotes the ordinal  $\alpha$  with the usual topology, then  $X \rightarrow (\text{top } \alpha)_\omega^1$  for all countable  $\alpha$  (KOMJÁTH and WEISS [1987]). However the following is open.

? **189. Question C4.** *Is it true that for all (countable) regular spaces  $Y$ , there is some regular space  $X$  such that  $X \rightarrow (Y)_1^\omega$ ?*

As a particularly interesting case of this we have

? **190. Question C5.** *Is there a regular space  $X$  such that  $X \rightarrow (\text{top } \omega_1)_\omega^1$ ?*

The diamond principle implies that  $|X| > \aleph_1$  (KOMJÁTH and WEISS [1987]), but this may be true without that assumption. Furthermore it is independent whether  $X$  can be  $\text{top } \omega_1$  (Silver, Shelah) or  $\{0, 1\}^{\omega_1}$  (Steprāns, Shelah).

We now introduce the notation, as in Ramsey theory,

$$X \rightarrow (Y)_\omega^n$$

for topological spaces  $X$  and  $Y$  and positive integer  $n$  to mean that whenever the  $n$  element subsets of  $X$ ,  $[X]^n$  are partitioned into countably many pieces, there is  $H \subseteq X$  homeomorphic to  $Y$  such that  $[H]^n$  is a subset of one piece of the partition.

? **191. Question C6.** *Is there a regular space  $X$  such that  $X \rightarrow (\text{top } \omega + 1)_\omega^2$ ?*

There is no such  $X$  of size  $\leq \aleph_\omega$ , and consistently there is no such  $X$  (HAJNAL, JUHÁSZ and WEISS [1990]). Here is an infinite schema of problems—one for each positive integer  $n$ .

? **192. Question C7.** *Do there exist regular non-discrete spaces  $X_n$  and  $Y_n$  such that  $X_n \rightarrow (Y_n)_\omega^n$ ?*

It is known that the answer is yes for  $n = 1$  or  $n = 2$  and that any possible  $Y_3$  must be countable (HAJNAL, JUHÁSZ and WEISS [19 $\infty$ ]). Furthermore the answer is consistently no for all  $n \geq 4$ . A negative answer to one of the previous problems would generate the following question.

**Question C8.** *What is the least positive integer  $n$  (if any exists) such that for all regular  $X$  there is a partition of  $[X]^n$  into countably many pieces, such that if  $[H]^n$  is contained in only one piece of the partition, then  $H$  is a discrete subspace of  $X$ ?* **193. ?**

This would mean that partitioning the  $n$ -element subsets of  $X$  “destroys” all the topological properties of  $X$ . We know from above that  $n > 2$ ; and it is consistent that  $n = 4$  (JUHÁSZ and SHELAH [19 $\infty$ ], WEISS [19 $\infty$ ]).

There are more problems of this type—for example into a different number of pieces, or partitioning a basic topological space, etc.; see WEISS [1990].

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## CHAPTER 6

### **Perfectly normal compacta, cosmic spaces, and some partition problems**

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## 1. Some Strange Questions

We discuss a circle of questions related to:

**Question 1.1.** *Is there in **ZFC**, an uncountable space<sup>1</sup> which does not contain a copy of one of the following:* **194. ?**

- (i) *an uncountable discrete space;*
- (ii) *an uncountable subspace of the reals;*
- (iii) *an uncountable subspace of the Sorgenfrey line?*

(Recall that the *Sorgenfrey line* is the real line with the right half-open interval topology.)

Let  $(\alpha)$  be the statement that every uncountable space *does* contain one of (i)–(iii). We will see in later sections that the question of the consistency of  $(\alpha)$  could be what lies at the heart of several longstanding problems concerning the structure of perfectly normal compacta, and problems concerning the characterization of continuous images of separable metric spaces. It is also closely related to some Ramsey-type theorems for topological spaces.

There are consistent counterexamples to  $(\alpha)$ . Indeed, hereditarily separable non-Lindelöf spaces ( $S$ -spaces) and hereditarily Lindelöf non-separable spaces ( $L$ -spaces), which exist assuming the Continuum Hypothesis (**CH**) or other special axioms of set-theory (see, e.g., ROITMAN [1984] or TODORČEVIĆ [1988]), contain counterexamples. For instance, if  $X$  is an  $S$ -space, then there is a “right-separated” subspace  $Y = \{y_\alpha\}_{\alpha < \omega_1} \subseteq X$ ; i.e.,  $y_\alpha \notin \overline{\{y_\beta : \beta > \alpha\}}$  for each  $\alpha < \omega_1$ . No uncountable subspace of  $Y$  is Lindelöf so  $Y$  cannot contain copies of uncountable subspaces of the real line or Sorgenfrey line, and of course  $Y$  contains no uncountable discrete subspace. S. TODORČEVIĆ [1983] showed  $S$ -spaces cannot be constructed in **ZFC**. Indeed the Proper Forcing Axiom (**PFA**), a strengthening of “Martin’s Axiom for  $\omega_1$  many dense sets” (**MA** <sub>$\omega_1$</sub> ) implies that there are no  $S$ -spaces. The consistency of no  $L$ -spaces is still not known, but a relevant partial result is that under **MA** +  $\neg$ **CH**, there are no first countable  $L$ -spaces (SZENTMIKLÓSSY [1978]).

A space  $X$  satisfies the *countable chain condition* (**ccc**) if every collection of pairwise-disjoint non-empty open subsets of  $X$  is countable. A subset  $A$  of  $X$  is *discrete* if the subspace topology on  $A$  is the discrete topology. Note that  $X$  is *hereditarily ccc* (i.e., every subspace is **ccc**) if and only if  $X$  contains no uncountable discrete subspace. So  $(\alpha)$  can be restated as: “*Every uncountable hereditarily ccc space contains a copy of an uncountable subspace of the real line or an uncountable subspace of the Sorgenfrey line*”.

Now every hereditarily **ccc** space either contains an  $S$ -space or an  $L$ -space, or is both hereditarily separable *and* hereditarily Lindelöf (see ROITMAN [1984]). So it seems likely that any **ZFC** counterexample to  $(\alpha)$  will be

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<sup>1</sup>All spaces are assumed to be regular and  $T_1$ .

hereditarily separable and hereditarily Lindelöf; certainly any first countable one will.

Note that each of the spaces (i)–(iii) in Question 1.1 is suborderable. (Recall that *suborderable* spaces are precisely the subspaces of linearly ordered spaces; or equivalently there is a linear ordering of the space such that each point has a base of open or half-open intervals, or is isolated.) Consider the following statement:

- ( $\beta$ ) Every uncountable space contains an uncountable suborderable subspace.

Assuming  $\mathbf{MA}_{\omega_1}$ , ( $\beta$ ) is equivalent to ( $\alpha$ ); this is because  $\mathbf{MA}_{\omega_1}$  destroys essentially the only counterexample to ( $\beta$ ), namely, a left-separated subspace of a Souslin line.

Another property that the three classes of spaces mentioned in Question 1.1 have in common is that they have a weaker metrizable topology; that is, they are *submetrizable*. Even the consistency of the following statement is unknown:

- ( $\gamma$ ) Every uncountable space contains an uncountable submetrizable subspace.

M. BROD [19 $\infty$ ] shows that  $X$  is a counterexample to ( $\gamma$ ) if and only if  $f(X)$  is countable whenever  $f: X \rightarrow \mathbb{R}$  is continuous. She also observes that OSTASZEWSKI'S [1976] space, constructed using axiom  $\diamond$  (a consequence of  $V = L$  stronger than  $\mathbf{CH}$ ), is a counterexample, and Roitman has observed that any Souslin line contains a counterexample. Any counterexample to ( $\gamma$ ) is of course a counterexample to ( $\alpha$ ) too; on the other hand, the author showed (GRUENHAGE [1989]) that if there is a counterexample to ( $\alpha$ ), then assuming  $\mathbf{MA}_{\omega_1}$ , there is a submetrizable one.

The spaces (i)–(iii) have a property somewhat stronger than submetrizability. A space is *cometrizable* if there is a weaker metric topology such that each point has a neighborhood base consisting of sets closed in the metric topology. The Sorgenfrey line, for example, is cometrizable with respect to the usual Euclidean topology:  $\{[a, x): x > a\}$  is a neighborhood base at  $a$ . Cometrizability is topologically interesting because most spaces in the literature obtained by modifying the topology of some metric space are cometrizable; it's a convenient way of ensuring that the resulting space is regular. It's also interesting for us because we can answer our question for this class of spaces: assuming  $\mathbf{PFA}$ , every cometrizable space contains either an uncountable discrete space, or a copy of an uncountable subspace of the real line or the Sorgenfrey line. Thus  $\mathbf{PFA}$  implies that the following statement ( $\delta$ ), which looks like a minor strengthening of ( $\gamma$ ), is equivalent to ( $\alpha$ ):

- ( $\delta$ ) Every uncountable space contains an uncountable cometrizable subspace.

The cometrizable assumption helps in the proof of ( $\alpha$ ) because it allows one to apply certain partition theorems for sets of reals - more on this in §3.

While the existence of  $S$ -spaces or  $L$ -spaces give counterexamples to ( $\alpha$ ), and can be useful in finding counterexamples to ( $\beta$ ) - ( $\delta$ ), the consistency of these statements is unknown even under assumptions that preclude the existence of  $S$ -spaces or  $L$ -spaces. For example, add a first-countability assumption; then under **PFA**, every counterexample must be hereditarily separable and hereditarily Lindelöf. Or just consider the statements ( $\alpha$ )–( $\delta$ ) restricted to spaces which are both hereditarily separable and hereditarily Lindelöf; we still have no answers.

In the following two sections, we will see more of the motivation behind these questions (especially 1.1), and we will also see that consistency proofs even for the restricted classes of spaces just mentioned above would have interesting applications.

## 2. Perfectly Normal Compacta

There is a wide variety of non-metrizable perfectly normal compact spaces, if one is willing to assume axioms of set-theory beyond **ZFC**. A Souslin line compactified by including first and last points is an example; many other examples can be constructed with the help of the **CH** (see, e.g., BURKE and DAVIS [1981] and FILIPPOV [1969]). But essentially the only known example in **ZFC** is Alexandrov's "double arrow" space  $D = [0, 1] \times 2$  with the lexicographic order topology<sup>2</sup>. Maybe  $D$  really is in some sense the only **ZFC** example. The fundamental problem is: Can one make a precise formulation of this and prove it? An early attempt at this was due to D.H. Fremlin, who asked: "Is it consistent that every compact perfectly normal space is a continuous image of  $D \times [0, 1]$ ?" This was answered in the negative by W.S. WATSON and W.A.R. WEISS [1988]. However, their counterexample contains a copy of  $D$ ; in fact, it's  $D$  together with countably many isolated points added in a certain way. Also, like  $D$ , it admits a  $\leq 2 - to - 1$  map onto a compact metric space. This led Fremlin to pose the following question, which is still unsolved:

**Question 2.1.** *Is it consistent that every perfectly normal compact space admits a  $\leq 2 - to - 1$  map onto a compact metric space?* **195. ?**

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<sup>2</sup>This space is also called the "split interval", because one can imagine obtaining it by splitting each  $x \in [0, 1]$  into two points  $x_0$  and  $x_1$ , and giving it the order topology, where  $x_i < y_j$  if  $x < y$  or if  $x = y$  and  $i < j$ .

The connection isn't immediately evident, but this is closely related to statement  $(\alpha)$  of the previous section. Using the fact that  $(\alpha)$  holds for cometrizable spaces, FREMLIN [1984] obtained the following partial result: assuming **PFA**, every perfectly normal compact space  $X$  admits a map  $f$  from  $X$  onto a compact metric space  $M$  such that the set  $\{x \in M: |f^{-1}(x)| \geq 3\}$  is first-category in  $M$ . Indeed, combining the main idea in Fremlin's proof with some ideas in GRUENHAGE [1988] shows that the statement of Question 2.1 is *equivalent*, under **PFA**, to " $(\alpha)$  holds for subspaces of perfectly normal compacta". Another form of the fundamental problem is:

? 196. **Question 2.2.** *Is it consistent that every non-metrizable perfectly normal compact space contains a copy of  $A \times 2$  with the lexicographic order topology for some uncountable  $A \subseteq [0, 1]$ ?*

Note that it wouldn't make sense to ask for containment of *all* of  $D$ : if  $A \subseteq [0, 1]$  is uncountable, then the quotient space  $D(A)$  of  $D$  obtained by identifying  $(x, 0)$  and  $(x, 1)$  for  $x \notin A$  is still a non-metrizable perfectly normal compactum.

In GRUENHAGE [1988] it is shown that if  $(\alpha)$  holds for subspaces of perfectly normal compacta, then under **PFA**, the answer to Question 2.2 is "yes". So it is also the case that a positive answer to 2.1 implies, under **PFA**, a positive answer to 2.2. I don't know if 2.1 and 2.2 are equivalent. Todorčević observed that a positive answer to 2.2 follows from  $(\alpha)$  alone, because  $(\alpha)$  implies the consequence of **PFA** used in GRUENHAGE [1988], namely that every function from an uncountable subset of  $\mathbb{R}$  to  $\mathbb{R}$  has a monotone restriction to some uncountable subset of its domain.

It is easier to see how  $(\alpha)$  gets involved in 2.2 than 2.1. Note that the subspaces  $A \times \{0\}$  and  $A \times \{1\}$  of  $A \times 2$  are homeomorphic to subspaces of the Sorgenfrey line. One can show by an inductive construction that a non-metrizable perfectly normal compact space contains an uncountable subspace  $Y$  such that  $Y$  contains no uncountable metrizable subspace. So if  $(\alpha)$  holds,  $Y$  must contain an uncountable Sorgenfrey subspace. See GRUENHAGE [1988] to see how to push this to get a copy of some  $A \times 2$ .

There are some more specific questions about perfectly normal compacta that stem from the same fundamental problem of our lack of **ZFC** examples.

? 197. **Question 2.3.** *Is it consistent that every perfectly normal locally connected compact space is metrizable?*

This question is due to M.E. RUDIN [1982]; its motivation comes from a line of results that begin with her solution to a problem of Alexandrov: *Is every perfectly normal manifold (= Hausdorff and locally  $\mathbb{R}^n$  for some  $n$ ) metrizable?* RUDIN and ZENOR [1976] constructed a counterexample assuming **CH**. A little later, Rudin showed that under **MA** +  $\neg$ **CH**, the answer to

Alexandrov's question is "yes". D. LANE [1980] extended this to show that, under  $\mathbf{MA} + \neg\mathbf{CH}$ , every perfectly normal locally connected, locally compact space is paracompact. It's still a question whether or not "paracompact" can be replaced by "metrizable" in Lane's theorem. Since paracompact spaces are metrizable iff they are locally metrizable, this boils down to Question 2.3.

A compact Souslin line is a counterexample to 2.3, and others have been constructed assuming  $\mathbf{CH}$  (e.g. FILIPPOV [1969], BURKE and DAVIS [1981]). In GRUENHAGE [1988], we showed that if  $(\alpha)$  holds (for subspaces of perfectly normal compacta), then the answer to 2.3 is "yes".

The following question appears in PRZYMUSIŃSKI [1984]:

**Question 2.4.** *If  $X$  and  $Y$  are compact and  $X \times Y$  is perfectly normal, must one of  $X$  and  $Y$  be metrizable?* **198. ?**

If  $X$  is perfectly normal and compact, and  $Y$  is compact metric, then it is easily shown that  $X \times Y$  is perfectly normal; so the question is whether this is the *only way* a product of two compact spaces can be perfectly normal.

RUDIN [1979] showed that, assuming axiom  $\diamond$ , there are two Souslin lines whose product is perfectly normal. TODORČEVIĆ [1986] showed that if  $A$  and  $B$  are disjoint subsets of  $[0, 1]$  such that there does not exist a one-to-one monotone function from any uncountable subset of  $A$  to  $B$ , then  $D(A) \times D(B)$  is perfectly normal. The existence of such sets  $A$  and  $B$  is implied by  $\mathbf{CH}$ , and is consistent with  $\mathbf{MA} + \neg\mathbf{CH}$ . It's not consistent with  $\mathbf{PFA}$ , so it's possible  $\mathbf{PFA}$  could imply a positive answer to 2.4.

As mentioned earlier, Todorčević observed that  $(\alpha)$  implies every uncountable function from  $\mathbb{R}$  to  $\mathbb{R}$  contains an uncountable monotone subfunction; it's not difficult to deduce from this that  $(\alpha)$  implies a positive answer to 2.4. (It is shown in GRUENHAGE [1988] that " $\mathbf{PFA} + (\alpha)$ " implies a positive answer to 2.4.)

### 3. Cosmic Spaces and Coloring Axioms

A space is *cosmic* if it is the continuous image of a separable metric space  $X$ . Equivalently,  $X$  is cosmic if it has a countable network, i.e., a countable collection  $\mathcal{N}$  of subsets of  $X$  such that if  $x \in U$  with  $U$  open, then  $x \in N \subseteq U$  for some  $N \in \mathcal{N}$ .

There are several unsolved problems asking if cosmicity of  $X$  is implied by certain conditions on  $X$  or its powers.

**Question 3.1.** *Is  $X$  cosmic if:*

- (a)  $X^\omega$  is hereditarily separable and hereditarily Lindelöf?
- (b)  $X^2$  has no uncountable discrete subspaces?
- (c)  $X$  is a Lindelöf semi-metric space?

**199. ?**

(d)  $X$  has the pointed **ccc**?

Recall that  $X$  is *semi-metric* if there is for each  $x \in X$  a base  $b(x, n)$ ,  $n < \omega$ , such that  $y \in b(x, n) \Leftrightarrow x \in b(y, n)$ , and  $X$  has the *pointed ccc* if whenever  $x_\alpha \in U_\alpha$ ,  $\alpha < \omega_1$ , where  $U_\alpha$  is open, then  $x_\alpha \in U_\beta$  and  $x_\beta \in U_\alpha$  for some  $\beta \neq \alpha$ . Note that cosmic spaces satisfy (a), (b) and (d); first countable cosmic spaces satisfy (c).

Question 3.1(a) is due to ARKHANGEL'SKIĬ [1978], (c) to HEATH [1966], and (d) to TKACHENKO [1978]. E. MICHAEL [1971] used **CH** to construct a subset of the plane with the “bow-tie” topology which is a counterexample to all four questions. C. CIESIELSKI [1987] constructed a counterexample to (a), (b), and (d) consistent with  $\mathbf{MA}_{\omega_1}$ .

Regarding (d), HAJNAL and JUHÁSZ [1982] showed that for spaces of size and weight  $\leq \aleph_1$ , if every finite power of  $X$  has the pointed **ccc**, then under  $\mathbf{MA}_{\omega_1}$ ,  $X$  is cosmic. TODORČEVIĆ [1988] later improved this by removing the cardinality restriction, assuming **PFA**. In any case, the following question of Juhász is relevant: “*Is the pointed ccc productive?*”. There are counterexamples consistent with  $\mathbf{MA}_{\omega_1}$  (see GRUENHAGE [1988]), but it's conceivable that **PFA** would imply positive answer, which would solve (d).

The following statement, closely related to  $(\alpha)$  in §1, is in some sense stronger than (a)–(d) above:

- ( $\epsilon$ ) A space  $X$  is cosmic iff  $X$  contains no uncountable discrete space and no uncountable subspace of the Sorgenfrey line.

With  $\mathbf{MA}_{\omega_1}$ , ( $\epsilon$ ) implies  $(\alpha)$ , for assuming  $\mathbf{MA}_{\omega_1}$ , every uncountable cosmic space contains an uncountable metrizable subspace. Consistency of ( $\epsilon$ ) implies positive answers to (c) and (d), and if consistent with **PFA**, would give positive answers to (a) and (b) as well. See GRUENHAGE [1989] for more details. Todorčević observed that it follows from his result mentioned above that under **PFA**,  $(\alpha)$  and ( $\epsilon$ ) are actually equivalent. Like  $(\alpha)$ , statement ( $\epsilon$ ) is true under **PFA** for the class of cometrizable spaces GRUENHAGE [1989]. On the other hand, TODORČEVIĆ [1988] showed that as long as  $\mathfrak{b}$ , the least cardinal of an unbounded family in  $\omega^\omega$ , is not equal to  $\omega_2$ , then there is a cometrizable counterexample to ( $\epsilon$ ) and 3.1(a), (b) and (d). Note that this subsumes the previously mentioned examples.

Fremlin and Todorčević noticed that the argument for “( $\epsilon$ ) holds for cometrizable spaces” given in GRUENHAGE [1989] can be translated to show that it follows from a certain partition relation for sets of reals. Let  $[X]^2$  denote the set of all unordered pairs of elements of  $X$ . Call  $U \subseteq [X]^2$  *open* if the corresponding set  $\{(x, y) : \{x, y\} \in U\}$  of ordered pairs is open in  $X^2$ . Following

TODORČEVIĆ [1988], we define the *Open Coloring Axiom* **OCA**<sup>3</sup> as follows:

- (**OCA**) Let  $X$  be a second countable space. If  $U \subseteq [X]^2$  is open, then either there exists an uncountable  $Y \subseteq X$  with  $[Y]^2 \subseteq U$ , or  $X = \bigcup_{n \in \omega} X_n$  with  $[X_n]^2 \cap U = \emptyset$  for all  $n$ .

See TODORČEVIĆ [1988] or FREMLIN [1984] for a proof that **OCA** implies  $(\epsilon)$ . In TODORČEVIĆ [1988], it is also shown that the following is equivalent to **OCA**:

- (**CSM**) Let  $X$  be a second countable space, and suppose for each  $x \in X$ , we have assigned a closed set  $F_x \subseteq X$ . Then either there exists an uncountable  $Y \subseteq X$  such that

$$y \neq y' \in Y \Rightarrow y' \notin F_y$$

or  $X = \bigcup_{n \in \omega} X_n$  such that

$$x \neq x' \in X_n \Rightarrow x \in F_{x'}, \text{ or } x' \in F_x.$$

Another version of **CSM** (also suggested by Todorčević) worth considering is the statement **CSM'**, where the conclusion of **CSM** is changed to either there is an uncountable  $Y \subseteq X$  such that

$$y \neq y' \in Y \Rightarrow y \notin F_{y'} \text{ or } y' \notin F_y$$

or  $X = \bigcup_{n \in \omega} X_n$  such that

$$x \neq x' \in X_n \Rightarrow x' \in F_x.$$

A natural way to try to improve the result that “ $(\epsilon)$  holds for cometrizable spaces” is to try to prove that **OCA** or **CSM** holds for more general spaces<sup>4</sup>, e.g., hereditarily **ccc** spaces. TODORČEVIĆ [1988] states that **CSM'** for hereditarily **ccc** spaces would imply  $(\epsilon)$  for first countable spaces. Both **CSM** and **CSM'** imply that the pointed **ccc** is finitely productive (since for any assignment of neighborhoods  $U_x$  to points  $x \in X$ , there would be an uncountable  $Y \subseteq X$  with  $y \in Y_{y'}$  for every  $y \neq y' \in Y$ ), so if consistent with **PFA**, would give a positive answer to 3.1(d).

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<sup>3</sup>In ABRAHAM, RUBIN and SHELAH [1985], a version of **OCA** restricted to spaces of size  $\aleph_1$  is denoted **SOCA1**, and a somewhat different statement is denoted **OCA**. However, **SOCA1** fails in **ZFC** without the cardinality restriction, because what we would call the “closed coloring axiom” is false in **ZFC**.

<sup>4</sup>**CSM** and **CSM'** always imply **OCA**; it's not known if any of the three statements are equivalent in general.



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## CHAPTER 7

### Open problems on $\beta\omega$

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## 1. Introduction

The aim of this paper is to collect some open problems on  $\beta\omega$ , the Čech-Stone compactification of the integers. It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Of many of the problems the origin is not known. For that reason we do not credit anybody for posing a certain problem. We would like to thank our colleagues who brought many of the problems to our attention.

In addition to this, we also comment on the status of the problems in the list from the second author's paper in the Handbook of Set-Theoretic Topology: VAN MILL [1984], hereafter referred to as the  $\beta\omega$ -Handbook Paper.

## 2. Definitions and Notation

Many notions are defined in the list, but many occur so frequently that we collect them in this section.

If  $f$  is a function from  $\omega$  to  $\omega$  then  $\beta f: \beta\omega \rightarrow \beta\omega$  is its Stone extension, i.e.,  $\beta f(p) = \{P : f^{-1}[P] \in p\}$ .

The *Rudin-Keisler* (pre-)order  $\leq_{\mathbf{RK}}$  on  $\beta\omega$  is defined as follows:  $p \leq_{\mathbf{RK}} q$  iff there is  $f \in {}^\omega\omega$  such that  $\beta f(q) = p$ . Two points  $p, q \in \beta\omega$  are called **RK**-equivalent, in symbols  $p \simeq_{\mathbf{RK}} q$ , if there is a permutation  $\pi: \omega \rightarrow \omega$  such that  $\beta\pi(p) = q$ . We denote the **RK**-equivalence class of  $p$  by  $[p]_{\mathbf{RK}}$ .

Let  $\mathcal{P}(\omega)$  denote the power set of  $\omega$ . Let  $fin$  denote the ideal of finite subsets of  $\omega$ . The quotient algebra  $\mathcal{P}(\omega)/fin$  is naturally isomorphic to the Boolean algebra of clopen subsets of  $\omega^* = \beta\omega \setminus \omega$ .

If  $A, B \subseteq \omega$ , then  $A$  is *almost contained in*  $B$ , abbreviated  $A \subseteq^* B$ , if  $|A \setminus B| < \omega$ .

We denote the character of a point  $p \in \omega^*$  by  $\chi(p)$ , thus

$$\begin{aligned} \chi(p) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base at } p \text{ in } \omega^*\} \\ &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq p \text{ generates } p\}. \end{aligned}$$

A point  $p \in \omega^*$  is called a *P-point*, if for every function  $f$  from  $\omega$  into itself, there is an element  $P \in p$  on which  $f$  is finite-to-one, or constant. Equivalently, the intersection of countably many neighborhoods of  $p$  in  $\omega^*$  is again a neighborhood of  $p$ . A *simple P-point* is a point with a linearly ordered local base. If in the above definition  $P$  can always be chosen so that  $f$  is one-to-one or constant on  $P$ , then  $p$  is called *selective*. Let  $\kappa$  be an infinite cardinal. A subset  $A$  of a space  $X$  is called a  $P_\kappa$ -set if the intersection of fewer than  $\kappa$  many neighborhoods of  $A$  is again a neighborhood of  $A$ . If  $A$  consists of one point then that point is called a  $P_\kappa$ -point. A *P-set* is a  $P_{\omega_1}$ -set.

For a discussion of **MA** and **PFA**, see WEISS [1984] and BAUMGARTNER [1984]. For information on the cardinals  $\mathfrak{a}$ ,  $\mathfrak{b}$ ,  $\mathfrak{c}$ ,  $\mathfrak{d}$  and others, see the contribution of Vaughan in this volume.

### 3. Answers to older problems

In this section we collect the problems from the  $\beta\omega$ -Handbook Paper that have been solved. We list them, with their answers, using the original numbering.

- 2 Are there points  $p$  and  $q \in \omega^*$  such that if  $f: \omega \rightarrow \omega$  is any finite-to-one map, then  $\beta f(p) \neq \beta f(q)$ ?

Let us abbreviate the following statement by **NCF** (Near Coherence of Filters):

for every  $p$  and  $q \in \omega^*$  there is a finite-to-one  $f: \omega \rightarrow \omega$  such that  $\beta f(p) = \beta f(q)$ .

So the question is whether **NCF** is false. Under **MA** it is. But not in **ZFC**: Shelah produced models such that for all  $p$  and  $q \in \omega^*$  there is a finite-to-one map  $f: \omega \rightarrow \omega$  such that  $\beta f(p) = \beta f(q)$  is a  $P$ -point of character  $\omega_1$  (observe that in this model **CH** is false, so that a point with character  $\omega_1$  has small character), see the papers BLASS and SHELAH [1987, 19 $\infty$ ]. The latter model is the model obtained by iterating rational perfect set forcing (MILLER [1984])  $\omega_2$  times; the former model is also obtained in an  $\omega_2$ -step iteration but the poset used is somewhat more difficult to describe. This iteration however can be modified to produce a model in which there are  $p$  and  $q$  in  $\omega^*$  with linearly ordered bases and with  $\chi(p) = \omega_1$  and  $\chi(q) = \omega_2$ . This answered a question of Nyikos who showed that if  $p$  and  $q$  in  $\omega^*$  are simple  $P$ -points with  $\chi(p) < \chi(q)$  then  $\chi(p) = \mathfrak{b}$  and  $\chi(q) = \mathfrak{d}$  and asked whether this situation is actually possible.

The consistency of **NCF** implies among other things that the Čech-Stone-remainder  $\mathbb{H}^*$  of the half-line  $\mathbb{H} = [0, \infty)$ , which is an indecomposable continuum (see BELLAMY [1971] and WOODS [1968]), has (consistently) only one composant. For details, see e.g., RUDIN [1970]. In fact **NCF** is equivalent to the statement that  $\mathbb{H}^*$  has exactly one composant (MIODUSZEWSKI [1978]). See the papers BLASS [1986] and [1987] for more information on **NCF**.

- 6 Is there a **ccc**  $P$ -set in  $\omega^*$ ?

In [1989] FRANKIEWICZ, SHELAH and ZBIERSKI announced the consistency of a negative answer.

Now a **ccc** subset of  $\omega^*$  is topologically quite small (it is nowhere dense for example), and it is also interesting to know what nowhere dense  $P$ -sets can look like. By way of an example one may wonder whether  $\omega^*$  can be realized as a nowhere dense  $P$ -subset of itself. The answer to this question is in the negative. JUST [19 $\infty$ ] recently showed the consistency of the statement that no nowhere dense  $P$ -set in  $\omega^*$  is homeomorphic to  $\omega^*$ . In fact, Just showed that “if  $A \subseteq \omega^*$  is a nowhere dense  $P$ -set and a continuous image of  $\omega^*$  then  $A$  is **ccc**” is consistent with **ZFC**.

## 8 Is the autohomeomorphism group of $\omega^*$ algebraically simple?

This problem was motivated by the question in VAN DOUWEN, MONK and RUBIN [1980] whether the automorphism group  $\text{Aut}(B)$  of a homogeneous Boolean algebra  $B$  is algebraically simple. The Boolean algebra of clopen subsets of  $\omega^*$  is clearly isomorphic to the quotient algebra  $\mathcal{P}(\omega)/\text{fin}$ , which is easily seen to be homogeneous. It can be shown that under **CH**,  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  is algebraically simple, see ŠTĚPÁNEK and RUBIN [1989]. However, VAN DOUWEN showed in [1990] that the group of *trivial* (see below) automorphisms of  $\mathcal{P}(\omega)/\text{fin}$  is not algebraically simple. It follows that in Shelah's model (see SHELAH [1982]) where every automorphism of  $\mathcal{P}(\omega)/\text{fin}$  is trivial,  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  is not algebraically simple. Independently, KOPPELBERG [1985] constructed a different example of a homogeneous Boolean algebra the automorphism group of which is not simple, under **CH**.

Let us take this opportunity to correct a small mistake in the  $\beta\omega$ -Handbook Paper which caused some confusion. On page 537, line 8 it states:

As remarked in Section 2.2, Shelah [1978] has shown it to be consistent that every autohomeomorphisms of  $\omega^*$  is induced by a permutation of  $\omega$ .

This is not true of course. If  $F$  and  $G$  are finite subsets of  $\omega$  then each bijection  $\pi: \omega \setminus F \rightarrow \omega \setminus G$  induces a homeomorphism  $\bar{\pi}$  of  $\omega^*$  and Shelah proved that consistently, all homeomorphisms of  $\omega^*$  are of this form. Let us call such homeomorphisms trivial or induced, and let  $\text{Triv}$  denote the subgroup of  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  consisting of all trivial automorphisms. So van Mill misquoted Shelah's result. All results in the  $\beta\omega$ -Handbook Paper depending on Shelah's result (such as Theorem 2.2.1) are correct, as can be seen by making trivial modifications to the proofs given.

Van Douwen's argument is now easy to summarize: if  $h \in \text{Triv}$  is represented as above then the parity of  $|F| + |G|$  depends only on  $h$ ; and the automorphisms for which the parity is even form a subgroup of  $\text{Triv}$  of index 2.

## 10 Is every first countable compactum a continuous image of $\omega^*$ ?

To put this question into perspective, note that by ARKHANGEL'SKIĬ's result from [1969], every first countable compactum has cardinality at most  $\mathfrak{c}$  and hence has weight at most  $\mathfrak{c}$ , and hence is—under **CH**—a continuous image of  $\omega^*$  by PAROVICHENKO's result in [1963]. Pertinent to this question is also the result of PRZYMUSIŃSKI [1982] that every perfectly normal compact space is a continuous image of  $\omega^*$ . Problem 10 was recently answered in the negative (necessarily consistently) by BELL [19 $\infty$ ] who modified an older construction of his, BELL [1982], to obtain the desired counterexample.

## 12 Is there a separable closed subspace of $\beta\omega$ which is not a retract of $\beta\omega$ ?

Such spaces were constructed by SHAPIRO [1985] and SIMON [1987]. Simon, using heavy machinery from independent linked families, directly constructed

a closed separable subspace of  $\beta\omega$  which is not a retract; Shapiro constructed a certain compact separable space and showed that not every copy of its absolute in  $\beta\omega$  could be a retract of  $\beta\omega$ . See question 24 for more information.

- 13** Let  $(*)$  denote the statement that every Parovichenko space is coabsolute with  $\omega^*$ . Is  $(*)$  equivalent to **CH**?

DOW [1984] answered this question in the negative by establishing the interesting fact that  $(*)$  follows from the continuum having cofinality  $\omega_1$ . This suggests a question that will be posed later on.

- 14** Let  $X$  be the Stone space of the reduced measure algebra of  $[0, 1]$ . Is it consistent that  $X$  is not a continuous image of  $\omega^*$ ?

This question was answered in the affirmative by FRANKIEWICZ [1985], using the oracle-cc method of SHELAH [1982].

- 17** Is there a  $p \in \omega^*$  such that  $\omega^* \setminus \{p\}$  is not  $C^*$ -embedded in  $\omega^*$ ?

Recall that  $X$  is  $C^*$ -embedded in  $Y$  if every continuous function from  $X$  to  $[0, 1]$  can be extended over  $Y$ . This question has been solved completely now. By an old result of GILLMAN [1966] it follows that under **CH**, for every  $p \in \omega^*$  the space  $\omega^* \setminus \{p\}$  is not  $C^*$ -embedded in  $\beta\omega$ . However, by a result of VAN DOUWEN, KUNEN and VAN MILL [1989] it is consistent with **MA** +  $\mathfrak{c} = \omega_2$  that for every  $p \in \omega^*$  the space  $\omega^* \setminus \{p\}$  is  $C^*$ -embedded in  $\beta\omega$ . In [1987] MALYKHIN announced the result that if one adds  $\mathfrak{c}^+$  Cohen reals to any model of set theory then one obtains a model in which  $\omega^* \setminus \{p\}$  is  $C^*$ -embedded in  $\omega^*$  for every  $p \in \omega^*$ . A proof may be found in DOW [1988a].

- 22** Is there a point  $p \in \omega^*$  such that some compactification of  $\omega \cup \{p\}$  does not contain a copy of  $\beta\omega$ ?

RYLL-NARDZEWSKI and TELGÁRSKY [1970] showed that if  $p$  is a simple  $P$ -point then the space  $\omega \cup \{p\}$  has a scattered (= every subspace has an isolated point) compactification. Thus the answer to this problem is in the affirmative under **MA**. On the other hand in [1987] MALYKHIN also announced that in the same model as mentioned above for every  $p \in \omega^*$  every compactification of  $\omega \cup \{p\}$  contains a copy of  $\beta\omega$ .

- 24** Is there a point  $p \in \omega^*$  such that if  $f: \omega \rightarrow \omega$  is any map, then either  $\beta f(p) \in \omega$  or  $\beta f(p)$  has character  $\mathfrak{c}$  in  $\beta\omega$ ?

Since under **MA** each point in  $\omega^*$  has character  $\mathfrak{c}$  in  $\omega^*$ , the answer to this question is trivially YES under **MA**. However, it is not YES in **ZFC** because in answer 2 we already remarked that Shelah has constructed models in which  $\mathfrak{c} = \omega_2$  and in which (in particular) for every  $p \in \omega^*$  there is a finite-to-one function  $f: \omega \rightarrow \omega$  such that  $\beta f(p)$  has character  $\omega_1$  in  $\omega^*$ . In fact **NCF** is equivalent to the statement that for every  $p \in \omega^*$  there is a finite-to-one  $f: \omega \rightarrow \omega$  such that  $\chi(\beta f(p)) < \mathfrak{d}$ , so that every model for **NCF** provides a negative answer to this question.

#### 4. Autohomeomorphisms

We consider the autohomeomorphism group of the space  $\omega^*$  or equivalently  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ . Recall answer **8**:

The autohomeomorphism group of  $\omega^*$  may, but need not, be algebraically simple.

This prompts us to add a few problems about  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ , the solutions of which may shed some light on the possible algebraic structure of this group. See also the contribution by Steprāns to this volume.

**Question 1.** *Can  $\text{Triv}$  be a proper normal subgroup of  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$ , and if yes what is (or can be) the structure of the factor group  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})/\text{Triv}$ ; and if no what is (or can be)  $[\text{Triv} : \text{Aut}(\mathcal{P}(\omega)/\text{fin})]$ ?* **200. ?**

For  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  we let  $I(h) = \{A \subseteq \omega : h|A \text{ is trivial}\}$ , where “ $h|A$  is trivial” means that there are a finite set  $F \subseteq A$  and a one-to-one  $f: A \setminus F \rightarrow \omega$  such that  $h(X) = f[X \setminus F]$  for  $X \subseteq A$ . Let us observe that  $h$  is trivial iff  $\omega \in I(h)$ , and that  $I(h)$  is an ideal.

To make the statements of some of the following questions a bit easier we shall call  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$ : *totally non-trivial* if  $I(h) = \text{fin}$ , *somewhere trivial* if  $I(h) \neq \text{fin}$  and *almost trivial* if  $I(h)$  is a tall ideal. Recall that an ideal  $I$  on  $\omega$  is *tall* if every infinite subset of  $\omega$  contains an infinite element of  $I$ .

It is not hard to show that under **CH** there is a totally non-trivial automorphism. Recently it was shown by SHELAH and STEPRĀNS in [1988] that **PFA** implies that every  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  is trivial, they also mention that Velickovic showed it to be consistent with **MA** +  $\neg\text{CH}$  that a non-trivial automorphism exists.

We ask the following questions:

**Question 2.** *Is it consistent with **MA** +  $\neg\text{CH}$  that a totally non-trivial automorphism exists?* **201. ?**

**Question 3.** *Is it consistent to have a non-trivial automorphism, while for every  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  the ideal  $I(h)$  is the intersection of finitely many prime ideals?* **202. ?**

**Question 4.** *Does the existence of a totally non-trivial automorphism imply that  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  is simple?* **203. ?**

**Question 5.** *Does the existence of a non-trivial automorphism imply that  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  is simple?* **204. ?**



The proof in SHELAH and STEPRĀNS [1988] suggests the following questions:

- ? 205. **Question 6.** *If every automorphism is somewhere trivial, is then every automorphism trivial?*
- ? 206. **Question 7.** *Is every ideal  $I(h)$  a  $P$ -ideal (if every automorphism is somewhere trivial)?*

An ideal  $I$  on  $\omega$  is said to be a  $P$ -ideal if whenever  $\{X_n : n \in \omega\}$  is a subfamily of  $I$  there is an  $X$  in  $I$  such that  $X_n \subseteq^* X$  for all  $n \in \omega$ .

For the next group of questions we make the following definitions: if  $\kappa$  is a cardinal and  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  call  $h$

$\kappa$ -weakly trivial if  $|\{p : h(p) \not\equiv_{\mathbf{RK}} p\}| < \kappa$  and,

$\kappa$ -quasi trivial if for every  $p \in \omega^*$  there is  $S_p \subseteq [p]_{\mathbf{RK}}$  such that

$$|S_p| < \kappa \wedge \forall q, q' \in [p]_{\mathbf{RK}} \setminus S_p : h(q) \simeq_{\mathbf{RK}} h(q') \text{ and } |\{p : S_p \neq \emptyset\}| < \kappa$$

Let  $W_\kappa = \{h : h, h^{-1} \kappa\text{-weakly trivial}\}$  and  $Q_\kappa = \{h : h, h^{-1} \kappa\text{-quasi trivial}\}$ . It is known that  $W_\kappa$  is a normal subgroup of  $Q_\kappa$ .

- ? 207. **Question 8.** *Is it consistent to have a cardinal  $\kappa$  such that every automorphism is  $\kappa$ -weakly trivial?*
- ? 208. **Question 9.** *Is it consistent to have a cardinal  $\kappa$  such that every automorphism is  $\kappa$ -quasi trivial?*
- ? 209. **Question 10.** *Is it consistent to have  $W_\kappa \neq Q_\kappa = \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  for some regular  $\kappa \leq \mathfrak{c}$ ?*
- ? 210. **Question 11.** *(MA +  $\neg\mathbf{CH}$ ) if  $p$  and  $q$  are  $P_\mathfrak{c}$ -points is there an  $h$  in  $\text{Aut}(\mathcal{P}(\omega)/\text{fin})$  such that  $h(p) = q$ ?*

Note: say  $p \equiv q$  iff  $\omega = \bigcup_n P_n = \bigcup_n Q_n$  (finite sets) such that

$$\forall A \in p \exists B \in q \forall n |A \cap P_n| = |B \cap Q_n|$$

and conversely. Clearly  $\mathbf{RK}$ -equivalent points are  $\equiv$ -equivalent. As a partial answer to Problem 11 the following was shown to be consistent:

**MA +  $\neg\mathbf{CH}$**  + “for all  $P_\mathfrak{c}$ -points  $p$  and  $q$ , if  $p \equiv q$  then there is an  $h \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$  such that  $h(p) = q$ .”

We ask:

- ? 211. **Question 12.** *Is  $\equiv$  different from  $\mathbf{RK}$ -equivalence in **ZFC**?*

## 5. Subspaces

In this section we deal with subspaces of  $\beta\omega$  and  $\omega^*$ . The following question is well-known.

**Question 13.** *For what  $p$  are  $\omega^*\setminus\{p\}$  and (equivalently)  $\beta\omega\setminus\{p\}$  non-normal?* **212. ?**

There are several simple proofs that, under **CH**, for any  $p \in \omega^*$ , the spaces  $\beta\omega\setminus\{p\}$  and  $\omega^*\setminus\{p\}$  are not normal, see e.g., RAJAGOPALAN [1972], WARREN [1972], and VAN MILL [1986]. It was shown by Beslagic and Van Douwen that **CH** may be relaxed to the equality  $\mathfrak{r} = \mathfrak{c}$ , here  $\mathfrak{r}$  is the least cardinality of a “reaping” family; this is a family  $\mathcal{R}$  of subsets of  $\omega$  such that for every subset  $X$  of  $\omega$  there is an  $R \in \mathcal{R}$  such that  $R \subseteq X$  or  $R \cap X = \emptyset$ .

However, for years no significant progress has been made on Problem 13. The best result so far is that if  $p \in \omega^*$  is an accumulation point of some countable discrete subset of  $\omega^*$ , then  $\omega^*\setminus\{p\}$  is not normal. In [1982] GRYZLOV showed that also points that are not an accumulation point of any countable subset of  $\omega^*$  may have this property.

Related to this question is the following:

**Question 14.** *Is it consistent that there is a non-butterfly point in  $\omega^*$ ?* **213. ?**

A butterfly point is a point  $p$  for which there are sets  $D$  and  $E$  such that  $\overline{D} \cap \overline{E} = \{p\}$ . For a non-butterfly point  $p$  the space  $\omega^*\setminus\{p\}$  is normal.

In connection with Answer 17 one may also ask

**Question 15.** *Is it consistent that  $\omega^*\setminus\{p\}$  is  $C^*$ -embedded in  $\omega^*$  for some but not all  $p \in \omega^*$ ?* **214. ?**

**Question 16.** *What spaces can be embedded in  $\beta\omega$ ?* **215. ?**

In [1973] LOUVEAU proved that under **CH**, a compact space  $X$  can be embedded in  $\beta\omega$  if and only if  $X$  is a compact zero-dimensional  $F$ -space of weight at most  $\mathfrak{c}$  (or, equivalently, the Stone space of a weakly countably complete Boolean algebra of cardinality at most  $\mathfrak{c}$ ). There is a consistent example in VAN DOUWEN and VAN MILL [1980] of a compact zero-dimensional  $F$ -space of weight  $\mathfrak{c}$  that cannot be embedded in any compact extremally disconnected space (that is, the Stone space of a complete Boolean algebra). So the **CH** assumption in Louveau’s result is essential. These remarks have motivated Problem 16.

**Question 17.** *Is **CH** equivalent to the statement that every compact zero-dimensional  $F$ -space of weight  $\mathfrak{c}$  is embeddable in  $\omega^*$ ?* **216. ?**

A compact space is *basically disconnected* if it is the Stone space of a  $\sigma$ -complete Boolean algebra.

? 217. **Question 18.** *Is there (consistently) a basically disconnected compact space of weight  $\mathfrak{c}$  that is not embeddable in  $\omega^*$ ?*

A natural candidate for such an example would be the Čech-Stone compactification of an appropriate  $P$ -space (= a space in which every  $G_\delta$ -set is open). However, DOW and VAN MILL showed in [1982] that such an example does not work. For more information, see the remarks following Problem 16

Motivated by Answer 6 we specialize Problem 16 to:

? 218. **Question 19.** *Describe the  $P$ -sets of  $\omega^*$*

Finally, to finish the questions on embeddings we ask:

? 219. **Question 20.** *Is there a copy of  $\omega^*$  in  $\omega^*$  not of the form  $\overline{D} \setminus D$  for some countable and discrete  $D \subseteq \omega^*$ ?*

Of course Just's result cited after Answer 6 blocks an "easy" way out of this problem: a  $P$ -set homeomorphic to  $\omega^*$  would certainly do the trick. However it may still be possible for example, to realize  $\omega^*$  as a weak  $P$ -set in  $\omega^*$ .

? 220. **Question 21.** *Is every subspace of  $\omega^*$  strongly zero-dimensional?*

We have no information on this problem.

? 221. **Question 22.** *Is every nowhere dense subset of  $\omega^*$  a  $\mathfrak{c}$ -set?*

A set  $D$  in a topological space is called a  $\kappa$ -set, where  $\kappa$  is a cardinal number, if there is a disjoint family  $\mathcal{U}$  of size  $\kappa$  of open sets such that  $D \subseteq \overline{U}$  for every  $U$  in  $\mathcal{U}$ . One may think of  $\kappa$ -sets as providing an indication of how non-extremally disconnected a space is (in an extremally disconnected space *no* nowhere dense set is even a 2-set). As is well-known,  $\omega^*$  is not extremally disconnected and we are asking whether it is, in a way, totally non-extremally disconnected ( $\mathfrak{c}$  is the largest cardinal we can hope for of course). It should be noted however that, as far as we know, it is also unknown whether every nowhere dense set in  $\omega^*$  is a 2-set. The reason we ask about  $\mathfrak{c}$ -sets is that all partial answers seem to point into the direction of  $\mathfrak{c}$ : BALCAR and VOJTÁŠ showed in [1980] that every one-point set is a  $\mathfrak{c}$ -set. This was improved in BALCAR, DOČKALOVÁ and SIMON [1984] to sets of density less than  $\mathfrak{c}$ . Furthermore the answer to the general question is known to be YES if either  $\mathfrak{a} = \mathfrak{c}$ ,  $\mathfrak{d} = \mathfrak{c}$ ,  $\mathfrak{d} = \omega_1$  or  $\mathfrak{b} = \mathfrak{d}$ . More information, including proofs of the above YES can be found in BALCAR and SIMON [1989].

A positive answer to question 22 would provide a negative answer to the following, natural sounding, question:

**Question 23.** *Is there a maximal nowhere dense subset in  $\omega^*$ ?*

**223. ?**

Here ‘maximal’ means that it is not nowhere dense in a larger nowhere dense subset of  $\omega^*$ . One can see with a little effort that no  $\mathfrak{c}$ -set can be a maximal nowhere dense set.

A compact space  $X$  is *dyadic* if it is a continuous image of a power of  $\{0, 1\}$ . The *absolute* of a compact space  $X$  is the Stone space of the Boolean algebra of regular open subsets of  $X$ .

**Question 24.** *Is there an absolute retract of  $\beta\omega$  that is not the absolute of a dyadic space?*

**223. ?**

Recall from answer 12 that not every separable closed subspace of  $\beta\omega$  is a retract of  $\beta\omega$ . On the other hand it was shown by MAHARAM in [1976] that such a subspace can be reembedded into  $\beta\omega$  in such a way that it is a retract of  $\beta\omega$ . This motivates the notion of an absolute retract of  $\beta\omega$ . A closed subspace  $X$  of  $\beta\omega$  is an Absolute Retract (**AR**) of  $\beta\omega$  if for every embedding  $h: X \rightarrow \beta\omega$ ,  $h[X]$  is a retract of  $\beta\omega$ . It can be shown that if  $X \subseteq \beta\omega$  is the absolute of a dyadic space then  $X$  is an **AR** of  $\beta\omega$  and SHAPIRO [1985] established the converse in case  $X$  is the absolute of some (compact separable) space of weight at most  $\omega_1$ .

## 6. Individual Ultrafilters

In this section we collect some questions that ask for individual points in  $\beta\omega$  or for special ultrafilters.

**Question 25.** *Is there a model in which there are no  $P$ -points and no  $Q$ -points?*

**224. ?**

Recall that  $p \in \omega^*$  is a  $Q$ -point if for every finite-to-one function  $f: \omega \rightarrow \omega$  there is an element  $E \in p$  such that  $f$  is one-to-one on  $E$ . We already mentioned that Shelah produced a model in which there are no  $P$ -points (WIMMERS [1982]). The continuum is  $\omega_2$  in this model. On the other hand there is also a model in which there are no  $Q$ -points (MILLER [1980]). In this model—Laver’s model for the Borel Conjecture (LAVER [1976])—the continuum is also  $\omega_2$ . The interest in Problem 25 comes from the fact that, by results from KETONEN [1976] and MATHIAS [1978], if  $\mathfrak{c} \leq \omega_2$  then there is either a  $P$ -point or a  $Q$ -point.

An ultrafilter  $p$  is called *rapid* if for every  $f \in {}^\omega\omega$  there exists a  $g \in {}^\omega\omega$  such that  $\forall n f(n) < g(n)$  and  $g[\omega] \in p$  (in words: the counting functions of the elements of  $p$  form a dominating family). Every  $Q$ -point is rapid.

? **225. Question 26.** *Is there a model in which there is a rapid ultrafilter but in which there is no  $Q$ -point?*

The reason we ask this question is that in every model without  $Q$ -points that we know of there are also no rapid ultrafilters.

In [1982] SHELAH showed that it is consistent that there is, up to permutation, only one selective ultrafilter. Of course one may then also ask:

? **226. Question 27.** *Is it consistent that there is, up to permutation, only one  $P$ -point in  $\omega^*$ ?*

? **227. Question 28.** *Is there a model in which every point of  $\omega^*$  is an  $R$ -point?*

A point  $p \in \omega^*$  is an  $R$ -point if there is an open  $F_\sigma$ -set  $U \subseteq \omega^*$  such that

- (i)  $p \in \overline{U}$ , and
- (ii)  $\forall A \in [U]^{<\mathfrak{c}} : p \notin \overline{A}$ .

Note that an  $R$ -point is not a  $P$ -point. So we are asking for a special model without  $P$ -points.  $R$ -points were introduced in VAN MILL [1983] but have played no role of importance so far. So this question is probably not very much of interest.

? **228. Question 29.** *Is there  $p \in \omega^*$  such that every compactification of  $\omega \cup \{p\}$  contains  $\beta\omega$ ?*

This question was motivated by an example in VAN DOUWEN and PRZY-MUSIŃSKY [1979]: there is a countable space with only one non-isolated point, every compactification of which contains a copy of  $\beta\omega$ . Compare this question also with Answer **22**: it is consistent that for every  $p \in \omega^*$  every compactification of  $\omega \cup \{p\}$  contains a copy of  $\beta\omega$ . There is however, as far as we know, no **ZFC**-construction of a point with these properties.

For the next question identify  $\omega$  with  $\mathbb{Q}$ .

? **229. Question 30.** *Is there  $p \in \omega^*$  such that  $\{A \in p : A \text{ is closed and nowhere dense in } \mathbb{Q} \text{ and also homeomorphic to } \mathbb{Q}\}$  is a base for  $p$ ?*

This question arose in the study of remote points. A point  $p$  in  $\beta X \setminus X$  is called a remote point of  $X$  if for every nowhere dense subset  $D$  of  $X$  one has  $p \notin \overline{D}$ . For us it is important to know that  $\mathbb{Q}$  has remote points (VAN DOUWEN [1981b] and CHAE and SMITH [1980]).  $\mathbb{Q}$  also has non-remote points: simply take  $p \in \overline{\mathbb{N}}$ , such a point is also a real ultrafilter on the set  $\mathbb{Q}$ .

Problem 30 asks for a not-so-simple non-remote point, which is still a real ultrafilter on  $\mathbb{Q}$ . The answer is known to be YES under  $\mathbf{MA}_{\text{countable}}$ . A related question is the following:

**Question 31.** *Is there a  $p \in \omega^*$  such that whenever  $\langle x_n : n \in \omega \rangle$  is a sequence in  $\mathbb{Q}$  there is an  $A \in p$  such that  $\{x_n : n \in A\}$  is nowhere dense?* **230. ?**

For the next question let

$$\mathbb{G} = \left\langle \langle f_\alpha : \alpha \in \omega_1 \rangle, \langle g_\alpha : \alpha \in \omega_1 \rangle \right\rangle$$

be a Hausdorff Gap in  ${}^\omega\omega$ , and let

$$I_{\mathbb{G}} = \{M : \exists h \in {}^M\omega \forall \alpha f_\alpha \restriction M <^* h <^* g_\alpha \restriction M\}$$

Under  $\mathbf{MA} + \neg\mathbf{CH}$  this ideal is tall.

**Question 32.** ( $\mathbf{MA} + \neg\mathbf{CH}$ ) *Are there  $\mathbb{G}$  and  $p$  ( $P$ -point, selective) such that  $p \subseteq I_{\mathbb{G}}^+$ ?* **231. ?**

This question is more delicate than it may seem: it is a theorem of Woodin, see DALES and WOODIN [1987], that under  $\mathbf{MA} + \neg\mathbf{CH}$  one can find for every  $p \in \omega^*$  an  $h \in {}^\omega\omega$  such that for all  $\alpha \in \omega_1$   $f_\alpha <_p h <_p g_\alpha$ , where  $f <_p g$  means that  $\{n : f(n) < g(n)\} \in p$ .

Now let  $p$  be a  $P_c$ -point, and find  $A \in p$  such that  $A \subseteq^* \{n : f_\alpha(n) < h(n) < g_\alpha(n)\}$  for all  $\alpha$ . It follows that  $A \in I_{\mathbb{G}}$ . Loosely speaking one can say that to a  $P_c$ -point  $\omega_1$  seems countable. What we are asking for here is an ultrafilter with some strong properties that, in spite of  $\mathbf{MA} + \neg\mathbf{CH}$ , considers  $\omega_1$  to be uncountable.

## 7. Dynamics, Algebra and Number Theory

For the next question identify  $\omega$  with  $\mathbb{Z}$ , and consider the shift  $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $\sigma(n) = n + 1$ . Denote its extension to  $\beta\mathbb{Z}$  also by  $\sigma$ , and likewise its restriction to  $\mathbb{Z}^*$ . For  $p \in \mathbb{Z}^*$  we let  $O_p$  denote its orbit  $\{\sigma^n(p) : n \in \mathbb{Z}\}$  and  $C_p$  is the closure of  $O_p$ .  $C_p$  is called an orbit closure. An orbit closure is called *maximal* if it is not a proper subset of any other orbit closure.

**Question 33.** *Is there a point in  $\omega^*$  that is not an element of any maximal orbit closure?* **232. ?**

It would also be of interest to know the answer to the following, related, question:

**Question 34.** *Is there an infinite strictly increasing sequence of orbit closures?* **233. ?**

The next question is related to Furstenberg's multiple recurrence theorem, FURSTENBERG [1981]. A convenient way to state this theorem is for us: if  $f$  and  $g$  are commuting continuous selfmaps of the Cantor set  ${}^\omega 2$  then there are a  $p \in \omega^*$  and an  $x \in {}^\omega 2$  such that  $p\text{-}\lim f^n(x) = p\text{-}\lim g^n(x) = x$ . The question is whether one can switch quantifiers, i.e.

? **234. Question 35.** *Is there a  $p \in \omega^*$  such that for every pair of commuting continuous maps  $f, g : {}^\omega 2 \rightarrow {}^\omega 2$  there is an  $x \in {}^\omega 2$  such that  $p\text{-}\lim f^n(x) = p\text{-}\lim g^n(x) = x$ ?*

There are ultrafilters  $p$  such that for every  $f$  there is an  $x$  such that  $x = p\text{-}\lim f^n(x)$ : take an idempotent in the semigroup  $\langle \omega^*, + \rangle$  (see below), pick  $y$  arbitrary and let  $x = p\text{-}\lim f^n(y)$ . An equivalent question is whether the Cantor cube  ${}^2$  satisfies the conclusion of Furstenberg's theorem. The answer is known to be yes under **MA**.

Let  $S_\omega$  denote the permutation group of  $\omega$ , it acts on  $\omega^*$  in the obvious way.

? **235. Question 36.** *For what nowhere dense sets  $A \subseteq \omega^*$  do we have  $\bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^*$ ?*

Let  $\mathfrak{n}$  be the smallest number of nowhere dense sets needed to cover  $\omega^*$ . In BALCAR, PELANT and SIMON [1980] it is shown that  $\mathfrak{c} < \mathfrak{n}$  is consistent, hence we can say “for all nowhere dense sets” in models for this inequality. However also  $\mathfrak{n} \leq \mathfrak{c}$  is consistent, and in such models we do not have an easy answer. Some nowhere dense sets satisfy the inequality in **ZFC**: singletons work since  $|\omega^*| > |S_\omega|$ . In addition GRYZLOV has shown in [1984] that in **ZFC** the following nowhere dense  $P$ -set also works:

$$A = \bigcap \{ X^* : X \subseteq \omega \text{ and } \delta(X) = 1 \},$$

where

$$\delta(X) = \lim_{n \rightarrow \infty} \frac{|X \cap n|}{n}$$

if this limit exists ( $\delta(X)$  is called the density of  $X$  in  $\omega$ ).

There is a natural nowhere dense set in  $\omega^*$  the permutations of which consistently cover  $\omega^*$ : identify  $\omega$  and  $\omega^2$ , and for every  $n \in \omega$  and  $f \in {}^\omega \omega$  put  $U(f, n) = \{ \langle k, l \rangle : k \geq n, l \geq f(k) \}$ . The set  $B = \bigcap \{ U(f, n)^* : f \in {}^\omega \omega \text{ and } n \in \omega \}$  is nowhere dense and  $\bigcup_{\pi \in S_\omega} \pi[B] = \omega^* \setminus \{ p : p \text{ is a } P\text{-point} \}$ . A probably more difficult question is:

? **236. Question 37.** *For what nowhere dense sets  $A \subseteq \omega^*$  do we have  $\bigcup \{ h[A] : h \in \mathcal{H}(\omega^*) \} \neq \omega^*$ ?*

Again singletons work, but now the reason is deeper:  $\omega^*$  is not homogeneous. Clearly the set  $B$  of the question 36 also satisfies  $\bigcup\{h[B] : h \in \mathcal{H}(\omega^*)\} = \omega^* \setminus \{p : p \text{ is a } P\text{-point}\}$ . As a start one may investigate the set  $A$  defined above.

For the next set of questions we consider binary operations on  $\beta\omega$ . If  $*$  is any binary operation on  $\omega$  then one can extend it in a natural way to  $\beta\omega$  as follows: first define  $p*n$  for  $p \in \omega^*$  and  $n \in \omega$  by  $p*n = p - \lim m*n$ ; then if  $q \in \omega^*$  we define  $p*q = q - \lim p*n$ . It is not hard to verify that this operation is continuous in the second coordinate. We shall be especially interested in the cases  $*$  = + and  $*$  =  $\times$ . In these cases  $\langle\beta\omega, *\rangle$  is a right-continuous semigroup. A lot is known about these semigroups, see HINDMAN [1979] and VAN DOUWEN [1980b], but some questions still remain.

**Question 38.** *Can  $\langle\beta\mathbb{N}, +\rangle$  be embedded in  $\langle\mathbb{N}^*, +\rangle$ ?* **237. ?**

**Question 39.** *Are there  $p, q, r$  and  $s$  in  $\mathbb{N}^*$  such that  $p + q = r \times s$ ?* **238. ?**

**Question 40.** *What are the maximal subgroups of  $\langle\beta\mathbb{N}, +\rangle$  and  $\langle\beta\mathbb{N}, \times\rangle$ ?* **239. ?**

The next two questions are from VAN DOUWEN [1981a], where one can find much more information on the topic of these problems. To begin a definition: a map  $h: X \times Y \rightarrow S \times T$  is said to be elementary if it is a product of two mappings or a product composed with (if possible) a reflection on  $S \times T$ . Let  $X$  be  $\omega^*$  or  $\beta\omega$ .

**Question 41.** *If  $h: X^2 \rightarrow X^2$  is a homeomorphism, is there a disjoint open cover  $\mathcal{U}$  of  $X$  such that for all  $U, V \in \mathcal{U}$  the map  $h|U \times V$  is elementary?* **240. ?**

**Question 42.** *If  $\varphi: X^2 \rightarrow X$  is continuous, is there a disjoint open cover  $\mathcal{U}$  of  $X$  such that for all  $U, V \in \mathcal{U}$  the map  $\varphi|U \times V$  depends on one coordinate?* **241. ?**

## 8. Other

The following question is one of the most, if not *the* most, interesting problems about  $\beta\omega$ .

**Question 43.** *Are  $\omega^*$  and  $\omega_1^*$  ever homeomorphic?* **242. ?**

In spite of its simplicity and the general gut reaction: NO!, it is still unanswered. A nice touch to this question is its Boolean Algebraic variant:

are the Boolean Algebras  $\mathcal{P}(\omega)/fin$  and  $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$  ever isomorphic?

This variant also makes sense in the absence of the Axiom of Choice (**AC**): the spaces  $\omega^*$  and  $\omega_1^*$  then need not exist (FEFERMAN [1964/65]); but the



algebras  $\mathcal{P}(\omega)/fin$  and  $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$  always do. It would be very interesting indeed if on the one hand in **ZFC** the spaces  $\omega^*$  and  $\omega_1^*$  are not homeomorphic, while on the other hand it would be consistent with **ZF** that  $\mathcal{P}(\omega)/fin$  and  $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$  are isomorphic.

In BALCAR and FRANKIEWICZ [1978] it is shown that if  $\omega^*$  and  $\kappa^*$  are homeomorphic for some uncountable regular  $\kappa$  then there is a  $\kappa$ -scale in  ${}^\omega\omega$ . As a consequence one obtains that if  $\mathfrak{h} < \mathfrak{d}$  the spaces  $\omega^*$  and  $\omega_1^*$  are not homeomorphic. The same conclusion follows from **MA**: if **CH** is true then  $|\omega^*| < |\omega_1^*|$  and if **CH** is false then there is no  $\omega_1$ -scale. The results from FRANKIEWICZ [1977] allow one to conclude that  $\omega_1^*$  and  $\omega_2^*$  are not homeomorphic and that then, in fact,  $\kappa^*$  and  $\lambda^*$  are not homeomorphic if  $\omega \leq \kappa < \lambda$  and  $\langle \kappa, \lambda \rangle \neq \langle \omega, \omega_1 \rangle$ . Some of the consequences of a positive answer have been shown to be consistent, see e.g., STEPRĀNS [1985].

Related to this question is the following:

- ? 243. **Question 44.** *Is there consistently an uncountable cardinal  $\kappa$  such that  $\omega^*$  and  $U(\kappa)$  are homeomorphic?*

Here,  $U(\kappa)$  is the subspace of  $\beta\kappa$  consisting of all uniform ultrafilters. Let us observe that for such a  $\kappa$  we would have  $\text{cof}(\kappa) = \omega$  and  $2^\omega = 2^\kappa$ , see VAN DOUWEN [19∞a] for more information, including a proof of the following curious fact: there is at most one  $n \in \omega$  for which there is a  $\kappa > \omega_n$  with  $U(\omega_n)$  and  $U(\kappa)$  homeomorphic.

Recall from question 36 that  $\mathfrak{n}$  is the minimal number of nowhere dense sets needed to cover  $\omega^*$ . For any dense-in-itself topological space  $X$  one can define  $n(X)$  ( $wn(X)$ ) as the minimal cardinality of a family of nowhere dense sets that covers  $X$  (has a dense union). The number  $wn(\omega^*)$  is equal to the cardinal  $\mathfrak{h}$  (see the article by Vaughan). It is straightforward to show that  $n(X^n) \geq n(X^m)$  and  $wn(X^n) \geq wn(X^m)$  whenever  $n \leq m$ . The general question is about the behaviour of the sequences  $\langle n(\omega^*) : n \in \mathbb{N} \rangle$  and  $\langle wn(\omega^*) : n \in \mathbb{N} \rangle$ . Some specific questions:

- ? 244. **Question 45.** *When do the sequences  $\langle n(\omega^*) : n \in \mathbb{N} \rangle$  and  $\langle wn(\omega^*) : n \in \mathbb{N} \rangle$  become constant?*
- ? 245. **Question 46.** *Is it consistent that  $n(\omega^*) > n(\omega^* \times \omega^*)$ , that  $wn(\omega^*) > wn(\omega^* \times \omega^*)$ ?*

What is known is that  $\mathfrak{n} > \mathfrak{c}$  implies  $n(\omega^*) = n(\omega^* \times \omega^*)$ .

Here we pose the question suggested by Answer 13:

- ? 246. **Question 47.** *Does the statement that all Parovichenko spaces are co-absolute (with  $\omega^*$ ) imply that  $cf(\mathfrak{c}) = \omega_1$ ?*

**Question 48.** *Let  $X$  be a compact space that can be mapped onto  $\omega^*$ . Is  $X$  non-homogeneous?* **247. ?**

VAN DOUWEN proved in [1978] proved that the answer to this question is in the affirmative provided that  $X$  has weight at most  $\mathfrak{c}$ . In general, the problem is unsolved. For more information, see the article by Kunen in this volume.

**Question 49.** *Is it consistent that every compact space contains either a converging sequence or a copy of  $\beta\omega$ ?* **248. ?**

Under various extra-set-theoretical assumptions compact spaces have been constructed that contain neither a converging sequence nor a copy of  $\beta\omega$ , but no **ZFC**-example is known. It is as far as we know also unknown what the effect of **MA** +  $\neg$ **CH** is on this problem. Of this question there is also a Boolean Algebraic variant: is it consistent that every infinite Boolean Algebra has either a countably infinite homomorphic image or a complete homomorphic image.

**Question 50.** *Is there a locally connected continuum such that every proper subcontinuum contains a copy of  $\beta\omega$ ?* **249. ?**

**Question 51.** *Is there an extremally disconnected normal locally compact space that is not paracompact?* **250. ?**

KUNEN and PARSONS proved in [1979] proved that if  $\kappa$  is weakly compact, then the space  $\beta\kappa \setminus U(\kappa)$  is normal but not paracompact. In addition, VAN DOUWEN [1979] proved that there is a locally compact basically disconnected (= the closure of every open  $F_\sigma$ -set is open) space which is normal but not paracompact. This is basically all we know about this problem.

**Question 52.** *Is every compact hereditarily paracompact space of weight at most  $\mathfrak{c}$  a continuous image of  $\omega^*$ ?* **251. ?**

This question is related to Answer 10: Przymusiński showed that every perfectly normal (= hereditarily Lindelöf) compact space is a continuous image of  $\omega^*$ , whereas the first-countable nonimage by Bell is hereditarily metacompact. Since perfectly normal compact spaces are (hereditarily) **ccc**, and since separable compact spaces are clearly continuous images of  $\omega^*$ , we are also led to ask:

**Question 53.** *Is every hereditarily **ccc** compact space a continuous image of  $\omega^*$ ?* **252. ?**

The answer is yes under **MA** $_{\omega_1}$  by SZENTMIKLOSSY'S result from [1978] that then compact hereditarily **ccc** spaces are perfectly normal, and under

**CH** by Parovichenko's theorem, since compact hereditarily **ccc** spaces are of size at most  $\mathfrak{c}$ , see e.g., HODEL [1984].

Identify  $\mathcal{P}(\omega)$  with  ${}^\omega 2$ , and define  $\leq_\alpha$  by  $p \leq_\alpha q$  iff there is a map  $f: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$  of Baire class  $\alpha$  such that  $f(q) = p$ .

- ? **253. Question 54.** Suppose that  $p \leq_\alpha q$  and  $q \leq_\alpha p$ . Are  $p$  and  $q$  **RK**-equivalent, or can they be mapped to each other by a Baire isomorphism of class  $\alpha$ ?
- ? **254. Question 55.** Do  $\leq_\alpha$ -minimal points exist, and can they be characterized?
- ? **255. Question 56.** Do  $\leq_\alpha$ -incomparable points exist?

For the next question identify  $\omega$  with  $\mathbb{Q}$ .

- ? **256. Question 57.** If  $I$  is the ideal of nowhere dense subsets of  $\mathbb{Q}$  can  $I$  be extended to a (tall)  $P$ -ideal?

See DOW [1990] for more information on this problem (a YES answer implies that the space of minimal prime ideals of  $C(\omega^*)$  is not basically disconnected).

The following question is probably more about forcing than about  $\beta\omega$ .

- ? **257. Question 58.** Is there a **ccc** forcing extension of  $L$ , in which there are no  $P$ -points?

Now we formulate some problems on characters of ultrafilters. It is easy to show that  $\omega_1 \leq \chi(p) \leq \mathfrak{c}$  for all  $p \in \omega^*$ . Furthermore in [1939] POSPÍŠIL has shown that there are  $2^{\mathfrak{c}}$  points in  $\omega^*$  of character  $\mathfrak{c}$ . This is the best one can say: under **MA** we have  $\chi(p) = \mathfrak{c}$  for all  $p \in \omega^*$  while by exercise VII A10 in KUNEN [1980] the existence of a  $p \in \omega^*$  with  $\chi(p) = \omega_1$  is consistent with any cardinal arithmetic.

There are several models in which one has a  $p \in \omega^*$  with  $\chi(p) < \mathfrak{c}$ , but these models have a few properties in common.

The first is that in all of these models there are  $P$ -points in  $\omega^*$ .

These constructions fall roughly speaking into two categories: in the first of these, and the models from KUNEN [1980] fall into this one, the ultrafilter of small character is built in an iterated forcing construction and is almost unavoidably a  $P$ -point.

In the constructions of the second category one normally starts with a model of **CH** and enlarges the continuum while preserving some ultrafilters from the ground model. Again the ultrafilters that are preserved are most of the time  $P$ -points. An extreme case of this are the models for **NCF** from BLASS and SHELAH [1987, 19 $\infty$ ]: there the ultrafilters that are preserved are precisely the  $P$ -points and in the final model we even have  $\chi(p) < \mathfrak{c}$  if and only if  $p$  is a  $P$ -point.

In HART [1989] the first author showed that in the model obtained by adding any number of Sacks-reals side-by-side there are many types of ultrafilters of character  $\omega_1$  including very many non  $P$ -points. Unfortunately all these ultrafilters were constructed using  $P$ -points. This leads us to our first problem:

**Question 59.** *Is there a model in which there are no  $P$ -points, but there is an ultrafilter of character less than  $\mathfrak{c}$ ?* **258. ?**

This is probably a very difficult problem and an answer to the following problem may be easier to give:

**Question 60.** *Is there a model in which there is an ultrafilter of character less than  $\mathfrak{c}$  without any  $P$ -point below it in the Rudin-Keisler order?* **259. ?**

The second property of these models is maybe not so obvious: in all models that we know of there seems to be only one character below  $\mathfrak{c}$ . In the majority of these models we have  $\mathfrak{c} = \omega_2$  so that an ultrafilter of small character automatically has character  $\omega_1$ . In various other models usually nothing is known about ultrafilters other than the ones constructed explicitly. Thus we get to our second problem: let  $\Xi = \{ \chi(p) : p \in \omega^* \text{ and } \chi(p) < \mathfrak{c} \}$ .

**Question 61.** *What are the possibilities for  $\Xi$ ; can  $\Xi$  be the set of all (regular) cardinals below  $\mathfrak{c}$ , with  $\mathfrak{c}$  large; what is  $\Xi$  in the side-by-side Sacks model?* **260. ?**

In [1989] FRANKIEWICZ, SHELAH and ZBIERSKI announce the consistency of “ $\mathfrak{c} > \omega_2$  and for every regular  $\kappa \leq \mathfrak{c}$  there is an ultrafilter of character  $\kappa$ ”.

Here we mention a well-known question on the Rudin-Keisler order, which has some partial answers involving characters of ultrafilters.

**Question 62.** *Is there for every  $p \in \omega^*$  a  $q \in \omega^*$  such that  $p$  and  $q$  are  $\leq_{\mathbf{RK}}$ -incomparable?* **261. ?**

It is known that there exist  $p$  and  $q$  in  $\omega^*$  such that  $p$  and  $q$  are  $\leq_{\mathbf{RK}}$ -incomparable (KUNEN [1972]). However, the full answer to this problem is not known yet, some partial positive results can be found in HINDMAN [1988] and BUTKOVIČOVÁ [1990b]; for example if  $p$  is such that  $\chi(r) = \mathfrak{c}$  for every  $r \leq_{\mathbf{RK}} p$  then there is a  $q$  that is  $\leq_{\mathbf{RK}}$ -incomparable with  $p$ , and if  $2^\kappa > \mathfrak{c}$  for some  $\kappa < \mathfrak{c}$  then such a  $q$  can be found for every  $p$  of character  $\mathfrak{c}$ . The ideas in BLASS and SHELAH [1987, 1990] may shed light on this problem.

The  $\pi$ -character of a point  $p$  in a space  $X$  is the minimum cardinality of a family of nonempty open sets such that every neighborhood of  $p$  contains one of them.

- ? **262. Question 63.** *Is there consistently a point in  $\omega^*$  whose  $\pi$ -character has countable cofinality?*

Under **MA** the  $\pi$ -character of any ultrafilter is  $\mathfrak{c}$ . It was shown by BELL and KUNEN in [1981] that there is always a  $p$  with  $\pi\chi(p) \geq \text{cof}(\mathfrak{c})$  and that it is consistent that both  $\mathfrak{c} = \aleph_{\omega_1}$  and  $\pi\chi(p) = \omega_1$  for all  $p \in \omega^*$ .

Another character problem is the following:

- ? **263. Question 64.** *Is it consistent that  $t(p, \omega^*) < \chi(p)$  for some  $p \in \omega^*$ ?*

The tightness  $t(p, X)$  of a point  $p$  in a space  $X$  is the smallest cardinal  $\kappa$  such that: whenever  $p \in \overline{A}$  there is a  $B \subseteq A$  with  $|B| \leq \kappa$  such that  $p \in \overline{B}$ .

For the next few questions we consider the product  $\omega \times I$ , the projection  $\pi: \omega \times I \rightarrow \omega$  and its Čech-Stone-extension  $\beta\pi$ . For  $p \in \omega^*$  we put  $I_p = \beta\pi^{-1}(p)$ . It is not too hard to show that  $I_p$  is a continuum, and in fact a component of the remainder of  $\omega \times I$ . Our first question is:

- ? **264. Question 65.** *Are there  $p$  and  $q$  with  $I_p$  and  $I_q$  not homeomorphic?*

$I_p$  has many cutpoints: for every  $f: \omega \rightarrow I$  the point  $f_p = p - \lim \langle n, f(n) \rangle$  is a cutpoint of  $I_p$  provided  $\{n : f(n) \neq 0, 1\} \in p$ . The question is whether there are any others.

- ? **265. Question 66.** *Are there cutpoints in  $I_p$  other than the points  $f_p$  for  $f: \omega \rightarrow I$ ?*

Under **MA**<sub>countable</sub> such points exist, and it is conjectured that there are none in Laver's model (LAVER [1976]) for the Borel Conjecture.

- ? **266. Question 67.** *How many subcontinua does  $I_p$  have?*

SMITH [1986] and VAN DOUWEN [1977] have a few.

The next few questions come from analysis. We refer the reader to the book by DALES and WOODIN [1987] for more information on, and references for, what follows.

It is an old problem of Kaplansky for what (if any) compact spaces  $X$  the algebra  $C(X)$  admits an incomplete norm. For the moment call  $X$  *incomplete* if  $C(X)$  does admit an incomplete norm. It is not overly difficult to show that if  $\beta\omega$  is incomplete then so is every space  $X$ . Moreover if some space is incomplete then so is  $\omega + 1$  (the converging sequence).

The problem itself is solved to a large extent. Dales and Esterle independently showed that under **CH** the space  $\beta\omega$ —and hence every space—is incomplete. Woodin showed that it is consistent with **MA** +  $\neg\text{CH}$  that  $\omega + 1$ —and hence every space—is not incomplete. What remains is the following question:

**Question 68.** *If  $C(\omega + 1)$  admits an incomplete norm then does  $C(\beta\omega)$  admit one too?* **267. ?**

To make the question maybe a bit more managable and also to be able to pose some more specialized problems we take the following facts from DALES and WOODIN [1987]: to begin observe that  $C(\beta\omega)$  is the same as  $\ell^\infty$ . For  $p \in \omega^*$  put  $\mathcal{M}_p = \{x \in \ell^\infty : p\text{-}\lim x = 0\}$  and  $\mathcal{I}_p = \{x \in \ell^\infty : \{n : x(n) = 0\} \in p\}$ . The quotient algebra  $\mathcal{M}_p/\mathcal{I}_p$  is denoted by  $A_p$ . Now one can show that  $\beta\omega$  is incomplete iff for some  $p \in \omega^*$  the algebra  $A_p$  admits a non-trivial seminorm. One can do a similar thing for  $\omega + 1$ . We write  $c_0/p$  for the algebra  $c_0/\mathcal{I}_p$ , where as usual  $c_0 = \{x \in \ell^\infty : \lim x = 0\}$ . Now  $\omega + 1$  is incomplete iff for some  $p \in \omega^*$  the algebra  $c_0/p$  admits a non-trivial seminorm.

We see that if there is a  $p \in \omega^*$  such that  $A_p$  is seminormable then there is a  $q \in \omega^*$  such that  $c_0/q$  is seminormable. Problem 68 now becomes: “if  $c_0/p$  is seminormable for some  $p$ , is there a  $q$  such that  $A_q$  is seminormable?” A stronger question is:

**Question 69.** *If  $p \in \omega^*$  and  $c_0/p$  is seminormable, is  $A_p$  seminormable?* **268. ?**

It would also be interesting to know the answer to the following:

**Question 70.** *If  $p \in \omega^*$  and  $A_p$  is seminormable, is  $c_0/p$  seminormable?* **269. ?**

Finally, to end this set, we mention a question connected to Woodin’s proof. First we define a partial order  $\ll$  on the algebras  $A_p$  and  $c_0/p$ : say  $a \ll b$  iff there is a  $c$  such that  $a = bc$ . If  $B$  is  $A_p$  or  $c_0/p$  we call  $B$  weakly seminormable iff there are a nonempty downward closed—wrt.  $\ll$ —subset  $S$  of  $B \setminus \{0\}$  and a strictly increasing map of  $\langle S, \ll \rangle$  into  $\langle {}^\omega\omega, <^* \rangle$ . It can be shown that if  $B$  is seminormable, it is also weakly seminormable. In addition if there is a  $p$  such that  $c_0/p$  is weakly seminormable, there is also a  $q$  such that  $A_q$  is weakly seminormable. Thus a positive answer to the following question would also answer question 68 positively.

**Question 71.** *If  $p \in \omega^*$  and  $A_p$  is weakly seminormable, is  $A_p$  seminormable, or is  $A_q$  for some other  $q$ ?* **270. ?**

We now turn to the Rudin-Frolík order  $\leq_{\mathbf{RF}}$  on  $\omega^*$ , which is defined as follows:  $p \leq_{\mathbf{RF}} q$  iff there is an embedding  $i: \beta\omega \rightarrow \beta\omega$  such that  $i(p) = q$ . A lot is known about this order but a few problems remain:

**Question 72.** *What are the possible lengths of unbounded  $\leq_{\mathbf{RF}}$ -chains?* **271. ?**

In [1985, 1984] BUTKOVIČOVÁ has shown that  $\omega_1$  and  $\mathfrak{c}^+$  are both possible. Another question is related to decreasing  $\leq_{\mathbf{RF}}$ -chains:

? **272. Question 73.** *For what cardinals  $\kappa$  is there a strictly decreasing chain of copies of  $\beta\omega$  in  $\omega^*$  with a one-point intersection?*

VAN DOUWEN [1985] showed that  $\mathfrak{c}$  works. It is readily seen that for any  $\kappa$  for which there is a positive answer to this question one gets a strictly decreasing  $\leq_{\mathbf{RF}}$ -chain of length  $\kappa$  without a lower bound.

However BUTKOVIČOVÁ [19 $\infty$ a] has shown that such chains exist for every infinite  $\kappa < \mathfrak{c}$ . What is needed, in case  $\kappa$  has uncountable cofinality, to produce such a chain, is a strictly decreasing sequence  $\langle X_\alpha : \alpha < \kappa \rangle$  of copies of  $\beta\omega$  and a point  $p$  in  $K = \bigcap_{\alpha < \kappa} X_\alpha$  which is not an accumulation point of any countable discrete subset of  $K$ . Butkovičová constructed such a sequence and such a point directly, but one naturally wonders whether this can be done for every sequence of copies of  $\beta\omega$ .

? **273. Question 74.** *If  $\kappa \leq \mathfrak{c}$  has uncountable cofinality and if  $\langle X_\alpha : \alpha < \kappa \rangle$  is a strictly decreasing sequence of copies of  $\beta\omega$  with intersection  $K$ , is there a point  $p$  in  $K$  that is not an accumulation point of any countable discrete subset of  $K$ ?*

## 9. Uncountable Cardinals

In this section we collect some questions on ultrafilters on uncountable cardinals, and we are mainly interested in uniform ultrafilters here. We let  $\kappa$  denote an arbitrary infinite cardinal. To begin we ask whether ultrafilters of small character may exist.

? **274. Question 75.** *Is there consistently an uncountable cardinal  $\kappa$  with a  $p \in U(\kappa)$  such that  $\chi(p) < 2^\kappa$ ?*

Let us note that for “small” uncountable cardinals there is no easy analogue of Kunen’s method mentioned above (see problem 59): to preserve the cardinals below  $\kappa$  one seems to need a  $\kappa$ -complete ultrafilter, and that brings us immediately to measurable cardinals. So we ask in particular:

? **275. Question 76.** *Is it consistent to have a measurable cardinal  $\kappa$  with a  $p \in U(\kappa)$  such that  $\chi(p) < 2^\kappa$ ?*

And to stay somewhat down to earth we also ask specifically:

? **276. Question 77.** *Is it consistent to have a uniform ultrafilter on  $\omega_1$  of character less than  $2^{\omega_1}$  e.g.,  $\omega_2$ ?*

A related and intriguing question is:

**Question 78.** *Is it consistent to have cardinals  $\kappa < \lambda$  with points  $p \in U(\kappa)$  and  $q \in U(\lambda)$  such that  $\chi(p) > \chi(q)$ ?* **277. ?**

A question with a topological background is the following:

**Question 79.** *If  $\kappa \geq \omega$  is nonmeasurable and  $\mathcal{F}$  is a countably complete uniform filter on  $\kappa^+$  then what is the cardinality of the set  $\{u \in U(\kappa^+) : \mathcal{F} \subseteq u\}$ ?* **278. ?**

If the cardinality is  $2^{2^{\kappa^+}}$  then there are almost Lindelöf spaces  $X$  and  $Y$  with  $X \times Y$  not even almost  $\kappa$ -Lindelöf. For  $\kappa = \omega$  the cardinality is indeed  $2^{2^{\omega_1}}$ , see BALCAR and ŠTĚPÁNEK [1986].

The next question is purely topological. To state it we must make some definitions. In general if  $A$  is a subset of a topological space  $X$  we let  $[A]_{<\kappa}$  denote the set  $\bigcup\{\overline{B} : B \in [A]^{<\kappa}\}$ . Furthermore if  $\kappa \subseteq X \subseteq \beta\kappa$  then  $\beta_X\kappa$  is the maximal subset of  $\beta\kappa$  for which every (continuous)  $f: \kappa \rightarrow X$  has a continuous extension  $\bar{f}: \beta_X\kappa \rightarrow X$ . The question is

**Question 80.** *Assume that  $\kappa$  is regular, that  $\kappa \subseteq X \subseteq \beta\kappa$  is such that  $[X]_{<\kappa} = X$  and  $\beta_X\kappa = X$ . Now if  $Y$  is a closed subspace of a power of  $X$ , is then also  $X$  a closed subspace of a power of  $Y$ ?* **279. ?**

The answer is yes for  $\kappa = \omega$ , see HUŠEK and PELANT [1974] for the proof and more information.

The following question is related to the analysis-type problems from the previous section. If  $p$  is a (uniform) ultrafilter on  $\kappa$  then we denote by  $\mathbb{R}_p$  the ultrapower of  $\mathbb{R}$  modulo the ultrafilter  $p$ . On it we define an equivalence relation  $\equiv$  by  $a \equiv b$  iff there is an  $n \in \mathbb{N}$  such that  $|a| < |nb|$  and  $|b| < |na|$ , here  $<$  is the natural linear order of  $\mathbb{R}_p$ . The equivalence classes under  $\equiv$  are called the Archimedean classes of  $\mathbb{R}_p$ . The question is whether the cardinality of  $\mathbb{R}_p$  can be larger than the cardinality of  $\mathbb{R}_p / \equiv$ , specifically:

**Question 81.** *Are there  $\kappa$  and  $p \in U(\kappa)$  such that  $|\mathbb{R}_p| > |\mathbb{R}_p / \equiv| = \kappa$ ?* **280. ?**

Dales and Woodin have shown that a positive answer is consistent relative to the existence of a large cardinal.

We finish with two questions about  $\omega_1$ .

**Question 82.** *Is there a  $C^*$ -embedded bi-Bernstein set in  $U(\omega_1)$ ?* **281. ?**

A bi-Bernstein set is a set  $X$  such that  $X$  and its complement intersect every uncountable closed subset of  $U(\omega_1)$ .



? 282. **Question 83.** Are there open sets  $G_1$  and  $G_2$  in  $U(\omega_1)$  such that  $\overline{G_1} \cap \overline{G_2}$  consists of exactly one point?

See DOW [1988b] for more information on  $\omega_1^*$ .

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## CHAPTER 8

### **On first countable, countably compact spaces III: The problem of obtaining separable noncompact examples**

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For a decade now, the problem in the title has held a special fascination for me. It is a problem that can be approached on many levels and from many angles, and has led me and others off on numerous unexpected tangents.

Before getting into that, a word about separation axiom: all through this paper “space” will mean “Hausdorff space”, but for first countable, countably compact spaces this implies regularity as well (ENGELKING [1977, p.398]). For  $T_1$  spaces there is an easy example: refine the cofinite topology on  $\omega_1$  by declaring initial segments to be open. It is the extra separation axiom that makes the title problem so difficult.

Set-theoretically, it has a strange dual personality. On the one hand, the construction of examples under **CH** is one of the easiest of all consistency results in topology; so easy that when Mary Ellen Rudin performed it in the early 60’s, she did not bother to publish it, and even now I do not know whether her construction was the first. On the other hand, unless there is a clever **ZFC** construction, we still do not seem to have the proper machinery for solving it. It is one of a handful of problems (of which the normality of  $\square^\omega\omega + 1$  is the best known) for which we have affirmative answers if either  $\mathfrak{c} = \aleph_1$  or  $\mathfrak{c} = \aleph_2$ . Most of the sophisticated forcing construction nowadays give these values to the continuum, and while there has been growing interest in “mixed supports”, which does give interesting new models of  $\mathfrak{c} \geq \aleph_3$ , little of it has been directed towards the “no man’s land” of the title problem. These are the models where  $\omega_1 < \mathfrak{p}$  and  $\mathfrak{b} < \mathfrak{c}$  (notation explained later). Even today our supply of such models is quite meager, especially of models of  $\omega_1 < \mathfrak{p}$  and  $\mathfrak{b} < \mathfrak{d}$ , from where a negative answer to the title question (if indeed there is one) is most likely to come. I will however describe one towards the end of this article.

Topologically, the problem has many “relatives”, some of which I described in a 1986 survey paper (NYIKOS [1988b]) for the proceedings of the Sixth Prague Topological Symposium. Here we will mostly be concerned with “sub-problems”: problems which ask for the construction of special kinds of spaces as in the title. For example, it is not known whether one can always construct locally compact or locally countable ones; in fact, we do not (although perhaps we ought to) know whether the problem is made any harder by imposing these extra conditions. (At present, it actually appears harder to construct examples that do *not* satisfy them!) On the other hand, here are some problems where the difficulties seem genuinely different:

**Problem 1.** *Is there a separable, countably compact, noncompact manifold?* **283.** ?

Most of our current difficulty with Problem 1 is embodied in:

**Problem 2.** *Is it consistent that there is a countably compact manifold of weight  $> \aleph_1$ ?* **284.** ?



In NYIKOS [1982] and [1984] I emphasized those aspects of the theory of nonmetrizable manifolds where there was a parallelism with the theory of locally compact, locally countable spaces; but Problem 2 is one place where extra “geometric” difficulties with manifolds disrupt our analogies.

To theory of ultrafilters figures prominently in another problem, which seems to be the special case of “the Scarborough-Stone problem” in which they seemed to be most interested (SCARBOROUGH and STONE [1966, Footnote 1]):

? **285. Problem 3.** *Is countable compactness productive for first countable spaces?*

Here we have negative answers in all models of  $\mathfrak{b} = \mathfrak{c}$  (VAN DOUWEN [1984, Example 13.1]), and in an assortment of models of  $\mathfrak{p} = \omega_1$  (NYIKOS and VAUGHAN [1987]), though by no means all. In one way it is a special case of the title problem: if there is a model where all separable, countably compact, first countable spaces are compact, then in that model we have an affirmative solution to Problem 3. (See the discussion following Problem 7 below.)

Two other problems have more to do with the nature of **CH** than with topology, but I felt they are important enough to have offered cash prizes in the 1986 survey for their solution:

? **286. Problem 4.** *Call a space an Ostaszewski space if it is locally compact, locally countable, countably compact, noncompact, and every open set is either countable or co-countable. Is **CH** alone enough to imply the existence of an Ostaszewski space? (\$200 for yes, \$50 for no.)*

As is well known, an Ostaszewski space is perfectly normal: normality comes from the fact that, of any of two disjoint closed sets, one is compact. It is also hereditarily separable: it is scattered, and every subspace has countably many (relatively) isolated points, which are dense in the space. (Scatteredness, which means that every nonempty subspace has at least one (relatively) isolated point, follows easily from local compactness and local countability.)

? **287. Problem 5.** *Is **CH** consistent with the statement that every perfectly normal, countably compact space is compact? (\$100 either way.)*

Back in [1977], SHELAH wrote that he had been trying to convince others of the importance of obtaining **MA**-like principles compatible with **CH**. He gave one such, a 2-coloring axiom known as “Shelah’s principle”. However, this axiom is compatible with  $\diamond$  and is thus not suitable for a negative answer to Problem 4, nor a positive answer to Problem 5. In [1982, p.237] SHELAH gives a more complicated axiom which is compatible with **CH** and implies all Aronszajn trees are special (SHELAH [1982, p.241]). C. Schindwein

has conjectured this axiom will give a negative answer to Problem 4 and an affirmative one to Problem 5.

As for the title problem, I have offered (NÝIKOS [1988b]) \$500 for a solution by August of 1996. In a way, this is a bet that the problem will still be around then; but it is a bet I would not mind losing.

The foregoing introduction only begins to explain why I am fascinated by these problems. I hope that by the end of this article, I will have been able to share this fascination with many readers.

## 1. Topological background

The title problem can be rephrased and extended using the following concepts.

**1.1. DEFINITION.** A space is (strongly)  $\omega$ -bounded if every countable (resp.  $\sigma$ -compact) subset has compact closure.

One obviously has: strongly  $\omega$ -bounded  $\Rightarrow$   $\omega$ -bounded  $\Rightarrow$  countably compact.

**Problem 6.** *Is it consistent that every first countable, countably compact space is strongly  $\omega$ -bounded?* **288. ?**

If one omits “strongly” from this question, one has the negative version of the title problem. There is some question about whether the two are actually equivalent:

**Problem 7.** *Is every  $\omega$ -bounded, first countable space strongly  $\omega$ -bounded?* **289. ?**

An affirmative answer to Problem 7 in **ZFC** has been claimed by a Yugoslav topologist, but this claim has not been confirmed.

An easy corollary of the Tikhonov theorem is that the class of  $\omega$ -bounded spaces and the class of strongly  $\omega$ -bounded spaces are both productive. Thus if Problem 6, or its variant with “strongly” dropped, has an affirmative solution, then it is consistent that Problem 3 does also.

For a deeper understanding of the title problem, one must know about Property **wD**:

**1.2. DEFINITION.** A space  $X$  satisfies *Property wD* if for every infinite closed discrete subspace  $D$  of  $X$  there is an infinite  $D' \subset D$  which expands to a discrete collection of open sets, i.e. there is a discrete family  $\{U_d : d \in D'\}$  of open sets such that  $U_d \cap D' = \{d\}$  for all  $d \in D'$ . A space  $X$  satisfies *Property D* if every countable closed discrete subspace expands to a discrete collection of open sets.

Countably compact spaces satisfy Property **D** (and *a fortiori* Property **wD**) by default. But much more importantly:

**1.3. THEOREM.** *Every first countable, countably compact space satisfies Property **wD** hereditarily.*

PROOF. Let  $X$  be first countable and countably compact, and let  $Y \subset X$ . If  $Y$  has an infinite closed discrete subspace  $D$ , then there is a point  $p \in X \setminus Y$  and a sequence  $\langle d_i \rangle \rightarrow p$  from  $D$ . Let  $\{U_n\}_{n=1}^\infty$  be a local base at  $p$ , with  $U_1 = X$  and (by regularity of  $X$ )  $\overline{U_{n+1}} \subset U_n$  for all  $n$ . For each  $d_i$  we pick an open neighborhood  $V_i$  which is a subset of every  $U_n$  containing  $d_i$  and missing the first (hence every)  $\overline{U_m}$  which does not contain  $d_i$ ; and by induction we can easily arrange for the  $V_i$  to be disjoint. Then  $\{V_i\}_{i=1}^\infty$  is a discrete collection.  $\square$

Every first countable, separable, countably compact noncompact space that has been constructed in one model or another, has always been built up by induction beginning with some well-known separable, first countable, noncompact space and successively adding limit points to kill off infinite closed discrete subspaces. It seems very likely that this will continue to be the only means of producing these spaces. If so, then it is absolutely essential to be able to insure that the intermediate subspaces satisfy **wD** hereditarily.

Now **wD** is a comparatively weak property; for instance, every realcompact space satisfies it (VAUGHAN [1978]). It is therefore nice to know (GILLMAN and JERISON [1960, 8.15]) that every first countable, realcompact space is hereditarily realcompact. True, a countably compact, realcompact space is compact (GILLMAN and JERISON [1960, 5H]) but that need not discourage us: we might still have a construction where the intermediate spaces are (hereditarily) realcompact, while the final space is not.

In fact, most of the **CH** constructions to date have actually built up the final space  $X_{\omega_1}$  as the union of an ascending sequence  $\langle X_\alpha : \alpha < \omega_1 \rangle$  of metrizable spaces, often countable ones. All known countably compact manifolds *can* be built up like this, because (see Problem 2) they are all of weight  $\leq \aleph_1$ , hence the union of  $\leq \aleph_1$  open metrizable, second countable subspaces. However, in some models the examples have to be “bigger” than this, and we still have little understanding of how to maintain hereditary realcompactness at and beyond the  $\omega_1$ st step.

For Property **wD** we have some good general results under certain set-theoretic hypotheses. We give some of these in Section 3. They come under the general strategy of “keeping everything nice until stage  $\mathfrak{c}$ ”. The other general strategy with which we have had a little success is “finishing up the construction quickly”. This will be used in the following section, and part of the time in Section 4.

## 2. The $\gamma\aleph$ construction.

In [1970], FRANKLIN and RAJAGOPALAN gave a simple example of a space as in the title, under a precise set-theoretic hypothesis. Their hypothesis, and

also the construction, were phrased in terms of the Stone-Čech remainder  $\mathbb{N}^*$  of the discrete space  $\mathbb{N}$  of positive integers.

Here we give a more elementary set-theoretic treatment. The hypothesis can be stated in one of two equivalent forms:  $\mathfrak{p} = \omega_1$  and  $\mathfrak{t} = \omega_1$ . Recall:

$$\mathfrak{p} = \min\{ |a| : a \text{ is a subbase for a filter of infinite subsets of } \omega, \\ \text{such that if } B \subset^* A \text{ for all } A \in a, \text{ then } B \text{ is finite} \}$$

Here we use  $B \subset^* A$  to mean that  $B \setminus A$  is finite and  $A \setminus B$  is infinite.

One obtains the definition of  $\mathfrak{t}$  by restricting the subbase  $a$  to be totally ordered by  $\subset^*$ . Obviously,  $\mathfrak{p} \leq \mathfrak{t}$ ; the question of whether equality holds is probably the most basic unsolved problem relating to the “small uncountable cardinals” of HECHLER [1972] and VAN DOUWEN [1984]. A simple “diagonal” argument shows  $\mathfrak{p}$  is uncountable, and ROTHBERGER showed in [1948] that  $\mathfrak{p} = \omega_1$  implies  $\mathfrak{t} = \omega_1$  (the converse is trivial); proofs of this and many other facts cited here about  $\mathfrak{p}$ ,  $\mathfrak{t}$ ,  $\mathfrak{b}$ , and  $\mathfrak{d}$  may be found in VAN DOUWEN [1984]. See also J. Vaughan’s article.

The Franklin-Rajagopalan example was a countably compact version (which exists iff  $\mathfrak{t} = \omega_1$ ) of  $\gamma\mathbb{N}$ :

**2.1. NOTATION.**  $\gamma\mathbb{N}$  is the generic symbol for any locally compact space  $X$  in which  $\mathbb{N}$  is a dense set of isolated points, and  $X \setminus \mathbb{N}$  is homeomorphic to  $\omega_1$  with the order topology.

We use a definition of  $\mathbb{N}$  that makes it disjoint from  $\omega_1$ , so that we will identify  $\gamma\mathbb{N} \setminus \mathbb{N}$  with  $\omega_1$ . The following is a recipe for constructing all versions of  $\gamma\mathbb{N}$ .

**2.2. EXAMPLE.** Let  $\langle A_\alpha : \alpha \in \omega_1 \rangle$  be a  $\subset^*$ -ascending sequence of infinite subsets of  $\mathbb{N}$ . (An easy “diagonal” argument allows one to construct such a sequence in **ZFC**.) Set  $A_{-1} = \emptyset$ . On the set  $\mathbb{N} \cup \omega_1$ , we impose the topology which has sets of the form  $\{n\}$  ( $n \in \mathbb{N}$ ) and  $U_n(\beta, \alpha]$  ( $n \in \mathbb{N}$ ,  $\beta \in \omega_1 \cup \{-1\}$ ,  $\alpha \in \omega_1$ ) as a base, where  $(\beta, \alpha]$  means  $\{\gamma \in \omega_1 : \beta < \gamma \leq \alpha\}$  and

$$U_n(\beta, \alpha] = (\beta, \alpha] \cup (A_\alpha \setminus A_\beta) \setminus \{1, \dots, n\} \quad \text{where} \quad \{1, \dots, n\} \subset \mathbb{N}.$$

It is easy to show by induction that  $U_1(-1, \alpha]$  is compact for each  $\alpha$  and hence  $\mathbb{N} \cup \omega_1$  with this topology is locally compact. Since  $A_\alpha \setminus A_\beta$  is infinite whenever  $\beta < \alpha$ , the points of  $\omega_1$  are nonisolated, hence  $\mathbb{N}$  is dense. Thus this gives a  $\gamma\mathbb{N}$ .

Conversely, given any  $\gamma\mathbb{N}$  and  $\alpha < \omega_1$ , we can use regularity of  $\gamma\mathbb{N}$  and compactness of  $[0, \alpha]$  to put  $[0, \alpha]$  and  $[\alpha + 1, \omega_1]$  into disjoint open sets  $U_\alpha$  and  $V$  respectively. Then  $U_\alpha$  (and also  $V$ ) will be closed as well, and we may as well assume  $U_\alpha$  is compact. Let  $A_\alpha = U_\alpha \cap \mathbb{N}$ . Then it is routine to show that  $\langle A_\alpha : \alpha \in \omega_1 \rangle$  is  $\subset^*$ -ascending and that the topology imposed on  $\mathbb{N} \cup \omega_1$  by the above recipe is the one we started out with.

Since every point of  $\gamma\mathbb{N}$  has a compact, countable neighborhood,  $\gamma\mathbb{N}$  is first countable. Of course, it is separable and noncompact.

**2.3. OBSERVATION.**  *$\gamma\mathbb{N}$  is countably compact if, and only if, no infinite subset of  $\mathbb{N}$  is almost disjoint from all the  $A_\alpha$ . (Two sets are said to be almost disjoint if their intersection is finite.) The existence of a  $\subset^*$ -ascending sequence  $\langle A_\alpha : \alpha < \omega_1 \rangle$  of subsets of  $\mathbb{N}$ , such that no infinite subset of  $\mathbb{N}$  is almost disjoint from all  $A_\alpha$ , is equivalent to  $\mathfrak{t} = \omega_1$ .  $\square$*

At the present time, models of  $\mathfrak{t} = \omega_1$  are the rule rather than the exception. Of course, **CH** implies  $\mathfrak{t} = \omega_1$ ; but also, the addition of uncountably many Cohen, or random, or Sacks, Laver, Mathias, Miller, Prikry-Silver or Matet reals to any model of set theory will also give a model of  $\mathfrak{t} = \omega_1$ . At the present state of the art, one almost has to list all possible  $\subset^*$ -ascending  $\omega_1$ -sequences of subsets of  $\mathbb{N}$  and force to extend them, as in the construction of models of **MA**, to get  $\mathfrak{t} > \omega_1$ . And then one must make an effort to avoid having  $\mathfrak{b} = \mathfrak{c}$ , which follows from **MA** and allows the construction in the following section.

Before that, here is a final series of observations about  $\gamma\mathbb{N}$ . What makes first countability possible is that it stops where it does. Using an ascending  $\tau$ -sequence where  $\tau > \omega_1$  is perfectly feasible and one can use it to construct “nice” **ZFC** examples of locally compact, normal spaces, some of them sequentially compact, as in the case of  $\delta\mathbb{N}$  in FRANKLIN and RAJAGOPALAN [1970] and various examples of  $\delta\mathbb{N}$  in NYIKOS and VAUGHAN [1987]. But of course, such spaces will not be first countable if they simply mimic the definition in Example 2.2. One can attempt to salvage first countability by leaving out the ordinals  $< \tau$  of uncountable cofinality, but then one loses countable compactness unless one had it for  $\mathbb{N} \cup \omega_1$  to begin with. There are ways of modifying the construction further in some models of set theory, but they involve extra work. See Section 5.

### 3. The Ostaszewski-van Douwen construction.

In 1973, Ostaszewski revolutionized set-theoretic topology by introducing an extremely simple and powerful technique of building locally compact spaces by transfinite induction. Assuming the axiom  $\diamond$ , he used it to construct what we called an Ostaszewski space in the introduction (Problem 4). The basic building block of the technique is given by (2)  $\rightarrow$  (1) in:

**3.1. LEMMA.** *Let  $X$  be a first countable, locally compact, zero-dimensional (that is, it has a base of clopen sets) space, and let  $D$  be a closed discrete subspace of  $X$ . The following are equivalent.*

- (1) *There is a first countable, locally compact, zero-dimensional space  $Y$  containing  $X$  in which  $D$  has a cluster point.*
- (2) *There is an infinite subset of  $D$  which expands to a discrete collection of open subsets of  $X$ .*

PROOF. (1)  $\rightarrow$  (2). As in 1.3.

(2)  $\rightarrow$  (1). By local compactness, we may take the sets in the collection to be compact and open, and the collection to be countable. Add a point  $y$  to  $X$  and let  $Y = X \cup \{y\}$ . A base for the topology on  $Y$  consists of all open subsets of  $X$ , together with all sets obtained by taking the union of  $\{y\}$  with the union of all but finitely many members of the discrete collection.  $\square$

Eric van Douwen fine-tuned the technique to give a recipe (3.3 below) for building all spaces of the following kind:

**3.2. DEFINITION.** A space is an *Ostaszewski-van Douwen space* if it is locally countable, regular, countably compact, noncompact, and separable.

One could also write “first countable” or “locally compact” in place of “regular” and have equivalent definitions. On the one hand, as pointed out at the beginning of this article, every first countable, countably compact space is regular, and of course so is every locally compact space. On the other hand, regularity plus countable compactness implies the existence of a local base, at each point, of countably compact neighborhoods; when local countability is added, local compactness follows, and with it first countability.

Like the subclass of Ostaszewski spaces, Ostaszewski-van Douwen spaces are scattered, and every locally compact scattered space, as well as every Tikhonov space that is of cardinality  $< \mathfrak{c}$ , is zero-dimensional.

It is not known whether an Ostaszewski-van Douwen space exists in **ZFC**; if one does, then the title problem would be solved by what has been written. The following recipe thus has an optimistic hypothesis built into it at each stage, one that we do not know how to guarantee in **ZFC**: the hypothesis is that the space constructed up to that point satisfies **wd** (1.2). This hypothesis does follow under various extra axioms, one of which is discussed after the construction.

**3.3. CONSTRUCTION.** Let  $\{A_\alpha : \alpha < \mathfrak{c}\}$  list all countably infinite subsets of  $\mathfrak{c}$ . Let  $\omega$  be given the discrete topology. For  $\alpha \geq \omega$ ,  $\alpha < \mathfrak{c}$ , assume there is defined on each infinite  $\beta < \alpha$  a locally compact, locally countable, noncompact topology, with the relative topology on  $\gamma < \beta$  the same as the one defined on  $\gamma$ . If  $\alpha$  is a limit ordinal, let the topology on  $\alpha$  be that whose base is the union of all bases on earlier  $\beta$ . In particular,  $\beta$  will be open in  $\alpha$ , and the induction hypothesis continues to hold for  $\alpha + 1$  in place of  $\alpha$ .

If  $\alpha = \beta + 1$ , we assume in addition that the topology on  $\beta$  satisfies **wd**. Let  $B_\alpha$  be the  $A_\xi$  of smallest index which is a closed discrete subspace of  $\beta$ . (If there is none, then  $\beta$  is countably compact and we are done.) Let  $C_\alpha$  be an infinite, co-infinite subset of  $B_\alpha$  which can be expanded to a discrete collection of open sets  $U_c$  ( $c \in C_\alpha$ ) as in the definition of **wd**. Choose each  $U_c$  so that it is compact and misses  $B_\alpha \setminus C_\alpha$ .

We make  $\beta$  the extra point in the one-point compactification of  $\bigcup\{U_c : c \in U_\alpha\}$ . In other words, we define the topology on  $\alpha = \beta + 1 = \beta \cup \{\beta\}$  as in the

proof of (2)  $\rightarrow$  (1) in 3.1, with  $\beta$  playing the roles of both  $X$  (as a set) and  $y$  (as a point). Then all the induction hypotheses are satisfied for  $\alpha + 1$  in place of  $\alpha$ . In particular, noncompactness follows from the fact that  $B_\alpha \setminus C_\alpha$  is closed discrete, and **wD** because it cannot be destroyed in a regular space by the addition of a single point.

This inductive construction either stops with **wD** violated at some limit ordinal, or else at some limit ordinal  $\alpha \leq \mathfrak{c}$ , necessarily of uncountable cofinality, where  $\alpha$  is countably compact but not compact. By induction it follows that  $\omega$  is dense in  $\alpha$ , so that in this latter case we have an Ostaszewski-van Douwen space. In particular, if the construction continues to stage  $\mathfrak{c}$ , the resulting space is countably compact because every countable subspace is a subset of some  $\alpha < \mathfrak{c}$ , and if it is closed discrete in  $\alpha$ , it will eventually become the least  $A_\xi$  considered at some stage  $\gamma < \mathfrak{c}$  unless it acquires a limit point in the meantime.  $\square$

The construction cannot stop before stage  $\mathfrak{p}$ , because every regular, separable, countably compact space of Lindelöf number  $< \mathfrak{p}$  is compact (HECHLER [1975] and NYIKOS [1982]), and because of the fact that  $\mathfrak{p} \leq \mathfrak{b}$  and Theorem 3.7 below. However, it can stop at stage  $\mathfrak{p}$  even in models of  $\mathfrak{p} < \mathfrak{b}$  (NYIKOS [1980b]).

This construction was independently arrived at by M. Rajagopalan in a heavily disguised form. This is the  $V$ -process of RAJAGOPALAN [1976], which is done in  $\beta\omega$ . In particular, “Lemma 1.4” of RAJAGOPALAN [1976] is the hypothesis that the space constructed up to that point satisfies **wD**.

How can we insure that **wD** is never violated during the construction? One way is to assume our universe satisfies  $\mathfrak{b} = \mathfrak{c}$ .

**3.4. DEFINITION.** Given functions  $f, g: \omega \rightarrow \omega$ , write  $f <^* g$  to mean there exists  $n$  such that  $f(k) < g(k)$  for all  $k > n$ .

$$\begin{aligned} \mathfrak{b} &= \min\{|F| : F \text{ is a } <^*\text{-unbounded (‘‘undominated’’)} \text{ subset of } {}^\omega\omega\} \\ \mathfrak{d} &= \min\{|F| : F \text{ is a } <^*\text{-cofinal (‘‘dominating’’)} \text{ subset of } {}^\omega\omega\} \end{aligned}$$

Obviously,  $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ . It is also known (VAN DOUWEN [1984]) that  $\mathfrak{t} \leq \mathfrak{b}$ .

It may not be immediately obvious what relevance these concepts have to property **wD**, so let’s think a little about how that property might be violated. The spaces in the above construction are all regular, so that an elementary induction allows us to expand any countable closed discrete subspace  $D$  to a *disjoint* collection  $\mathbf{U} = \{U_d : d \in D\}$  of open sets, with  $V_d \cap D = \{d\}$ . Now if  $U$  is not discrete, the points that witness this (in others words, the points  $x$  such that every neighborhood of  $x$  meets infinitely many members of  $\mathbf{U}$ ) are outside  $\bigcup \mathbf{U}$  and can even be assumed (by another application of regularity) to be outside  $\bigcup \{\overline{U} : U \in \mathbf{U}\}$ . If we shrink the open sets still further, we do not add new witnesses and may kill off some. That is, if  $U'_d \subset U_d$  for all  $d \in D$ , and every neighborhood of  $x$  meets infinitely many  $U'_d$ , then every

neighborhood of  $x$  meets infinitely many  $U_d$ , but not necessarily conversely. Using regularity, it is easy to kill off any single witness or even any countable set of witnesses by a judicious shrinkage of the  $U_d$ .

This shrinkage is coded using  $({}^\omega\omega, <^*)$  in the proof of 3.7 below, whose idea is essentially van Douwen's. First we recall:

**3.5. DEFINITION.** A space  $X$  is *pseudonormal* if, whenever  $F_1$  and  $F_2$  are disjoint closed sets, one of which is countable, then there are disjoint open subsets  $U_i$  such that  $F_i \subset U_i$  for  $i = 1, 2$ .

**3.6. PROPOSITION.** *Every pseudonormal space is regular and satisfies Property **D**.*

PROOF. Regularity is obvious, so that if  $D$  is a closed discrete subspace, it can be expanded to a disjoint open collection  $\mathbf{U} = \{U_d : d \in D\}$ . Let  $V$  and  $U$  be disjoint open sets containing  $D$  and the complement of  $\bigcup \mathbf{U}$ , respectively. Then  $\{V \cap U_d : d \in D\}$  is a discrete open expansion of  $D$ .  $\square$

**3.7. THEOREM.** *Every regular, first countable space of Lindelöf number  $< \mathfrak{b}$  is pseudonormal.*

PROOF. Let  $X$  be as in the hypothesis and let  $D$  and  $F$  be closed subspaces of  $X$ , with  $D = \langle d_n : n \in \omega \rangle$ . Assign to each  $d \in D$  a nested countable local base,  $U_{n+1}(d) \subset U_n(d)$  for all  $n \in \omega$ , such that the closure of  $U_0(d)$  misses  $F$ . Assign to each  $x \in F$  an open  $V_x \ni x$  such that  $\overline{V_x}$  misses  $D$ , and a function  $f_x : \omega \rightarrow \omega$  such that  $U_{f_x(n)}(d_n)$  misses  $V_x$ .

CLAIM. If  $f_x <^* g$ , then no point of  $F \cap V_x$  is in the closure of  $U = \bigcup \{U_{g(n)}(d_n) : n \in \omega\}$ .

Once the claim is proven, let  $F' \subset F$  be a set of cardinality  $< \mathfrak{b}$ , such that  $\{V_x : x \in F'\}$  covers  $F$ . Let  $g \in {}^\omega\omega$  be such that  $f_x <^* g$  for all  $x \in F'$ . Then  $U$  and  $X \setminus \overline{U}$  are disjoint open sets containing  $D$  and  $F$ , respectively.

PROOF OF CLAIM. Let  $k$  be such that  $f_x(n) < g(n)$  for all  $n \geq k$ . Then  $\bigcup \{U_{g(n)}(d_n) : n \geq k\}$  misses  $V_x$ , hence so does its closure, and the closure of  $\bigcup \{U_{g(n)}(d_n) : n < k\}$  misses  $F$ .  $\square$

**3.8. COROLLARY (VAN DOUWEN [1984, 12.2]).** *Every regular, first countable space of cardinality  $< \mathfrak{b}$  satisfies Property **D**.*  $\square$

Hence if  $\mathfrak{b} = \mathfrak{c}$ , every initial stage of Construction 3.3 satisfies **wD**, and we have:

**3.9. COROLLARY (van Douwen).** *If  $\mathfrak{b} = \mathfrak{c}$ , there is an Ostaszewski-van Douwen space.*  $\square$

**3.10. OBSERVATION.** *The proof of 3.7 goes through if we replace countability of  $D$  by the hypothesis that  $D$  is the union of countably many closed subsets  $D_n$ , each of which has a countable family  $\mathbf{U}$  of closed neighborhoods such*



that every neighborhood of  $D_n$  contains some member of  $\mathbf{U}$ . Such is the case, for example, if  $D$  is a  $\sigma$ -compact subset of a locally metrizable space.

We will use this observation in 6.

Now we show that any Ostaszewski-van Douwen space  $X$  can be produced by Construction 3.3. Since  $X$  is scattered, it can be well-ordered,  $X = \{x_\xi : \xi < \tau\}$ , so that every initial segment is open, and so that  $\{x_n : n \in \omega\}$  is the set of all isolated points. Also, since  $|X| \leq \mathfrak{c}$  (by separability and first countability)  $X$  has a base of countable, compact open sets, we can take  $\tau = |X| \leq \mathfrak{c}$ . For instance, first take an arbitrary well-ordered  $\{y_\alpha : \alpha < \tau\}$  where  $\{y_n : n \in \omega\}$  is the set of all isolated points. Let  $x_n = y_n$  for all  $n \in \omega$ , and if  $X_\alpha$  has been defined, let  $\xi$  be the least ordinal such that  $y_\xi \in X_\alpha$ . Let  $K$  be a countable clopen neighborhood of  $y_\xi$ , and well-order  $K \setminus X_\alpha$  so that initial segments are open,  $K \setminus X_\alpha = \{x_{\alpha+\eta} : \eta < \delta\}$  for some countable ordinal  $\delta$ . This defines  $X_{\alpha+\delta}$ , and  $y_\xi \in X_{\alpha+\delta}$ , and the induction can continue. The whole space will be used up exactly at stage  $\tau$ .

Now if  $\alpha \geq \omega$ , then  $x_\alpha$  is in the closure of  $X_\alpha$ . Let  $\{V_n : n \in \omega\}$  be a strictly decreasing local base at  $x_\alpha$ , consisting of countable, compact open subsets of  $X_{\alpha+1} = X_\alpha \cup \{x_\alpha\}$ . Let  $D_n^\alpha = V_n \setminus V_{n+1}$ . Then  $\{D_n^\alpha : n \in \omega\}$  is a family of disjoint countable compact open sets, discrete in  $X_\alpha$ . Since  $\tau$  is a limit ordinal,  $X_{\alpha+1}$  still has infinite closed discrete subspaces.

It follows that  $X$  could have been obtained by Construction 3.3. All it takes is a fortuitous choice of  $\{A_\alpha : \alpha < \mathfrak{c}\}$ . For example, for each  $\alpha < \tau$  we can carefully pick  $A_{\omega\alpha}$  to be a choice set for  $\{D_n^{\omega+\alpha} : n \in \omega\}$ , while if  $\gamma = \omega\alpha + m$  ( $0 < m < \omega$ ) then we can pick  $A_\gamma$  to be a subset of  $X_{\alpha+1}$  which is not closed discrete.

Precisely because of its complete generality, Construction 3.3 tells next to nothing about the inner structure of the spaces constructed. In fact, it can just as easily be a recipe for disaster. If there is a separable, locally compact, locally countable space of cardinality  $< \mathfrak{c}$  which does not satisfy **wD**, it too can be constructed by 3.3; the argument just given goes through. Of course, if  $\mathfrak{p} = \omega_1$ , we can cleverly choose  $\{A_\alpha : \alpha < \mathfrak{c}\}$  and the expansions so as to arrive at a countably compact  $\gamma\mathbb{N}$  by stage  $\omega_1$ . What we do not have at present, and may never have, is an equally clever choice for higher values of  $\mathfrak{p}$ .

Lacking these choices, we can always try imposing some extra structure on the construction in the hope that Property **wD** will somehow continue to hold until we arrive at a countably compact space. The  $V$ -process, as described in RAJAGOPALAN [1976], has an extra parameter involving the topology on  $\beta\omega$ . Unfortunately, no new way of insuring the **wD** property has emerged from it to date.

This is also the case with the technique of NYIKOS [19 $\infty$ b], which works smoothly if one assumes **wD** at each stage of the construction but which cannot guarantee it. This technique makes the remainder (set of noniso-

lated points) a clopen subspace of some prefabricated locally compact, locally countable,  $\omega$ -bounded space  $Y$  of cardinality  $\mathfrak{c}$ . (Under **CH**, for instance, one could take  $Y = \omega_1$  and wind up with  $\gamma\mathbb{N}$ .) The existence of such  $Y$  is a modest requirement: to negate it, one must assume the existence of inner models with measurable cardinals. The construction begins with the discrete space  $\omega$  and proceeds by tearing off countable clopen chunks from  $Y$  and attaching them to the subspace built from the earlier chunks. To play devil's advocate: one could just be pulling down isolated singletons from  $Y$  for the first  $\mathfrak{b}$  stages, and then be stuck with the following space.

**3.11. EXAMPLE (VAN DOUWEN [1984, 12.2]).** Let  $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$  be a  $<^*$ -unbounded,  $<^*$ -wellordered family of increasing functions from  $\omega$  to  $\omega$ . Let  $X$  be a space whose underlying set is  $\omega \times (\omega + 1) \cup \{ \{f_\alpha\} : \alpha < \mathfrak{b} \}$  where the relative topology on  $\omega \times (\omega + 1)$  is the usual one, and a neighborhood of  $\{f_\alpha\}$  is any subset of  $x$  which contains  $\{f_\alpha\}$  together with a cofinite subset of  $f_\alpha$ . Then  $X$  is locally compact, locally countable, separable, and fails to satisfy **wD** since no infinite subset of  $\omega \times \{\omega\}$  can be expanded to a discrete open collection. So  $X$  cannot be embedded in a first countable, countably compact space.

Other inductive constructions of separable, first countable, countably compact noncompact spaces can get stuck through having something like 3.11 embedded as a subspace. For instance, one can modify 3.3 to produce spaces in which every point has a clopen neighborhood homeomorphic to the Cantor set  $2^\omega$ . All it takes is to start out with  $2^\omega \times \omega$  and then follow 3.3, with the difference that the subspace at stage  $\alpha$  has underlying set  $(2^\omega \times \omega) \cup [\omega, \alpha)$  in place of  $[0, \alpha) = \alpha$ . As long as  $\mathfrak{b} = \mathfrak{c}$  everything is OK, because the Lindelöf number at initial stages is  $< \mathfrak{b}$ , but otherwise, what is to keep 3.11 from creeping in and ruining **wD**? A slim chance is tendered by the fact that this modification of 3.3 is not the only recipe for constructing first countable, separable, countably compact, noncompact spaces which are locally homeomorphic to  $2^\omega$ . In the successor stages, we can take advantage of the fact that  $\bigcup \{U_c : c \in C_\alpha\}$  is homeomorphic to *any* dense proper open subset of the Cantor set, and compactify it accordingly. If we are not interested in local homogeneity, this idea can easily be modified using any first countable, zero-dimensional compact spaces.

This may be the place to mention a friendly debate I had with van Douwen before VAN DOUWEN [1984] was in final form. His 12.2 is as general as he ever got along the lines of Theorem 3.7 above: he used cardinality in place of Lindelöf degree. When I pointed out that this extra generality could be obtained with just minor changes in the proof, he responded that he could see no new application for the extra generality. (I never did mention 3.10 to him, but his response would probably have been the same.)

In a certain sense he was correct: the constructions outlined just now have not, to date, given us solutions to the title problem in any model except

the ones where Ostaszewski-van Douwen spaces are known to occur. Similar constructions of manifolds (Section 6) have fared even worse so far. Still, 3.7 does make these other constructions possible under  $\mathfrak{b} = \mathfrak{c}$ , and that seemed justification enough to me.

#### 4. The “dominating reals” constructions.

For almost ten years, the models in which separable, first countable, countably compact noncompact spaces were known to live all satisfied either  $\mathfrak{p} = \omega_1$  or  $\mathfrak{b} = \mathfrak{c}$ . Then it was noticed by JUHÁSZ, SHELAH and SOUKUP in [1988], and by NYIKOS in [1900a] that the addition of dominating reals is at least as effective in producing pseudonormality as having Lindelöf number  $< \mathfrak{b}$ . Moreover, the arguments are so similar that the construction in this section should have been discovered much earlier.

**4.1. DEFINITION.** Let  $M$  be a model of **ZFC**. A function  $g: \omega \rightarrow \omega$  is *dominating over  $M$*  or a *dominating real over  $M$*  if  $f <^* g$  for all  $f \in {}^\omega \omega \cap M$ .

There are many forcing constructions of dominating reals over a given model  $M$ . A very simple **ccc** method is in BAUMGARTNER and DORDAL [1985]. Also, any time one begins with a model  $M$  of  $\mathfrak{d} = \kappa$  and uses **ccc** forcing to produce a model of  $\mathbf{MA} + \mathfrak{c} > \kappa$ , one necessarily adds dominating reals to  $M$ .

**4.2. LEMMA.** Let  $M_\alpha$  be a transitive model of **ZFC** and let  $M_\beta$  be a transitive submodel of  $M_\alpha$  satisfying **ZFC**, such that in  $M_\alpha$  there is a dominating real over  $M_\beta$ . If  $X$  is a regular first countable space with base  $\mathbf{B}$ , and  $D$  and  $F$  are disjoint closed subsets of  $X$  in  $M_\beta$  with  $D$  countable, then in  $M_\alpha$  there are disjoint open sets  $U$  and  $V$  in the topology whose base is  $\mathbf{B}$ , such that  $D \subset U$ ,  $F \subset V$ .

**PROOF.** Let  $\langle d_n : n \in \omega \rangle \in M_\beta$  be a listing of  $D$  as in the proof of 3.7, and define  $U(d) = \langle U_n(d) : n \in \omega \rangle$  as in that proof, so that  $\langle U(d) : d \in D \rangle \in M_\beta$ . Also define  $\langle V_x : x \in F \rangle \in M_\beta$  as in that proof.

Let  $g \in M_\alpha$  be a dominating over  $M_\beta$ . Continue to follow the proof of 3.7, except to have  $F' = F$  and omitting the condition  $|F'| < \mathfrak{b}$ . Then no element of  $F$  is in the closure of  $\bigcup \{ U_{g(n)}(d_n) : n \in \omega \} = U$ , and  $U$  is an element of  $M_\alpha$  by the axioms of replacement and union. Let  $V = X \setminus \overline{U}$ .  $\square$

**4.3. COROLLARY.** If  $M_\alpha$ ,  $M_\beta$ ,  $X$ ,  $\mathbf{B}$ , and  $D$  are as in 4.2, and  $D$  is discrete, then in  $M_\alpha$  there is a discrete family of open sets expanding  $D$ .

**PROOF.** Back in  $M_\beta$ , let  $\langle U(d) : d \in D \rangle$  be a disjoint open expansion, and let  $F = X \setminus \bigcup \{ U(d) : d \in D \}$ . Then  $F \in M_\beta$ , and if  $U$  is as in 4.2, then  $\{ U \cap U(d) : d \in D \}$  is in  $M_\alpha$  and is a discrete expansion of  $D$ .  $\square$

An interesting consequence of 4.3 is that if  $X \in M_\beta$  is a first countable, pseudocompact space that is not countably compact, then in  $M_\alpha$  it loses its

pseudocompactness. On the other hand, a first countable, countably compact space might retain its countable compactness (and hence its pseudocompactness). For example, in BAUMGARTNER and DORDAL [1985] there are models  $M_\beta$  and  $M_\alpha$  as in 4.2, with **CH** holding in  $M_\beta$ , such that any countably compact  $\gamma\mathbb{N}$  in  $M_\beta$  remains countably compact in  $M_\alpha$ . This follows from BAUMGARTNER and DORDAL [1985, Corollary 3.4] and Observation 2.3. More generally, the question of whether a proper forcing preserves an Ostaszewski-van Douwen space in which the subspace of nonisolated points is  $\omega$ -bounded, reduces to the question of whether new closed discrete subspaces arise inside the subspace of isolated points. (See comments following the proof of 4.4 below.) There are some structural results about ideals in NYIKOS [1980b] which might help with this reduced question.

At the opposite extreme, there is a construction, assuming  $\mathfrak{b} = \mathfrak{c}$ , of an Ostaszewski-van Douwen space which is destroyed by the addition of any real: a separable, locally compact, locally countable quasi-perfect preimage of  $[0, 1]$  has this property because there is nothing “above” the added real, and Example 13.4 of VAN DOUWEN [1984] has all these properties when  $X = [0, 1]$ , except for separability. However, the closure of the isolated points over  $\mathbb{Q}$  has all the desired properties.

Also, many people have independently observed that an Ostaszewski space (Problem 4) cannot be destroyed by the addition of Cohen reals, although Fremlin has an example (unpublished) of one that is destroyed by the addition of a single random real.

As our insight into the title problem continues to deepen, it will probably become increasingly important to know what sorts of forcing preserve what sorts of first countable, countably compact spaces. This theme will be taken up after the proof of the following theorem, and also in the last section.

**4.4. THEOREM.** *Let  $M$  be a transitive model of **ZFC** with a sequence  $\langle M_\alpha : \alpha < \kappa \rangle$ , of transitive submodels, each satisfying **ZFC**, such that  $M_\beta \subset M_\alpha$  whenever  $\alpha < \kappa$ , and*

- (i) *For each  $A \in M$ ,  $\langle A \cap M_\alpha : \alpha < \kappa \rangle \in M$ .*
- (ii)  *$M \models \kappa$  is a limit ordinal of uncountable cofinality.*
- (iii) *Each countable set of ordinals in  $M$  is in  $M_\alpha$  and countable in  $M_\alpha$  for some  $\alpha \in \kappa$ .*
- (iv) *If  $\beta < \alpha$ , then  $M_\alpha$  has a dominating real over  $M_\beta$ .*

*Then there is an Ostaszewski-van Douwen space in  $M$ . Moreover, if  $\alpha < \kappa$  and  $X \in M_\alpha$  is a locally compact, locally countable space with base **B**, then in  $M$  there is a locally compact, locally countable, countably compact space containing  $X$  such that the relative topology on  $X$  has base **B**.*

**PROOF.** If  $M_\alpha \cap \mathcal{P}(\omega)$  is countable in  $M$  for all  $\alpha$ , then by (iii)  $M$  satisfies **CH** and the first conclusion holds. So suppose  $M_\alpha \cap \mathcal{P}(\omega)$  is uncountable in  $M$  for some  $\alpha$ , and let  $\mathcal{a}$  be a maximal almost disjoint (MAD) family of

infinite subsets of  $\omega$  in  $M_\alpha$  (that is, it is maximal in  $M_\alpha$ ; in  $M_{\alpha+1}$  it will lose its maximality!) of cardinality  $\mathfrak{c}$  in  $M_\alpha$ . Let  $X_\alpha \in M_\alpha$  be the  $\Psi$ -space based on  $a$ ; that is, its points are elements of  $\omega \cup a$ , and basic neighborhoods of  $A \in a$  are cofinite subsets of  $A \cup \{A\}$  which include the point  $A$ .

Then  $X_\alpha$  is locally compact, locally countable, and uncountable in  $M$ , and  $\omega$  is a dense subspace. Once we prove the “moreover” part, and  $Y$  is any locally compact, locally countable, countably compact space containing  $X_\alpha$ , then the closure of  $X_\alpha$  in  $Y$  will be an Ostaszewski-van Douwen space; in particular, it will be noncompact since it is uncountable.

So let  $X$  be as in the “moreover” part. We may assume the underlying set of  $X$  is some ordinal  $\lambda_\alpha$ . In  $M$ , let  $\preceq$  be a well-ordering of a “sufficiently large” set in  $M$  (to include as elements sets of every kind defined below; one can run through the argument to establish just how large to make the set). Whenever a set in  $M$  is defined in the sequel, we assume it is the  $\preceq$ -first member of its kind, so as to make the space we construct be a member of  $M$ .

Let  $\langle D_\xi : \xi \in \gamma_\alpha \rangle$  be a one-to-one listing in  $M_\alpha$  of an almost disjoint family of countably infinite closed discrete subspaces of  $X_\alpha = X$ , such that every countably infinite closed discrete subspace of  $X_\alpha$  in  $M_\alpha$  meets some  $D_\xi$  in an infinite set. Let  $\langle \mathbf{V}_x : x \in X \rangle$  be a listing in  $M_\alpha$  of local bases  $\mathbf{V}_x$  at each  $x$ , with  $\mathbf{V}_x = \langle V_n(x) : n \in \omega \rangle$  being nested (i.e.  $V_{n+1}(x) \subset V_n(x)$  for all  $n, x$ ) and each  $V_n(x)$  compact open, hence countable.

CLAIM. In  $M_{\alpha+1}$ , there is a family  $\langle \mathbf{U}_\xi : \xi \in \gamma_\alpha \rangle$  of discrete expansions,  $\mathbf{U}_\xi$  a discrete expansion of  $D_\xi$  to a family of  $V_n(x)$ ’s, such that if  $\xi \neq \eta$ , then at most finitely many intersections  $U \cap V$  with  $U \in \mathbf{U}_\xi$ ,  $V \in \mathbf{U}_\eta$ , are nonempty.

Once the claim is proven, let  $X_{\alpha+1} = X \cup (\lambda_\alpha + \gamma_\alpha)$ , with a local base at  $\lambda_\alpha + \xi$  being all sets of the form  $\{\lambda_\alpha + \xi\} \cup (\bigcup \mathbf{U}_\xi) \setminus \bigcup F$ , where  $F \subset \mathbf{U}_\xi$  is finite. Then  $X_{\alpha+1} \in M_{\alpha+1}$ , and  $X_{\alpha+1}$  is a locally compact and locally countable.

PROOF OF CLAIM. Let  $\langle h_\xi : \xi \in \gamma_\alpha \rangle \in M_\alpha$  be a set of bijections  $h_\xi : \omega \rightarrow D_\xi$ . For each  $\xi, \eta \in \gamma_\alpha$ , let  $f_{\xi\eta} : \omega \rightarrow \omega$  be such that  $V_{f_{\xi\eta}(n)}(h(n))$  does not meet  $V_{f_{\eta\xi}(m)}(h_\eta(m))$  unless  $h_\xi(n) = h_\eta(m)$ ; do this so that  $\langle f_{\xi\eta} : \xi, \eta \in \gamma_\alpha \rangle$  is in  $M_\alpha$ . Now if  $g : \omega \rightarrow \omega$  is dominating over  $M_\alpha$ , and  $\xi, \eta \in \gamma_\alpha$ , let  $m$  be such that  $h_\xi(k)$  and  $h_\eta(j)$  are distinct for all  $j, k \geq m$  and  $f_{\xi\eta}(k)$ ,  $f_{\xi\xi}(k)$ ,  $f_{\eta\eta}(k)$  and  $f_{\eta\xi}(k)$  are all less than  $g(k)$  for  $k \geq m$ . Then  $V_{g(j)}(h_\sigma(j))$  and  $V_{g(k)}(h_\tau(k))$  are disjoint for all  $j, k \geq m$  and for all choices of  $\sigma, \tau \in \{\xi, \eta\}$  except for  $\sigma = \tau$  and  $j = k$ . Hence if we take  $g \in M_{\alpha+1}$  and let

$$\mathbf{U}_\xi = \{V_{g(n)}(h_\xi(n)) : n \in \omega\}$$

for all  $\xi < \gamma_\alpha$ , then the  $\mathbf{U}_\xi$ ’s satisfy all the properties in the claim. (Discreteness can be verified as in the proof of 4.2 and 4.3 using the fact that each  $V_n(x_n)$  is closed as well as open, or one could apply 4.3 directly to shrink  $\mathbf{U}_\xi$  further to a discrete expansion, all within  $M_{\alpha+1}$ .) END OF PROOF OF CLAIM.

Now we can repeat the argument for  $M_{\alpha+1}$  and  $M_{\alpha+2}$  in place of  $M_\alpha$  and

$M_{\alpha+1}$ , letting  $\lambda_{\alpha+1} = \lambda_\alpha + \gamma_\alpha$ , and otherwise just changing the subscripts. The general step in going from any  $M_\gamma$  to  $M_{\gamma+1}$ , to produce  $X_{\gamma+1}$ , is the same. If  $\gamma$  is a limit ordinal, let  $\lambda_\gamma = \sup\{\lambda_\beta : \beta < \gamma\}$  and let the topology on  $X_\gamma = \lambda_\gamma$  be the one whose base is the union of the bases on the preceding  $X_\beta$ 's. Then  $Y = X_\kappa$  is the desired space. It is certainly locally compact and locally countable, and by induction it even has  $X_\alpha$  as a dense subspace. If  $A$  is a countably infinite subset of  $X_\kappa$ , then by (ii) and (iii) there exists  $\gamma < \kappa$  such that  $A \subset X_\gamma$ ,  $A \in M_\gamma$ , and  $M_\gamma \models A$  is countable. Then  $A$  will have an accumulation point in  $X_{\gamma+1}$ , and this establishes countable compactness of  $Y$ . (If  $A$  is closed discrete in  $X_\gamma$ , then  $A \cap D_\xi$  is infinite for some  $D_\xi$  defined wrt  $X_\gamma$ , and then  $D_\xi$  converges to  $\lambda_\gamma + \xi$ .)  $\square$

Local countability was actually quite important in this argument; without it, local compactness could easily be lost. For example, if we start out with  $X_\alpha = [0, 1] \cap M_\alpha$  with the usual topology, then  $X_\alpha$  would be locally compact in  $M_\alpha$  but would have no points of local compactness in  $M_{\alpha+1}$ , because of the nonconvergent Cauchy sequences one can define with the help of the added reals. There are natural ways of adding points to recover local compactness, of course, but they complicate the definition considerably. On the other hand, if  $M$  and  $N$  are models of **ZFC**, and  $M$  is a submodel of  $N$ , then a compact, countable space  $X$  in  $M$  retains both its properties in  $N$ . Indeed,  $X$  is homeomorphic to a countable ordinal, and this property is upwards-absolute.

Conditions (i) through (iii) in 4.4 hold in any iterated forcing of uncountable (as seen in the final model  $M$ ) cofinality, in which supports are countable (again as seen in  $M$ ). If the forcing, taken as a whole, is proper (as is countable-support iteration of proper posets, or finite-support iteration of **ccc** posets), then the ground model and  $M$  are in agreement on which sets are uncountable (although cardinals greater than  $\omega_1$  may be collapsed). Indeed, if  $M$  is a model and  $P \in M$  is proper, and  $G$  is an  $M$ -generic subset of  $P$ , then every countable set of ordinals in  $M[G]$  is contained in a countable set  $A$  in  $M$  (SHELAH [1982, p.81]).

This also explains why an  $\omega$ -bounded, locally compact, locally countable space cannot be destroyed by proper forcing: we may assume the underlying set is an ordinal, and then every countable subspace in the extension is contained in a compact, scattered, countable subset in the ground model.

**4.5. EXAMPLE.** Let  $M_0$  be a countable transitive model of **ZFC**+**CH**, and let  $\kappa$  be a regular uncountable cardinal in  $M_0$ . Let  $\langle P_\alpha : \alpha \leq \kappa \rangle$  be an iterated **ccc** forcing with finite supports, so that for each  $\alpha < \kappa$ ,  $M_{\alpha+1} \models \mathbf{MA} + \mathfrak{c} = \aleph_{\alpha+1}$ , where  $M_{\alpha+1} = M_0[G_{\alpha+1}]$  for some  $G_{\alpha+1}$  generic over  $P_{\alpha+1}$ . The conditions of 4.4 are satisfied by  $\langle M_\alpha : \alpha < \kappa \rangle$  and  $M = M_\kappa$ . In particular, since  $M_{\beta+1}$  satisfies **MA**, any set of fewer than  $\aleph_{\beta+1}$  functions from  $\omega$  to  $\omega$  in  $M_{\beta+1}$  (such as  $M_\beta \cap {}^\omega\omega$ ) is dominated by some  $g \in M_{\beta+1}$ . So there is an Ostaszewski-van Douwen space in  $M_\kappa$ . Also, any collection of fewer than  $\kappa$  functions in  $M_\kappa$  appears in some earlier model and so is dominated in

$M_\kappa$ . On the other hand, it is easy to see that there is a dominating family of cardinality  $\kappa$  in  $M_\kappa$ , so that  $\mathfrak{b} = \mathfrak{d} = \kappa$ . Similarly one shows that  $\mathfrak{p} = \kappa$  in  $M_\kappa$ , whereas  $\mathfrak{c} = \aleph_\kappa$ . So if  $\kappa > \aleph_1$ , then  $\mathfrak{p} > \aleph_1$  in the final model and  $\mathfrak{b} = \mathfrak{d} < \mathfrak{c}$ .

A similar model, but with a regular cardinal  $\lambda > \kappa$  in place of  $\aleph_\kappa$ , is described in VAN DOUWEN [1984, 5.1]. (*Caution:* In this reference, the  $M_\eta$ ,  $f_\eta$  etc. should be indexed by  $\eta < \kappa$ , not  $\eta < \lambda$ .)

The fact that  $\mathfrak{b} = \mathfrak{d}$  in 4.5 and VAN DOUWEN [1984, 5.1] is no accident: in fact, it is easy to see that if  $M$  is as in Theorem 4.4, then there is a *scale* in  $M$ , i.e. a dominating family well-ordered by  $<^*$ , of the same cofinality as  $\kappa$ ; and an elementary fact is that the existence of a scale of cofinality  $\lambda$  is equivalent to  $\mathfrak{b} = \mathfrak{d} = \lambda$ . This gives rise to a natural question:

**? 290. Problem 8.** *Is  $\mathfrak{b} = \mathfrak{d}$  enough to imply the existence of an Ostaszewski-van Douwen space? A first countable, separable, countably compact, non-compact space?*

Another natural question is how essential Property (iv) is to Theorem 4.4. In one sense it is indispensable: in the presence of the first three conditions,  $\langle M_\alpha : \alpha < \kappa \rangle$  must have a cofinal subsequence satisfying (iv) in order for the “Moreover” part to hold. For if ever we have an  $M_\beta$  over which no  $g: \omega \rightarrow \omega$  in  $M$  dominates, then in  $M_\beta$  there is a separable, locally compact, locally countable space which cannot be embedded into an Ostaszewski-van Douwen space (see next example). Moreover, a simple coding argument shows that if there is a dominating  $G$  over  $M_\beta$  in  $M$ , then there is one in some  $M_\alpha$ ,  $\alpha < \kappa$ : any  $g$  is a subset of  $\omega \times \omega$ , and there is a bijection  $\psi: \omega \rightarrow \omega \times \omega$  in  $M_0 \dots$

**4.6. EXAMPLE.** Let  $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$  be a  $<^*$ -dominating family of increasing functions from  $\omega$  to  $\omega$  which is almost disjoint (when each is considered as a subset of  $\omega \times \omega$ ). Such a family could be produced, for example, by beginning with a dominating family  $\langle g_\alpha : \alpha < \mathfrak{d} \rangle$ , letting  $\langle A_\alpha : \alpha \in \mathfrak{d} \rangle$  be a 1–1 listing of an almost disjoint family of infinite subsets of  $\omega$ , and then replacing  $g_\alpha$  by an increasing  $f_\alpha$  which exceeds  $g_\alpha$  on every coordinate, and whose range is a subset of  $A_\alpha$ .

Let  $X$  be a space whose underlying set is  $\omega \times (\omega + 1) \cup \{ \{f_\alpha\} : \alpha < \mathfrak{d} \}$  where the relative topology on  $\omega \times (\omega + 1)$  is the usual one, and a neighborhood of  $\{f_\alpha\}$  is any subset of  $x$  which contains  $\{f_\alpha\}$  together with a cofinite subset of  $f_\alpha$ . Then  $X$  is locally compact, locally countable, and fails to satisfy **wd** since no infinite subset of  $\omega \times \{ \omega \}$  can be expanded to a discrete open collection. (Compare 3.11.)

Now suppose  $X$  lives in some model  $M$  of **ZFC**, where the  $f_\alpha$  are now understood to be dominating in  $M$ , and of course  $\mathfrak{d}$  means  $\mathfrak{d}^M$ .

**4.7. CLAIM.** *Let  $M$  be a transitive submodel of  $N$ , where  $N$  is a transitive model of **ZFC**. Then the following are equivalent.*

- (i)  $X$  is pseudonormal in  $N$ .
- (ii)  $X$  satisfies **wD** in  $N$ .
- (iii) Some infinite subset of  $\omega \times \{\omega\}$  expands to a discrete collection of open sets.
- (iv) There is in  $N$  a real dominating over  $M$ .

**PROOF.** By Lemma 4.2, (iv)  $\rightarrow$  (i), and (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii) is trivial since  $\omega \times \{\omega\}$  remains closed discrete in  $N$ .

Finally, suppose  $A$  is an infinite subset of  $\omega$  such that, in  $N$ ,  $A \times \{\omega\}$  expands to a discrete collection  $\{U_{\langle a, \omega \rangle} : a \in A\}$  of open sets. Define  $g: A \rightarrow \omega$  by letting  $g(a)$  be the least  $n$  such that  $\langle a, k \rangle \in U_{\langle a, \omega \rangle}$  for all  $k \geq n$ , and for each  $i \in \omega$  let  $f(i) = g(a)$  for the least  $a \in A$  such that  $a \geq i$ . We will be done as soon as we show  $f$  is dominating over  $M$ .

If  $f$  is not dominating over  $M$ , then there exists  $\alpha$  such that  $B = \{n : f(n) \leq f_\alpha(n)\}$  is infinite. Since  $f_\alpha$  is increasing, it follows that for each  $b \in B$ , the least member of  $A$  that is  $\geq b$  is in  $B$  also. So  $f(n) \leq f_\alpha(n)$  on an infinite subset of  $A$ , whence every neighborhood of  $\{f_\alpha\}$  meets infinitely many  $U_{\langle a, \omega \rangle}$ , a contradiction.  $\square$

Of course,  $X$  could have been any cardinality in the interval  $[\mathfrak{d}, \mathfrak{c}]$ , and the claim would have gone through.

An amusing question is whether  $X$  can be rigged so that  $X \setminus (\omega \times \{\omega\})$  is pseudonormal in  $M$ . (This is true of Example 3.11, see VAN DOUWEN[1984, proof of 11.4(e)].) This is equivalent to the more general-sounding question:

**Problem 9.** *Call a space  $X$   $\Psi$ -like if it has a countable dense set  $D$  of isolated points, is locally compact and first countable, and  $X - D$  is discrete. Is there a pseudonormal  $\Psi$ -like space of cardinality  $\mathfrak{d}$ ? More generally, what is the maximum cardinality of a pseudonormal  $\Psi$ -like space? (From 3.11, it is clearly  $\geq \mathfrak{b}$ .)* **291. ?**

To show the equivalence, note that in 4.6,  $X$  (and hence  $X \setminus \omega \times \{\omega\}$ ) is  $\Psi$ -like. Conversely, given a  $\Psi$ -like space  $Y$  of cardinality  $\mathfrak{d}$  which is pseudonormal, identify its countable dense set of isolated points with  $\omega$ , and choose for each nonisolated point  $p_\alpha$  ( $\alpha \in \mathfrak{d}$ ) a compact open neighborhood  $U_\alpha$  with  $p_\alpha$  as its sole nonisolated point. Let  $A_\alpha = U_\alpha \setminus \{p_\alpha\}$ . Then  $\langle A_\alpha : \alpha \in \mathfrak{d} \rangle$  is almost disjoint, and we can define  $\langle f_\alpha : \alpha \in \mathfrak{d} \rangle$  and  $X$  as in 4.6. Given a countable subfamily  $\langle f_\beta : \beta \in B \rangle$ , let  $S$  be a countable set of  $\omega$  which (by pseudonormality) is almost disjoint from each  $A_\alpha$  satisfying  $\alpha \notin B$ , and such that  $A_\beta \subset^* S$  for all  $\beta \in B$ . For each  $\beta \in B$ , let  $h_\beta$  be  $f_\beta$  minus the finitely many  $\langle i, j \rangle \in f_\beta$  such that  $j \notin S$ . Then  $\bigcup \{h_\beta \cup \{\{f_\beta\}\} : \beta \in B\}$  is an open subset of  $X \setminus (\omega \times \{\omega\})$  which does not have any  $\{f_\alpha\}$ ,  $\alpha \notin B$ , in its closure.



Incidentally, there are  $\Psi$ -like spaces satisfying  $\mathbf{wD}$  which are of cardinality  $\mathfrak{c}$  (the maximum number possible). One is the Cantor tree, which is hereditarily realcompact since it admits a 1–1 continuous function into the plane (NYIKOS [1989] and GILLMAN and JERISON [1960, 8.17]). One can use it to construct an  $X$  as above so that  $X \setminus \omega \times \{\omega\}$  satisfies  $\mathbf{wD}$  and, if desired, is of cardinality  $\mathfrak{c}$ .

## 5. Linearly ordered remainders

In this section we extend the ideas behind  $\gamma\mathbb{N}$  to try to produce other first countable, countably compact spaces with  $\mathbb{N}$  as a dense set of isolated points and ordered remainder.

The beginning is simple. Suppose we have a  $\gamma\mathbb{N}$  which is not countably compact, with  $\mathbf{A} = \{A_\alpha : \alpha \in \omega_1\}$  an associated sequence in  $\mathbb{N}$ . Can we produce a first countable, countably compact space by adding more points? One condition under which the answer is yes is if  $\mathbf{A}$  is the bottom half of some tight  $(\omega_1, \omega_1^*)$ -gap.

**5.1. DEFINITION.** Let  $\kappa$  and  $\lambda$  be ordinals. A  $(\kappa, \lambda^*)$ -pre-gap, in  $\mathcal{P}(\mathbb{N})$  is a pair  $\langle \mathbf{A}, \mathbf{B} \rangle$  where  $\mathbf{A}$  and  $\mathbf{B}$  are  $\subset^*$ -totally ordered subsets of  $\mathcal{P}(\mathbb{N})$ , of cofinality  $\kappa$  and coinitality  $\lambda$ , respectively, and if  $a \in \mathbf{A}$ ,  $B \in \mathbf{B}$ , then  $A \subset^* B$ . (A set with a largest element is understood to have cofinality 1.)

A  $(\kappa, \lambda^*)$ -pre-gap is a *gap* if there is no set  $C$  such that  $A \subset^* C \subset^* B$  for all  $A \in \mathbf{A}$ ,  $B \in \mathbf{B}$ . A set  $D$  is said to be *beside the gap*  $\langle \mathbf{A}, \mathbf{B} \rangle$  if it is almost disjoint from all  $A \in \mathbf{A}$  and satisfies  $D \subset^* B$  for all  $B \in \mathbf{B}$ . A *tight gap* is a pre-gap that has no infinite set beside it.

Despite the similarity in the definitions, there is a big difference between gaps and tight gaps: while a tight gap is a gap, and there is an  $(\omega_1, \omega_1^*)$ -gap in  $\mathbf{ZFC}$  (as shown by Hausdorff), the existence of a tight  $(\omega_1, \omega_1^*)$ -gap is easily seen to imply  $\mathfrak{t} = \omega_1$  and is, in fact, equivalent to it (BLASZCZYK and SZYMAŃSKI [1982]).

ERIC VAN DOUWEN used in [1976] an  $(\omega_1, \omega_1^*)$ -gap to produce a first countable, countably paracompact, non-normal space with  $\mathbb{N}$  as a dense subset and remainder the topological direct sum of two copies of  $\omega_1$ . The construction (5.2 below) is a straightforward modification of the one for  $\gamma\mathbb{N}$ . In NYIKOS and VAUGHAN [1983] we used the same construction with a tight  $(\omega_1, \omega_1^*)$ -gap to produce a countably compact version of van Douwen's space; the two properties are equivalent just as one has countable compactness of  $\gamma\mathbb{N}$  associated with a “ $\mathfrak{t} = \omega_1$ -witnessing sequence” (Observation 2.3).

**5.2. CONSTRUCTION.** Let  $\langle \mathbf{A}, \mathbf{B} \rangle$  be an  $(\omega_1, \omega_1^*)$ -pre-gap, with  $\mathbf{A} = \{A_\alpha : \alpha < \omega_1\}$  and  $\mathbf{B} = \{B_\alpha : \alpha < \omega_1\}$  listed in an ascending and descending order, respectively. Let  $\omega_1^* = \omega_1 \times \{-1\}$  with the reverse of the usual order, i.e. if  $\beta < \alpha$  then  $\langle \beta, -1 \rangle > \langle \alpha, -1 \rangle$ . Let  $Y = \mathbb{N} \cup \omega_1 \cup \omega_1^*$ . Points of  $\mathbb{N}$  are

isolated, basic neighborhoods of  $\alpha \in \omega_1$  are just as in 2.2 (so that  $\mathbb{N} \cup \omega_1$  as a subspace is a version of  $\gamma\mathbb{N}$ ), while a neighborhood of  $\langle \alpha, -1 \rangle$  is any subset of  $Y$  containing a set of the form

$$V_n[\alpha, \beta) = \{ \langle \gamma, -1 \rangle : \beta < \gamma \leq \alpha \} \cup (B_\beta \setminus B_\alpha) - \{1, \dots, n\}.$$

(Note that  $\langle \alpha, -1 \rangle$  is *not* in the closure of  $B_\alpha$ , whereas  $\alpha$  is in the closure of  $A_\alpha$ .)

The space  $Y$  is locally compact, locally countable, and countably paracompact, is non-normal iff  $\langle \mathbf{A}, \mathbf{B} \rangle$  is a gap (VAN DOUWEN [1976]), and countably compact iff  $\langle \mathbf{A}, \mathbf{B} \rangle$  is a tight gap (NYIKOS and VAUGHAN [1983]).

The remainder  $Y \setminus \mathbb{N}$  has a natural linear order the same as the order  $\subset^*$  on  $\mathbf{A} \cup \mathbf{B}$ , and the topology is the order topology. Since our main interest is in what happens if  $\mathfrak{t} > \omega_1$ , we need to consider larger remainders (see the end of Section 2), but we will keep them linearly ordered, and each will have a  $\subset^*$ -totally ordered sequence in  $\mathbb{N}$  to go with it. Such setups are dealt with at length in NYIKOS [1988a], but with little attention to first countability.

First countability in a LOTS entails that if an ascending sequence (resp. descending sequence) in the remainder has uncountable cofinality, it cannot have a supremum (resp. infimum). On the other hand, countable compactness does require suprema and infima for countable sequences. So every ascending sequence of uncountable cofinality “upstairs” (in  $Y \setminus \mathbb{N}$ ) will have to be met by a descending sequence of uncountable cofinality. At the same time, we will have some associated subsets of  $\mathbb{N}$ , “downstairs”, totally ordered by  $\subset^*$ , and if  $\mathfrak{t} > \omega_1$ , then an ascending  $\omega_1$ -sequence and a descending  $\omega_1^*$ -sequence cannot form a tight gap, and if there is nothing “upstairs,” between the points associated with them, then any infinite set that is “beside the gap” will fail to have a limit point.

So if  $\mathfrak{t} > \omega_1$ , then every ascending  $\omega_1$ -sequence in  $Y \setminus \mathbb{N}$  will, in the two examples below, have to be met by a descending sequence of cofinality  $> \omega_1$ . For the sake of simplicity we take this to mean cofinality  $\omega_2$  in Example 5.3 below, which is a candidate for a space of the form  $Y \setminus \mathbb{N}$ , but which may also be of independent interest.

**5.3. EXAMPLE.** Let  $\Gamma = \{ \gamma \in \omega_2 : \gamma \text{ is a limit ordinal of uncountable cofinality} \}$ . For each  $n \in \omega - \{0\}$  let  $X_n$  be the set of all sequences  $\langle \gamma_1, \dots, \gamma_{n-1}, \alpha \rangle$  of length  $n$  whose first  $n-1$  terms are in  $\Gamma$  and whose last term is in  $\omega_2 - \Gamma$ , and let  $X_\omega = {}^\omega\Gamma$ . Let  $X = \bigcup \{ X_\alpha : 1 \leq \alpha \leq \omega \}$ . The order on  $X$  is “odd forward, even reverse” lexicographical order. That is, given two sequences  $\sigma$  and  $\tau$ , if the first  $k$  on which they disagree is odd, and  $\sigma(k) < \tau(k)$ , then  $\sigma < \tau$ ; if  $k$  is even and  $\sigma(k) < \tau(k)$ , then  $\sigma > \tau$ . Let  $X$  be given the order topology.

If  $\sigma \in X_\omega$ , then a local base for  $\sigma$  is all intervals of the form  $(\sigma^{2n}, \sigma^{2n+1})$  where  $\sigma^i = \langle \sigma(1), \dots, \sigma(i), 0 \rangle$  for all  $i$ . If  $\sigma \in X_{2n+1}$ , then a local base at  $\sigma$

consists of the sets  $(\langle \sigma(1), \dots, \sigma(2n), \beta \rangle, \sigma]$  where  $\beta < \sigma(2n+1)$ . If  $\sigma \in X_{2n}$ , we can use sets  $[\sigma, \langle \sigma(1), \dots, \sigma(2n-1), \beta \rangle)$  where  $\beta < \sigma(2n)$ . These sets are open, so that  $X$  is first countable, and its isolated points are those whose last term is a successor ordinal. It is also easy to see that  $X_\omega$  is dense in itself while each  $X_n$  is scattered. An argument like the one which follows shows that the subspace of all  $\sigma$  whose every term is in  $\bar{\Gamma}$  is also dense in itself and closed.

Also,  $X$  is countably compact. First suppose  $\langle \sigma_n \rangle_n$  is increasing. *Case 1.* For some  $k \in \omega$ , there are infinitely many  $\sigma_n$  which differ in the  $k$ th coordinate. Let  $k$  be the least such coordinate. Since the sequence is increasing,  $k$  is odd. Let  $\alpha = \sup\{\sigma_n(k) : n \in \omega\}$ . Then the sequence converges to  $\langle \gamma_1, \dots, \gamma_{k-1}, \alpha \rangle$  where  $\gamma_1, \dots, \gamma_{k-1}$  “is agreed upon by all but finitely many  $\sigma_n$ ”. *Case 2. Otherwise.* Then, for each coordinate  $k$ , the sequence  $\langle \sigma_n(k) \rangle_n$  is eventually constant; let  $\sigma(k)$  denote this constant value. Then  $\langle \sigma(k) \rangle_k \in X_\omega$ , and it is the limit of the sequence.

A similar argument works for an infinite descending sequence, and of course every infinite subset of a LOTS has either an infinite (strictly) ascending sequence or an infinite descending sequence.

A similar argument establishes that every ascending (resp. descending) sequence of order type  $\omega_1$  is met by a descending (resp. ascending) sequence of order type  $\omega_2$ : now Case I always holds, and the elements of the form  $\langle \gamma_1, \dots, \gamma_{k-1}, \alpha, \beta \rangle$  ( $\beta \in \omega_2 - \Gamma$ ) are coinitial (resp. cofinal) in the part of  $X$  above (resp. below) the range of the sequence. It is also not hard to show that an ascending  $\omega_2$ -sequence is met by a descending  $\omega_1$ -sequence unless it is cofinal in  $X$ , while every descending  $\omega_2$ -sequence is met by an ascending  $\omega_1$ -sequence.

So  $X$  is the right “shape” for embedding in a separable, first countable, countably compact space, but it could be the wrong size (if **CH** it is too large, if  $\mathfrak{t} > \omega_2$  it is too small), and other factors, discussed later, may conspire against countable compactness. However, if  $\mathfrak{t} \geq \omega_2$  one can construct with relative ease a prototype having all desired properties except perhaps countable compactness.

**5.4. EXAMPLE** ( $(\mathfrak{t} > \omega_1)$ ). Let  $E_n$  ( $n \in \omega - \{0\}$ ) be the set of all sequences of the form  $\sigma = \langle \gamma_1, \dots, \gamma_n \rangle$ ,  $\gamma_i \in \Gamma$  for all  $i$ . Let  $E = \bigcup_{n=1}^{\infty} E_n$ . Extend the above order on  $X$  to  $X \cup E$  by decreeing that if  $\sigma \in E_n$  and  $n$  is even, then  $\sigma < \tau$  for all  $\tau$  extending  $\sigma$ , while if  $n$  is odd, then  $\sigma > \tau$  for all  $\tau$  extending  $\sigma$ , otherwise the ordering is defined as before (“odd forward, even reverse”). For instance, using parentheses to denote sequences,

$$(0) < (\omega_1, \omega_1) < (\omega_1, \omega_1, 0) < (\omega_1, 0) < (\omega_1).$$

Let  $A$  be a  $\subset^*$ -totally ordered family of subsets of  $\mathbb{N}$ , indexed by  $\bigcup_{n=1}^{\infty} (X_n \cup E_n)$  in order-preserving fashion. Such a family might be defined by induction on  $n$ , as follows. By  $\mathfrak{t} \geq \omega_2$ , there is a  $\subset^*$ -ascending  $\omega_2$ -sequence which

can be indexed by  $X_1 \cup E_1$ . Suppose we have defined all sets indexed by  $X_i \cup E_i$  for all  $i \leq n$ . Each  $\sigma \in X_{n+1} \cup E_{n+1}$  is of the form  $\langle \gamma_1, \dots, \gamma_n, \alpha \rangle$  where  $\alpha \in \omega_2$ . Those with fixed  $\gamma_1, \dots, \gamma_n$  form an interval of  $\bigcup_{i=1}^{n+1} X_i \cup E_i$ , and if  $n$  is odd, the greatest member of this interval is  $\langle \gamma_1, \dots, \gamma_n, 0 \rangle$ , whose immediate successors are  $\tau = \langle \gamma_1, \dots, \gamma_n \rangle$  and then  $\langle \gamma_1, \dots, \gamma_n + 1 \rangle$ . The set  $\Sigma = \{ \langle \gamma_1, \dots, \gamma_{n-1}, \alpha \rangle : \alpha < \gamma_n \}$  is cofinal in the part of  $\bigcup_{i=1}^{n+1} X_i \cup E_i$  preceding this interval. Now  $\mathbf{B} = \{ A_\sigma : \sigma \in \Sigma \}$  has been defined as has  $A_\tau$ , and  $\langle \mathbf{B}, \{ A_\tau \} \rangle$  is not a gap because  $\mathfrak{t} \geq \omega_2$ , so there is a set  $B$  such that  $A_\sigma \subset^* B \subset^* A_\tau$  for all  $\sigma \in \Sigma$ . Again applying  $\mathfrak{t} \geq \omega_2$ , we can choose a  $\subset^*$ -descending  $\omega_2$ -sequence  $\langle C_\alpha : \alpha \in \omega_2 \rangle$  in  $A_\tau \setminus B$  and then let  $A_{\gamma_1, \dots, \gamma_n, \alpha} = B \cup C_\alpha$ . (We omit  $\langle \rangle$  in the subscripts for simplicity.)

If  $n$  is even, the argument is dual: orders are reversed, and “greatest”, “successors”, “cofinal”, “preceding”, and “descending” are replaced by their opposites,  $A_\tau \setminus B$  is replaced by  $B \setminus A_\tau$ , and  $B \cup C_\alpha$  is replaced by  $A_\tau \cup C_\alpha$ .

Let  $Y$  be the space whose underlying set is  $X \cup \mathbb{N}$ , in which points of  $\mathbb{N}$  are isolated, and a local base at each  $\sigma \in X$  is as follows. (Interval notation will always denote intervals in  $X$ .)

If  $\sigma = \langle \gamma_1, \dots, \gamma_{2n}, \alpha \rangle$ , and  $\alpha > 0$ , a basic open set at  $\sigma$  is of the form  $(\tau, \sigma] \cup A$  where  $\tau = \langle \gamma_1, \dots, \gamma_{2n}, \beta \rangle$ ,  $\beta < \alpha$ , and  $A$  is a cofinite subset of  $A_\sigma \setminus A_\tau$ , while if  $\alpha = 0$  then a basic open set at  $\sigma$  is of the form  $\{\sigma\} \cup A$  where  $A$  is a cofinite subset of  $A_\sigma \setminus A_{\gamma_1, \dots, \gamma_{2n}}$ .

If  $\sigma \in X_\omega$ , then for a local base at  $\sigma$  we use sets of the form

$$[\langle \gamma_1, \dots, \gamma_{2n}, 0 \rangle, \langle \gamma_1, \dots, \gamma_{2n+1}, 0 \rangle] \cup A$$

where  $A$  is a cofinite subset of

$$A_{\gamma_1, \dots, \gamma_{2n+1}} \setminus A_{\gamma_1, \dots, \gamma_{2n}}.$$

With this choice and the earlier choices for  $\sigma \in X_n$ , we have a base of clopen sets. For example, if  $\sigma = \langle \gamma_1, \dots, \gamma_{2n}, \alpha \rangle$  and  $\tau = \langle \gamma_1, \dots, \gamma_{2n}, \beta \rangle$ ,  $\beta < \alpha$  then

$$(\tau, \sigma] = [\langle \gamma_1, \dots, \gamma_{2n}, \beta + 1 \rangle, \sigma] = (\tau, \rho) \quad \text{where} \quad \rho = \langle \gamma_1, \dots, \gamma_{2n}, \alpha + 1 \rangle.$$

And if  $\alpha = 0$ , then  $\{\alpha\} = (\tau, \rho)$  where  $\tau = \langle \gamma_1, \dots, \gamma_{2n} + 1 \rangle$ , and  $\rho$  is as before. Note also that if  $\sigma \in X$ , then  $\sigma \in A_\sigma$  iff  $\sigma$  is of odd length iff  $\overline{A_\alpha}$  is a neighborhood of  $\sigma$ .

It is easy to see that  $Y$  is first countable and separable. It is not locally compact, nor is its closed subspace  $X$ , because no point of  $X_\omega$  has a compact neighborhood.

**5.5. PROPOSITION.**  *$Y$  is countably compact if, and only if, the following conditions hold:*

- (i) *The pair  $\langle \{ A_\alpha : \alpha < \omega_2 \}, \{ \mathbb{N} \} \rangle$  is a tight  $(\omega_2, 1)$ -gap.*
- (ii) *For each choice of  $\langle \gamma_1, \dots, \gamma_{2n} \rangle$ ,  $n > 0$ , the sets indexed by*

$$\{\langle \gamma_1, \dots, \gamma_{2n}, \alpha \rangle : \alpha < \omega_2\}$$

and the sets indexed by

$$\{\langle \gamma_1, \dots, \gamma_{2n-1}, \beta \rangle : \beta < \gamma_{2n}\}$$

together form a tight  $(\omega_2, \omega_1^*)$ -gap.

(iii) For each choice of  $\langle \gamma_1, \dots, \gamma_{2n+1} \rangle$ ,  $n \geq 0$ , the sets indexed by

$$\{\langle \gamma_1, \dots, \gamma_{2n}, \beta \rangle : \beta < \gamma_{2n+1}\}$$

and those indexed by

$$\{\langle \gamma_1, \dots, \gamma_{2n+1}, \alpha \rangle : \alpha < \omega_2\}$$

together form a tight  $(\omega_1, \omega_2^*)$ -gap.

In other words, in the language of NYIKOS [1988a], where  $*$  denotes Stone-Čech remainder in  $\beta\mathbb{N}$ ,  $\{A^* : A \in \mathbf{A}\}$  is a clopen  $\omega_1$ -tunnel through  $\beta\mathbb{N} \setminus \mathbb{N}$ .

PROOF. If any of the three conditions fails, then an infinite subset  $S$  of  $\mathbb{N}$  beside the pre-gap will be closed discrete. For example, if  $S$  is beside the pre-gap in (ii) and  $\sigma \in X$ , then either there will be some  $\beta < \gamma_{2n}$  such that  $\langle \gamma_1, \dots, \gamma_{2n-1}, \beta \rangle < \sigma$ , whence  $S \subset^* A_{\gamma_1, \dots, \gamma_{2n-1}, \beta}$  and so  $\sigma \notin \overline{S}$ ; or else there will be  $\alpha < \omega_2$  such that  $\sigma < \langle \gamma_1, \dots, \gamma_{2n}, \alpha \rangle$ , and the closure of  $A_{\gamma_1, \dots, \gamma_{2n}, \alpha}$  will be a neighborhood of  $\sigma$  having finite intersection with  $S$ .

Conversely, suppose the three conditions are met. We already know  $X$  is countably compact, so it suffices to show that every infinite subset  $S$  of  $\mathbb{N}$  has a limit point in  $X$ . By (i), there is a least  $\alpha_1 < \omega_2$  such that  $A_{\alpha_1} \cap S = S_1$  is infinite. If  $\alpha_1 \in \Gamma$ , then  $\langle \alpha_1 \rangle$  is a limit point of  $S$ . If  $\alpha_1 \in \Gamma$ , we continue as follows.

Suppose  $\alpha_1, \dots, \alpha_n$  and  $S_n \subset A_{\alpha_1, \dots, \alpha_n}$  have been defined and  $n$  is odd and  $\alpha_n \in \Gamma$ , pick the least ordinal  $\alpha_{n+1}$  such that  $A_{\alpha_1, \dots, \alpha_n, \alpha_{n+1}}$  omits infinitely many points of  $S_n$ . If  $\alpha_{n+1} \notin \Gamma$ , then  $\langle \alpha_1, \dots, \alpha_{n+1} \rangle$  is a limit point of  $S_n$ . If  $\alpha_{n+1} \in \Gamma$ , let  $S_{n+1} = S_n \setminus A_{\alpha_1, \dots, \alpha_{n+1}}$ .

If  $n$  is even and  $\alpha_n \in \Gamma$ , let  $\alpha_{n+1}$  be the least ordinal such that  $S_n = A_{\alpha_1, \dots, \alpha_{n+1}} = S_{n+1}$  is infinite. If  $\alpha_{n+1} \notin \Gamma$ , we are done as before. If  $\alpha_{n+1} \in \Gamma$ , we continue.

If we have been forced to choose  $\alpha_n$  for all  $n \in \omega$ , then every neighborhood of  $\langle \alpha_n \rangle_n$  contains infinitely many points of  $S$ .  $\square$

The crux of the matter is, whether it is consistent with  $\mathfrak{t} = \omega_2$  that an  $\mathbf{A}$  satisfying 5.5 exists. It is *not* consistent with higher values of  $\mathfrak{t}$  because of (i), and while consistency with  $\mathfrak{t} = \omega_1$  is not out of question, it is not of much interest as far as the title problem goes.

The method of constructing  $\mathbf{A}$  outlined near the beginning of 5.4 only produces pre-gaps. If the following problem has an affirmative solution, a simple modification of the technique would produce (tight) gaps.

? 292. **Problem 10.** *Is it consistent with  $\mathfrak{t} = \omega_2$  that every  $\subset^*$ -ascending  $\omega_1$ -*

sequence of subsets of  $\mathbb{N}$  is the bottom half of some  $(\omega_1, \omega_2^*)$ -gap? some tight  $(\omega_1, \omega_2^*)$ -gap?

This is easily seen to be equivalent to the “dual” problem of whether every  $\subset^*$ -descending  $\omega_1$ -sequence is the top half of some (tight)  $(\omega_2, \omega_1^*)$ -gap. Hence an affirmative solution could give new models where the title problem has a positive solution.

Even without “tight”, the best we can hope for in Problem 10 is a consistency result. As shown in BAUMGARTNER [1984], **PFA**, which implies all the “small uncountable cardinals” considered here, including  $\mathfrak{c}$ , are equal to  $\omega_2$ , also implies there are no  $(\omega_1, \omega_2^*)$ -gaps. (Caution: Baumgartner uses “gap” to mean what is here called a “pre-gap”, and “unfilled gap” to mean what is here called a “gap”. Also in the “natural candidate” following BAUMGARTNER [1984, 4.1] one needs to insert the following clause in the definition of  $P(\bar{a}, \bar{b})$ : “and  $a_\alpha - n \subset b_\beta - n$  for all  $a_\alpha \in x, \beta \in y$ .” Also, in the proof of BAUMGARTNER [1984, 4.2(c)], where  $\xi$  is defined to be  $\min(z_\alpha)$ , that should be the definition of  $\eta$  also.)

The following axiom, formally less demanding than the “tight” version of Problem 10, is equivalent to the existence of an **A** satisfying 5.5. On the one hand, every **A** as in 5.5 behaves like **C** below; on the other hand, an induction very similar to that in 5.4 constructs a desired **A**  $\subset$  **C**:

**5.6. AXIOM.** There is a  $\subset^*$ -chain **C** in  $\mathcal{P}(\mathbb{N})$  with a cofinal subset  $\{A_\alpha : \alpha < \omega_2\}$  which forms a tight gap with  $\{\mathbb{N}\}$ , such that:

- (i) For all  $\gamma \in \Gamma$ ,  $\langle \{A_\alpha : \alpha < \gamma\}, \{C \in \mathbf{C} : \alpha < \gamma(A_\alpha \subset^* C)\} \rangle$  is a tight  $(\omega_1, \omega_2^*)$ -gap;
- (ii) Given any tight  $(\omega_1, \omega_2^*)$ -gap  $\langle \mathbf{C}', \mathbf{C} \setminus \mathbf{C}' \rangle$  with  $\mathbf{C}' \subset \mathbf{C}$ , there is a  $\subset^*$ -descending family  $\{B_\alpha : \alpha < \omega_2\}$  in **C** coinitial with  $\mathbf{C} \setminus \mathbf{C}'$  such that, for all  $\gamma \in \Gamma$ ,

$$\langle \{C \in \mathbf{C} : \alpha < \gamma(C \subset^* B_\alpha)\}, \{B_\alpha : \alpha < \gamma\} \rangle$$

is a tight  $(\omega_2, \omega_1^*)$ -gap

- (iii) (The dual statement of (ii)).

There are some fairly obvious generalizations of 5.3 through 5.6 to regular cardinals  $\kappa > \omega_2$ . The simplest kinds have  $\Gamma = \{\alpha \in \kappa : \alpha \text{ is of uncountable cofinality}\}$  and then simply substitute  $\kappa$  for  $\omega_2$  in the definitions of  $X$  and  $Y$ . The analogue of Problem 10 is open for them too, and again the best we can hope for is a consistency result, thanks to some models of **MA** +  $\mathfrak{c} = \kappa$  in unpublished work of Kunen. Of course, the really interesting models are those where  $\mathfrak{b} < \mathfrak{c}$ , and there Problem 10 and its analogue for higher  $\kappa$  seem to be completely open.

The maximum generality that can easily be achieved by the ideas in this section and NYIKOS [1988a] is represented by the following theorem, whose

proof is left to the interested reader. For the definition of a clopen  $\omega_1$ -tunnel through  $\beta\mathbb{N} \setminus \mathbb{N}$ , see NYIKOS [1988a].

**5.7. THEOREM.** *If there is a clopen  $\omega_1$ -tunnel  $\mathbf{C}$  through  $\beta\mathbb{N} \setminus \mathbb{N}$  such that, if  $\mathbf{C}' \subset \mathbf{C}$ , then  $\mathbf{C}'$  has a supremum (resp. infimum) in  $\mathbf{C}$  iff  $\mathbf{C}'$  has countable cofinality (resp. coinitality), then there is a first countable, zero-dimensional, countably compact noncompact space  $X$  with  $\mathbb{N}$  as a dense set of isolated points and  $X \setminus \mathbb{N}$  totally ordered.*

These conditions are easily met if  $\mathfrak{t} = \omega_1$  by Example 2.2, but it is still not known whether they are compatible with  $\mathfrak{t} > \omega_1$ .

## 6. Difficulties with manifolds

The dual personality of the title problem becomes even more acute when it is restricted to manifolds. (By a manifold, we mean a connected space in which every point has a neighborhood homeomorphic to  $\mathbb{R}^n$  for some finite  $n$ .) On the one hand, the construction of a separable, countably compact, noncompact manifold under **CH** is almost as easy as that of an Ostaszewski-van Douwen space (NYIKOS [1984, Example 3.10]). On the other hand, there is Problem 2, and, worse yet:

? **293. Problem 11.** *Is it consistent that there is a pseudocompact manifold of weight  $> \aleph_1$ ?*

Unless the answer is “yes” in *every* model of  $\mathfrak{b} > \omega_1$ , we cannot say “yes” to:

? **294. Problem 12.** *Is there a **ZFC** example of a pseudocompact manifold which is not countably compact?*

The reason is that a pseudocompact space in which there is an infinite closed discrete subspace cannot satisfy **wD**, so by Theorem 3.7, if  $\mathfrak{b} > \aleph_1$ , then the weight must exceed  $\aleph_1$  also. In contrast, if  $\mathfrak{b} = \aleph_1$ , then we know there is a pseudocompact manifold that is not countably compact (Example 6.3).

Our extra difficulties stem from the fact that Property **wD** is no longer enough for extending a manifold to a larger one (as opposed to extending it simply to a first countable space). If we do not want to raise dimension, then *there must also be a closed copy of  $[0, 1]$  in the manifold we want to extend.* Even that is not always enough (Example 6.3).

A precise condition is (2) in:

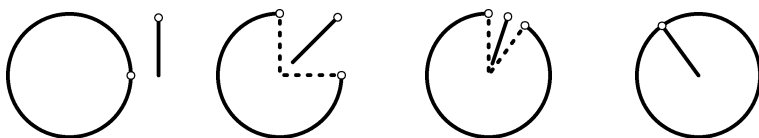
**6.1. THEOREM.** *Let  $X$  be an  $n$ -manifold. Then  $(1) \leftrightarrow (2) \rightarrow (3)$  in:*

(1) *There is an  $n$ -manifold  $Y$  containing  $X$  as a proper subspace.*

- (2) *There is, in  $X$ , a closed copy of  $\overline{B^n} - \{ \langle 1, 0, \dots, 0 \rangle \}$  where  $\overline{B^n}$  is the closed Euclidean  $n$ -ball.*  
 (3) *There is a closed copy of  $[0, 1]$  in  $X$ .*

PROOF. If  $n = 1$  then (2) and (3) are saying the same thing, and there are exactly four 1-manifolds by the definition of “manifold” adopted here.  $\mathbb{R}$  and the open long ray obviously satisfy both (1) and (2);  $S^1$  and the long line clearly satisfy neither. For the remainder of the proof we assume  $n \geq 2$ .

(2)  $\rightarrow$  (1). There is a well-known general method of adding a copy  $I$  of  $[0, 1]$  to  $B = \overline{B^n} - \{ \langle 1, 0, \dots, 0 \rangle \}$  so that the resulting space is homeomorphic to  $B$  by a homeomorphism leaving the “boundary” of  $B$  (i.e.  $B - E^n$  where  $E^n$  is the open unit  $n$ -ball) pointwise fixed. The method is suggested by the following pictures. As may be surmised, the original copy of  $B$  is dense in the whole space.



Details may be found in NYIKOS [1984, Section 3] or RUDIN and ZENOR [1976]. Now if  $X$  contains a closed copy of  $B$ , then we can perform this operation and let  $Y = X \cup I$  with the obvious topology.

(1)  $\rightarrow$  (3): Let  $y \in Y \setminus X$ . By “invariance of domain”,  $X$  is open in  $Y$ , and since  $X$  is not closed in  $Y$  we may take  $y \in \overline{X}$ . Let  $U$  be a neighborhood of  $y$  homeomorphic to  $\mathbb{R}^n$ , and let  $x \in X \cap U$ . Let  $f: [0, 1] \rightarrow U$  be an embedding such that  $f(0) = x$ ,  $f(1) = y$ . Then there is a (unique)  $r > 0$  such that  $f(s) \in X$  for all  $s \in [0, r)$  but  $f(r) \notin X$ . Then  $f \rightarrow [0, r)$  is a relatively closed copy of  $[0, 1]$  in  $X$ .

(1)  $\rightarrow$  (2): Let  $U$ ,  $y$ , and  $f$  be as above and let  $I = f \rightarrow [0, r)$ . We may assume  $f$  was defined so that there is a homeomorphism  $g$  of  $U$  with the open unit  $n$ -ball such that the image of  $\text{ran}(f)$  is a straight line segment. The image of  $X$  under  $g$  is then an open neighborhood of  $g \rightarrow f \rightarrow [0, r)$ . Let  $r_n = r(1 - 1/n)$  for each  $n \in \mathbb{N}$ . For each  $n$  there exists  $\varepsilon_n > 0$  so that the  $\varepsilon_n$ -ball around each point of  $g \rightarrow f \rightarrow [r_n, r_{n+1}]$  is a subset of  $g \rightarrow X$ . Let  $h: [0, r] \rightarrow [0, 1]$  be a continuous monotone function such that  $0 < h(p) < \varepsilon_n$  whenever  $p \in [r_n, r_{n+1}]$ ,  $h(r) = 0$ . Then the union of the closed  $h(p)$ -balls around the points  $g(f(p))$ ,  $p \in [0, r)$ , is the desired copy.  $\square$

**6.2. COROLLARY.** *If  $M$  is a pseudocompact  $n$ -manifold, then  $M$  cannot be properly extended to an  $n$ -manifold.*

PROOF. This is well known for  $n = 1$ : the only pseudocompact 1-manifolds are  $S^1$  and the long line. If  $n \geq 2$ , then  $B = \overline{B^n} - \{ \langle 1, 0, \dots, 0 \rangle \}$  has an



infinite discrete family of open sets, and they will be discrete in any manifold containing  $B$  as a closed subspace. Hence (2) fails for  $X = M$ .  $\square$

The following example, besides showing (3) need not imply (2), is the best example to date of a pseudocompact manifold that is not countably compact.

**6.3. EXAMPLE ( $\mathfrak{b} = \omega_1$ ).** Let  $C$  be the closed long ray, obtained by intersecting copies of  $(0, 1)$  between successive countable ordinals:  $C = \{\alpha + r : \alpha \in \omega_1, r \in [0, 1)\}$ . Let  $M$  be the extended plane (i.e. the one-point compactification of  $\mathbb{R}^2$ ) with the origin removed. The underlying set of  $X$  will be  $M \cup C$ .

Intuitively, we replace the origin by  $C$ , attaching it to  $M$  so that it becomes a continuation of the negative  $x$ -axis. To get at points  $p$  on  $[0, \omega) \subset C$ , we can approach them along straight line segments whose angle with the positive  $x$ -axis approaches 0 as  $p \rightarrow \omega$ . The angle of approach will be the same whether we approach from the positive half-plane or the negative half plane. To approach  $\omega$  and points beyond, we approach the origin along curves that are asymptotic to the positive  $x$ -axis: the more sharply asymptotic, the further up  $C$  the point approached.

Given  $p \in C$ , all we need do is specify an arc in the upper half-plane  $H^+$  along which the point is to be approached. Its mirror image in the lower half-plane  $H^-$  will also approach  $p$ . Then, given a basic neighborhood  $(q, r) \subset C$  of  $p$ , we extend to an open neighborhood of  $p$  by taking all points within  $\varepsilon$  of the origin (for some  $\varepsilon > 0$ ) that lie between the arcs approaching  $q$  and  $r$  in  $H^+$ , and doing the same in  $H^-$ . If  $r \geq \omega$ , there will be some crossing of the arcs, but the one attached to  $r$  in  $H^+$  will eventually be to the right of the one attached to  $q$  in  $H^+$ , and we can wlog confine ourselves to  $\varepsilon$  that put us in this region.

The negative  $x$ -axis approaches  $0 \in C$ . Any  $r \in (0, 1]$  is approached along  $\pi + 3\pi r/4$ . If  $r \in [1, \omega)$  it is approached along the line of slope  $1/r$ .

For  $\omega$  and beyond, we use a  $<^*$ -unbounded,  $<^*$ -well-ordered family of increasing functions in  ${}^{\mathbb{N}}\mathbb{N}$ ,  $\{f_\alpha : \alpha < \mathfrak{b} = \omega_1\}$ , with  $f_k(h) = kn$  for all  $k \in \omega - \{0\}$ . In  $\mathbb{R}^2$ , let  $g_\alpha$  be (the graph of) a continuous increasing function, with  $g_\alpha(1/n) = 1/f_\alpha(n)$  for all  $n$ . Let  $h_\alpha = g_\alpha|_{\mathbb{R}^+}$ . Note that  $h_i$  approaches  $i \in C$  for  $0 < i < \omega$ . We also have  $h_\alpha$  approach  $\alpha \in C$  for all  $\alpha \in \omega_1 - \{0\}$ . For points in the interval  $(\alpha, \alpha + 1)$  we interpolate between  $h_\alpha$  and  $h_{\alpha+1}$  in orderly fashion.

To see that the resulting manifold  $X$  is pseudocompact, we show that every infinite sequence  $\langle x_n \rangle$  in  $M \setminus$  (the positive  $x$ -axis) has a cluster point, and use the fact that this subspace is dense in  $X$ .

If  $\langle x_n \rangle$  meets the negative  $x$ -axis  $A$  in infinite many terms, it has a cluster point in  $A \cup \{\infty\} \cup \{0\}$  ( $0 \in C$ ). By symmetry we may assume  $x_n \in H^+$  for all  $n$ . Then either (I) there exists  $\varepsilon > 0$  such that  $\|x_n\| \geq \varepsilon$  for infinitely many  $n$ , when there is a cluster point in the complement of the  $\varepsilon$ -ball centered on

the origin or (II) infinitely many  $x_n$  lie to the left of some  $h_\alpha$ , when there is a cluster point in  $M \cup [0, \alpha]$ . Instead, if both conditions failed, there would be an infinite sequence of  $x_n$  with declining positive abscissae, converging to 0, and all but finitely many below the graph of any given  $h_\alpha$ . But this is easily seen to violate  $<^*$ -unboundedness of the  $f_\alpha$ .

It is also easy to see that any closed initial segment of the positive  $x$ -axis is a closed copy of  $(0, 1]$  in  $X$ .

**Problem 13.** *If  $X$  is an  $n$ -manifold of weight  $< \mathfrak{b}$  which contains a closed copy of  $[0, 1)$ , does  $X$  have a proper extension to an  $n$ -manifold?* **295. ?**

Observation 3.10 sheds some light on this problem, taking us from the weaker to the stronger of the following two conditions:

**6.4. DEFINITION.** A closed copy  $I$  of  $[0, 1)$  in an  $n$ -manifold  $X$  is *pre-tame* if there is an neighborhood of  $I$  homeomorphic to  $\mathbb{R}^n$ , by a homeomorphism taking  $I$  to a closed ray.  $I$  is *strongly pre-tame* if there is a closed neighborhood of  $I$  homeomorphic to  $\overline{B}^n - \{\langle 1, 0, \dots, 0 \rangle\}$ , by a homeomorphism taking  $I$  to  $\{\langle r, 0, \dots, 0 \rangle : 0 \leq r < 1\}$ .

Of course, Theorem 6.1 implies that an  $n$ -manifold has a proper extension to an  $n$ -manifold iff it contains a strongly pre-tame closed copy of  $[0, 1)$ . It is easy to see that a strongly pre-tame copy of  $I$  is pre-tame, and the non-negative  $x$ -axis in Example 6.3 is pre-tame but not strongly pre-tame. On the other hand, as suggested above:

**6.5. LEMMA.** *A pre-tame copy of  $[0, 1)$  in a manifold  $X$  of weight  $< \mathfrak{b}$  is strongly pre-tame.*

PROOF. Let  $U$  be a neighborhood witnessing pre-tameness of  $I$ . By Observation 3.10 there is an open neighborhood  $V$  of  $I$  whose closure in  $X$  is a subset of  $U$ . An argument like that in  $(1) \rightarrow (2)$  of Theorem 6.1, with  $V$  playing the role of  $g \rightarrow X$  there, produces a neighborhood homeomorphic to  $\overline{B}^n - \{\langle 1, 0, \dots, 0 \rangle\}$  which is closed in  $U$ , hence in  $X$ .  $\square$

**6.6. COROLLARY.** *An  $n$ -manifold of weight  $< \mathfrak{b}$  has a proper extension to an  $n$ -manifold iff it contains a closed pre-tame copy of  $[0, 1)$ .*

Thus Problem 13 can be re-phrased: if an  $n$ -manifold of weight  $< \mathfrak{b}$  contains a closed copy of  $[0, 1)$ , must it contain a closed pre-tame copy? I do not know the answer even if “of weight  $< \mathfrak{b}$ ” is dropped.

Unfortunately, Problem 13 is not the only hurdle to getting proper extensions of  $n$ -manifolds of weight  $< \mathfrak{b}$ . By shifting our base of operations into the first quadrant, we can modify 6.3 to produce a 2-manifold of weight  $\aleph_1$  without a closed copy of  $[0, 1)$ .

**6.7. EXAMPLE.** Our underlying set will be  $M \cup C$  as in 6.3, but the construction is carried out in **ZFC**. This time, it will be the negative half of the main diagonal (the ray  $\theta = 5\pi/4$ ) that approaches  $0 \in C$ , and  $r \in (0, 1] \subset C$  is approached at angles  $\frac{\pi}{4}(5 \pm 3r)$ .

In the first quadrant we make use of an  $(\omega_1, \omega_1^*)$ -gap in  ${}^{\mathbb{N}}\mathbb{N}$ , i.e. a pair  $\langle \mathbf{A}, \mathbf{B} \rangle$  of families of functions  $\langle f_\alpha : \alpha < \omega_1 \rangle$ ,  $\langle g_\alpha : \alpha < \omega_1 \rangle$  so that  $f_\beta <^* f_\alpha <^* g_\alpha <^* g_\beta$  whenever  $\beta < \alpha$ , and such that for no  $f: \mathbb{N} \rightarrow \mathbb{N}$  is it true that  $f_\alpha <^* f <^* g_\alpha$  for all  $\alpha < \omega_1$ . We may assume each  $f_\alpha$  and  $g_\alpha$  is increasing. For a detailed discussion on how to produce such a gap, see DALES and WOODIN [1987, p.104 ff].

Let  $h_1$  be the positive  $y$ -axis and  $k_1$  the positive  $x$ -axis; these both approach  $1 \in C$ . For  $\alpha > 1$  let  $h_\alpha$  and  $k_\alpha$  be continuous increasing functions from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ , satisfying  $h_\alpha(1/n) = 1/f_\alpha(n)$ ,  $k_\alpha(1/n) = 1/g_\alpha(n)$ . Let the graphs of  $h_\alpha$  and  $k_\alpha$  approach  $\alpha \in C$ , and interpolate naturally to approach points of  $(\alpha, \alpha + 1) \subset C$ .

There can be no closed copy of  $(0, 1]$  in the resulting manifold. Such a copy would have to approach the origin, but this can only be done in a way that eventually takes us between (the graphs of) any given  $h_\alpha$  and  $k_\alpha$ , but that is impossible. (The reason it can only be done in this way is that if we take out from  $X$  the part that is strictly below  $h_\alpha$  and strictly above  $k_\alpha$  from the point where they no longer touch one another on the way to the origin, then what is left is a countably compact space: a compact manifold-with-boundary with a copy of the long ray glued on.) The reason this is impossible is that if we take, for each  $n$ , a point  $\langle 1/n, y_n \rangle$  where the alleged copy touches the vertical line  $x = 1/n$ , let  $k_n$  be the integer closest to  $1/y_n$  whenever  $y_n > 0$ , and leave  $k_n$  undefined otherwise, then let  $f(n) = k_n$  when  $k_n$  is defined and  $f(n) = 1$  otherwise, then we will have  $f_\alpha <^* f <^* g_\alpha$  for all  $\alpha$ .

Under **CH** it is possible to choose  $f_\alpha$  and  $g_\alpha$  so that  $X$  is countably compact. On the other hand, if  $\mathfrak{p} > \omega_1$ , then  $X$  will have infinite closed discrete subspaces (see paragraph following construction 3.3), and then because  $\mathfrak{b} > \omega_1$  these can be even expanded to discrete collections of open sets. Also, in every model of **ZFC** one can modify the construction so as to leave a discrete collection of open disks converging to the origin. So, if we want to build a pseudocompact manifold of weight  $> \aleph_1$ , we seem to be faced with the devilish task of insuring that no such subspace as our manifold arises before the final stage.

There is one loophole: it *is* possible to add a limit point to any given closed discrete subspace of Example 6.7 by going to a third dimension: the usual embedding of  $M$  in the one-point compactification of  $\mathbb{R}^3$  can be extended to an embedding of  $X$  into  $\mathbb{R}^3 \cup C$  with a natural topology: “fatten” basic neighborhoods vertically, but to approach points of  $C$ , make both the  $y$ - and  $z$ -coordinates of the sequence converge to 0. Now modify that embedding so that some discrete family of open sets in  $H^+$ , rather than staying in the

$xy$ -plane, becomes a family of bumps of height 1 with their bottoms on the  $xy$ -plane coinciding with their boundaries. The tops of these bumps then converge to  $\langle 0, 0, 1 \rangle$ .

One problem with this loophole is that there are analogues of 6.7 in dimensions  $n \geq 3$ , such as its product with  $S^{n-2}$ , and we do eventually have to settle upon a dimension for the final manifold. Another problem is that we do not know how to find limit points for all closed discrete subspaces of 6.7 even by going to higher dimensions. If we relax the definition of “manifold” to allow Hilbert cube manifolds, where “invariance of domain” no longer holds, it might be possible to overcome these difficulties, but that remains to be seen.

Besides these problems, there is even the one of whether there is a separable, countably compact, noncompact manifold in all models of  $\mathfrak{p} = \omega_1$ , or even  $\mathfrak{b} = \omega_1$ . The answer would be affirmative if we knew how to solve Problem 14 below.

**6.8. DEFINITION.** Given a function  $f \in {}^{\mathbb{N}}\mathbb{N}$ , define

$$f^{\uparrow} = \{ \langle i, j \rangle \in \mathbb{N} \times \mathbb{N} : j \geq f(i) \}$$

$$f^{\downarrow} = \{ \langle i, j \rangle \in \mathbb{N} \times \mathbb{N} : j \leq f(i) \}.$$

Call a pair  $\mathbf{A}, \mathbf{B}$  of families in  ${}^{\mathbb{N}}\mathbb{N}$  a *tight*  $(\sigma, \tau^*)$ -gap if  $\mathbf{A} = \langle f_{\alpha} : \alpha < \sigma \rangle$ ,  $\mathbf{B} = \langle g_{\beta} : \beta < \tau \rangle$ , are such that  $(\langle f_{\alpha}^{\downarrow} : \alpha < \sigma \rangle, \langle g_{\beta}^{\uparrow} : \beta < \tau \rangle)$  is a tight gap in  $\mathbb{N} \times \mathbb{N}$ .

**Problem 14.** Does  $\mathfrak{p} = \omega_1$  or  $\mathfrak{b} = \omega_1$  imply there is a tight  $(\omega_1, \omega_1^*)$ -gap in  ${}^{\mathbb{N}}\mathbb{N}$ ? **296. ?**

It is not difficult to construct such a gap under **CH**, but the general problem remains open. Note that Example 6.7 is countably compact iff the gap used there is tight.

## 7. In the No Man's Land

Alan Dow has recently shown me a model of  $\omega_1 < \mathfrak{p}$  and  $\mathfrak{b} < \mathfrak{d}$ , in which it is not yet known whether there is a separable, first countable, countably compact, noncompact space. Here it is, with a slight modification by Amer Bešlagić.

**7.1. MODEL.** Beginning with any model of **ZFC**, let  $M_1$  be obtained by adding  $\aleph_2$  Cohen reals. Then let  $M$  be a model of **MA**( $\omega_1$ ) obtained from  $M_1$  by an iterated **ccc** forcing of cofinality  $\kappa \geq \aleph_3$  with finite supports, such that every poset at successors stages is of cardinality  $\leq \aleph_1$ . (One can simply imitate

the usual Martin-Solovay method of bringing about **MA**(KUNEN [1980, pp. 278–281]), but with the size of each  $Q_\alpha$  restricted to  $\leq \aleph_1$ .) Dow has shown:

(\*) In  $M$ , the initial set of  $\aleph_2$  Cohen reals is not dominated.

Here, by a slight abuse of language, we are identifying a Cohen real  $A$  with the unique order-preserving bijection  $f: \omega \rightarrow A$ . All that is needed to show (\*) is that  $|Q_\alpha| \leq \aleph_1$  for all  $\alpha$  and that supports are countable (which is not the same as saying “countable supports”!).

From (\*) it follows that  $\mathfrak{b} = \aleph_2$  in  $M$ ; it cannot be less because **MA**( $\omega_1$ ) holds in  $M$ . On the other hand, because supports are finite, new Cohen reals are added cofinally often and are not majorized by any function added at an earlier stage; from this, and the fact that **ccc** forcing preserves cofinalities, we get  $\mathfrak{d} \geq \kappa \geq \aleph_3$  in  $M$ . From **MA**( $\aleph_1$ ) follows  $\mathfrak{p} > \omega_1$  (hence  $\mathfrak{p} = \aleph_2$ ).

? **297. Problem 15.** *Is there a first countable, separable, countably compact, non-compact space in  $M$ ?*

One possibility is that the ground model, or some intermediate model, contains an Ostaszewski-van Douwen space of cardinality  $\geq \aleph_2$ . Could it be that at least one such space is preserved during the forcing, just as the unboundedness of the initial set of  $\aleph_2$  Cohen reals is?

In some cases, the answer is Yes. When the ground model satisfies  $\mathfrak{p} = \mathfrak{c} = \aleph_2$  and is obtained from a model of **GCH** by finite-support **ccc** forcing, then the ground model contains an Ostaszewski-van Douwen space with  $\omega$ -bounded remainder that is preserved in  $M$ . Details will appear in a future paper, after some other models have been investigated.

For instance, if the ground model satisfies  $2^{\aleph_1} = \aleph_2$ , and the iteration is of length  $\aleph_3$ , then  $\mathfrak{p} = \aleph_2$ , hence  $\mathfrak{b} = \aleph_2 = \mathfrak{c}$ , will be satisfied cofinally often in the iteration, giving us natural candidates for a Yes answer there too. We could, of course, try for a No answer by stretching out the iteration and keeping  $\mathfrak{b} = \aleph_1$  until the first  $\aleph_3$  stages are past. However, the fact that supports are finite means that the names for the size- $\aleph_1$  posets at successors stages “live” on  $\aleph_1$  or fewer coordinates of the iteration. Picking carefully the posets that kill off various unbounded families, of size  $\aleph_1$ , it will often be possible to build up inner models of  $\mathfrak{b} = \mathfrak{c} = \aleph_2$ , and then be faced with the question of whether there is anything in  $M$  that kills off all the Ostaszewski-van Douwen spaces that live there.

Our knowledge of what preserves, or what kills, various Ostaszewski-van Douwen spaces, is still very spotty, and less developed than our knowledge of how to create them. A concerned effort to improve it may well be repaid by the illumination it brings into this no man’s land.

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## CHAPTER 9

### Set-theoretic problems in Moore spaces

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## 1. Introduction

The problems presented in this paper reflect the author's view of Moore spaces as the historical testing ground for topological pathology. Moore spaces were initially but a first stage towards R. L. Moore's successful topological characterization of the plane MOORE [1932]. However, through his extensive use of this concept in teaching "non-metric" topology to several generations of gifted students, Moore spaces have become one of the most widely studied class of spaces in topology. Their continuing relevance has been due to the depth of positive theory shared with metric spaces, together with the wealth of non-trivial counterexamples which differentiate them. Recall that many of the now standard structures in general topology (e.g., subparacompact spaces (BING [1951] and MCAULEY [1956]), the collectionwise normal and collectionwise Hausdorff properties (BING [1951]),  $Q$ -sets (BING [1951] and HEATH [1964]), base of countable order theory (WORRELL and WICKE [1965], and Pixley-Roy constructions (PIXLEY and ROY [1969]) were direct products of the study of non-metrizable Moore spaces. Furthermore, the rebirth of "set-theoretic" topology in the past two decades was certainly due in a large part to the *normal Moore space conjecture* JONES [1937] and its partial solutions (e.g., TALL [1969], FLEISSNER [1974], REED and ZENOR [1976], NYIKOS [1980], FLEISSNER [1982b, 1982a]).

Of course, as anyone who cares to look through Moore's notebooks at the University of Texas will easily agree, most of Moore's own interests concerned the geometric and continua theory of Moore spaces. The work of his students G. T. Whyburn, R. L. Wilder, F. B. Jones, G. S. Young, R. H. Bing, E. E. Moise, R. D. Anderson, B. J. Ball, H. Cook and others reflects this legacy.

In this paper, the author has chosen to concentrate on a very short and admittedly very personal list of problems which continue to interest and to defy him after several years pursuit.

### *Organization of the paper*

The next six sections contain problems related to the topics indicated. In section 8, we discuss recent solutions to two problems which the author had intended to include in the paper. References are given in section 9.

## 2. Normality

We now know that the normal Moore space conjecture is in the hands of the gods and large cardinals (see TALL [1984] and FLEISSNER [1984]). However, there remain several related problems which we can still hope either to establish in **ZFC** or at least show to be both consistent and independent, without the use of large cardinals.

? 298. **Problem 2.1.** *Is each normal Moore space submetrizable?*

In REED and ZENOR [1976], the author and Phil Zenor showed that in **ZFC** each normal, locally compact, locally connected Moore space is metrizable. The key idea in the proof of this result was that each normal Moore space of cardinality  $\leq \mathfrak{c}$  is submetrizable, i.e., admits a one-to-one continuous map onto a metric space. Using the same techniques, it follows that a normal Moore space is submetrizable if and only if it has a  $\sigma$ -disjoint separating open cover. Recall that a *separating cover*  $\mathcal{H}$  of the space  $S$  is a cover such that for all  $x, y \in S$  such that  $x \neq y$ , there exists  $H \in \mathcal{H}$  such that  $x \in H$  but  $y \notin H$ .

It is easy to see that a space with a  $\sigma$ -disjoint separating open cover has cardinality  $\leq \lambda^\omega$ , where  $\lambda$  is the cellularity. Hence, for example, the existence of a normal Moore space with cardinality greater than  $\mathfrak{c}$  but cellularity  $\mathfrak{c}$  would provide a negative answer to the above question. By similar arguments, a normal Moore space would be non-submetrizable if it contained a discrete subset  $X$  of cardinality  $> \mathfrak{c}$  such that no subset of  $X$  of cardinality  $> \mathfrak{c}$  could be screened by open sets. Under  $V = L$ , there exist no normal Moore spaces of either type, since under this assumption each normal Moore space is collectionwise Hausdorff and has a  $\sigma$ -discrete  $\pi$ -base (see 2.5). However, in [1984, p. 66], JUHÁSZ gives a first countable space under **(CH+ an  $\omega_2$ -Souslin line)** with cardinality greater than  $\mathfrak{c}$  but with cellularity  $\mathfrak{c}$ . The author's Moore space machine (REED [1974c]) over this space produces a Moore space with the same properties. However, whereas Juhász's space is normal, the Moore space will not in general be normal. The Pixley-Roy space on a  $Q$ -set (PIXLEY and ROY [1969] is an example under **(MA +  $\neg$ CH)** of a normal Moore space of cardinality  $\omega_1$  but with cellularity  $\omega$ . However, note that if a Moore space is formed by a Pixley-Roy construction on a first countable space  $X$ , the Moore space will have cardinality  $\leq \lambda^\omega$ , where  $\lambda$  is its cellularity.

The above observations by the author have recently been analyzed and significantly extended by MIRIAM BROD in [1987, 1990]. For example, she has characterized submetrizable spaces as those which admit a  $\sigma$ -discrete separating cover of co-zero sets. Also, she has made progress on deciding the consistency of normality for Moore spaces such as the one over the above-mentioned Juhász space.

Note that the full strength of normality is needed for a positive result. In WAGE, FLEISSNER and REED [1976] and REED [1980], the author gave an example (under **MA +  $\neg$ CH**) of a countably paracompact, separable Moore space of cardinality  $< \mathfrak{c}$  which is not submetrizable.

? 299. **Problem 2.2.** *Is it consistent with **ZFC** that the square of each normal Moore space is normal?*

In [1976] HOWARD COOK gave an example (under **MA +  $\neg$ CH**) of a normal

Moore space whose square is not normal. This paper was never published. The author announced in REED [1986] that under  $\mathbf{MA} + \neg\mathbf{CH}$ , the Moore space over  $X$  (in the sense of REED [1976]) is such an example, for all  $X$  in the class  $\mathcal{C}$  of spaces formed via the intersection topology w.r.t. the real line and the countable ordinals. Furthermore, the proof technique to show that the squares are not normal also yields a simple proof that Cook's original example works. These results were presented at the 1986 Prague Symposium.

**Problem 2.3.** *If  $X$  is a normal, locally compact Moore space, must the square of  $X$  be normal?* **300. ?**

A positive answer to this question seems likely in  $\mathbf{ZFC}$ . In [1974] FLEISSNER has shown that under  $V = L$ , normal locally compact Moore spaces are metrizable. Hence, one might look for a negative result under  $\mathbf{MA} + \neg\mathbf{CH}$ . However, in [1976], ALSTER and PRZYMUSIŃSKI showed that under  $\mathbf{MA}$ , if  $X$  is a cometrizable  $T_2$ -space (i.e., admits a *regular*, one-to-one continuous map onto a separable metric space) and the cardinality of  $X$  is less than  $\mathfrak{c}$ , then  $X^\omega$  is normal. The author observed in REED [1974d] that if  $X$  is a normal, locally compact Moore space of cardinality  $\leq \mathfrak{c}$  then  $X$  is cometrizable, and hence under  $\mathbf{MA}$  the square of  $X^\omega$  is normal if the cardinality of  $X$  is strictly less than  $\mathfrak{c}$ . This result was in turn extended by Alster and Przymusiński (see [1977]) who showed that under  $\mathbf{MA}$ , if  $X$  is a normal, locally compact, separable Moore space then  $X^\omega$  is normal.

**Problem 2.4.** *If the square of  $X$  is a normal Moore space, is  $X$  submetrizable?* **301. ?**

From the above remarks, it is clear that the product question and the submetrizability question are linked. COOK has shown in [1976] that if  $X^2$  is a normal Moore space, then  $X$  is continuously semi-metrizable and hence has the three link property (equivalently a regular  $G_\delta$ -diagonal (ZENOR [1972])). It is easy to see that each submetrizable space also has a regular  $G_\delta$ -diagonal. However, in REED [1980] (see WAGE, FLEISSNER and REED [1976]), the author gave an example of a continuously semi-metrizable Moore space which is not submetrizable. Note that a space with the three link property must have cardinality  $\leq \lambda^\omega$  where  $\lambda$  is the cellularity, hence the method for constructing possible counterexamples discussed in 2.1 is ruled out here. Also, note that each of the known consistent examples of normal Moore spaces whose squares are not normal are of small cardinality and submetrizable, but not cometrizable. Hence, a reasonable conjecture would be that under  $\mathbf{MA}$ , the square of a normal Moore space  $X$  of cardinality  $< \mathfrak{c}$  is normal if and only if  $X$  is cometrizable (see PRZYMUSIŃSKI [1977]).

? 302. **Problem 2.5.** *Is it consistent with **ZFC** that each normal Moore space is completable?*

Recall that a Moore space  $X$  is complete if and only if it has a development  $\{\mathcal{G}_n\}$  such that if  $\{M_n\}$  is a non-increasing sequence of closed sets in  $X$  such that for each  $n$ ,  $M_n$  is contained in an element of  $\mathcal{G}_n$ , then  $\bigcap\{M_n\} \neq \emptyset$ . In Moore's original terminology, this was Axiom 1<sub>4</sub>, hence a complete Moore space was a space satisfying Axioms 0 and all four parts of Axiom 1. A metrizable Moore space is complete if and only if it admits a complete metric. Furthermore, a completely regular Moore space is complete if and only if it is Čech complete CREEDE [1971]. Each complete Moore space has the Baire property.

A Moore space is completable if it can be embedded in a complete Moore space. The first example of a non-completable Moore space was given in Mary Ellen Rudin's Thesis (see (ESTILL) RUDIN [1950]). In [1965] FITZPATRICK showed that each completable Moore space has a dense metrizable subspace, hence the **ccc**, non-separable Moore space constructed by PIXLEY and ROY in [1969] is also non-completable. A separable, non-completable Moore space was constructed by the author in REED [1972a]. In [1974], PRZYMUSIŃSKI and TALL showed that the Pixley-Roy space on a  $Q$ -set is a normal Moore space. Hence, under **MA** +  $\neg$ **CH**, there exists a normal, non-completable Moore space. A necessary and sufficient condition for a Moore space to be completable was given by WHIPPLE in [1966].

An important concept in the analysis of completions of Moore spaces is that of a  $\pi$ -base. Recall that a  $\pi$ -base for a space is a collection of non-empty open sets such that each non-empty open set in the space contains a member of the collection. In [1967], FITZPATRICK showed that a Moore space has a  $\sigma$ -discrete  $\pi$ -base if and only if it has a development satisfying Moore's metrization criterion at each point of a dense metrizable subset; this was later extended by WHITE who showed in [1978] that a Moore space has a dense metrizable subset if and only if it has a  $\sigma$ -disjoint  $\pi$ -base. In REED [1971a], the author gave an example of a Moore space with a  $\sigma$ -disjoint base but without a  $\sigma$ -discrete  $\pi$ -base.

In REED [1974b] the author showed that a Moore space can be densely embedded in a developable  $T_2$ -space with the Baire property if and only if it has a  $\sigma$ -discrete  $\pi$ -base. Thus, the normal Moore space under **MA** +  $\neg$ **CH** from PRZYMUSIŃSKI and TALL [1974] cannot even be densely embedded in a developable  $T_2$ -space with the Baire property. However, in [1974] FLEISSNER showed that under  $V = L$ , each normal Moore space is collectionwise Hausdorff. In [1967] FITZPATRICK showed that each normal collectionwise Hausdorff Moore space has a  $\sigma$ -discrete  $\pi$ -base. Hence, under  $V = L$ , we have that each normal Moore space can be densely embedded in a developable  $T_2$ -space with the Baire property.

**Problem 2.6.** *Is it true that a Moore space can be densely embedded in a Moore space with the Baire property if and only if it has a  $\sigma$ -discrete  $\pi$ -base?* **303. ?**

**Problem 2.7.** *Can each normal Moore space of weight  $\leq \mathfrak{c}$  be embedded in a separable Moore space?* **304. ?**

In [1969] JIM OTT raised the question as to whether each Moore space of weight  $\leq \mathfrak{c}$  can be embedded in a separable Moore space. In REED [1976], the author gave several partial solutions, including a positive answer for locally compact Moore spaces of weight  $\leq \omega_1$ . In [1980] VAN DOUWEN and PRZYMUSIŃSKI extended the same proof technique to obtain a positive answer for all Moore spaces of weight  $\leq \omega_1$ . Furthermore, they gave an example consistent with **ZFC** of a Moore space of weight  $\mathfrak{c}$  which cannot be embedded in a separable Moore space. Hence Ott's original question was shown to be independent of and consistent with **ZFC**.

Again the question with regard to normality is related to submetrizability. In [1977] PRZYMUSIŃSKI showed that each cometrizable Moore space is a closed subspace of a separable Moore space. Hence, each normal, locally compact Moore space of weight  $\leq \mathfrak{c}$  can be embedded as a closed subset of a separable Moore space.

### 3. Chain Conditions

The study of chain conditions in Moore spaces is a very rewarding area for those like the author who think that a theorem is but a poor consolation for the lack of an interesting counterexample. The story of chain conditions in Moore spaces is the search for interesting, non-trivial counterexamples.

The first example of a **ccc**, nonseparable Moore space was given in (ESTILL) RUDIN [1950] (one of the most horrendous constructions known to mankind). Fortunately, a simple example (in fact, arguably the most natural non-metrizable Moore space) was given by PIXLEY and ROY in [1969]. Examples of Moore spaces with the DCCC but not the **ccc** were given by the author in REED [1974a], where he also noted that DFCC Moore spaces (DFCC is equivalent to pseudocompactness for completely regular Moore spaces) are separable. In VAN DOUWEN and REED [19 $\infty$ ], Eric van Douwen and the author showed that the productivity of the **ccc** in Moore spaces is consistent with and independent of **ZFC**, whereas there is a DCCC Moore space in **ZFC** whose square is not DCCC.

**Problem 3.1.** *Is each starcompact Moore space compact?* **305. ?**

Several questions concerning chain conditions in Moore spaces were raised in REED [1974a]. Most of these questions were answered in VAN DOUWEN and

REED [19 $\infty$ ]. However, van Douwen and the author were unable to decide if there existed a 3-separable (equals DCCC) Moore space which was not 2-separable (see REED [1971a] and REED [1974a] for details). The following definitions arose in the attempt to analyze this question.

A space  $X$  is said to be *n-starcompact* if for every open cover  $\mathcal{U}$  of  $X$ , there is some finite subset  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\text{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$ .

A space  $X$  is said to be *strongly n-starcompact* if for every open cover  $\mathcal{U}$  of  $X$ , there is some finite subset  $B$  of  $X$  such that  $\text{st}^n(B, \mathcal{U}) = X$ .

A space  $X$  is said to be *n-star Lindelöf* if for every open cover  $\mathcal{U}$  of  $X$ , there is some countable subset  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\text{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$ .

A space  $X$  is said to be *strongly n-star Lindelöf* if for every open cover  $\mathcal{U}$  of  $X$ , there is some countable subset  $B$  of  $X$  such that  $\text{st}^n(B, \mathcal{U}) = X$ .

It is easy to see that if  $X$  is strongly  $n$ -starcompact then  $X$  is  $n$ -starcompact, and if  $X$  is  $n$ -starcompact then  $X$  is strongly  $n + 1$ -starcompact. A similar hierarchy holds for the star-Lindelöf properties. For  $T_3$ -spaces, strongly 1-starcompact equals countably compact and  $n$ -starcompact equals the DFCC, for  $n \geq 2$ . For Moore spaces, 1-star Lindelöf equals separable and  $n$ -star Lindelöf equals the DCCC for  $n \geq 2$ .

The existence of a 3-separable Moore space which is not 2-separable is equivalent to the existence of a 2-star-Lindelöf Moore space which is not strongly 2-star-Lindelöf. The author finally found such a Moore space in 1989 and presented it at the Oxford Topology Symposium. Actually, it was a space he had constructed in 1987 as an example of a DCCC Moore space with a  $\sigma$ -locally countable base (hence  $\sigma$ -para-Lindelöf) which is not separable.

In investigating the above properties, van Douwen and the author were able to give examples to show that all but three of the possibly distinct classes in Moore spaces were in fact distinct. Ian Tree, a D.Phil. student at Oxford, has recently pointed out the fact that strongly 2-starcompact is equivalent to 2-starcompact for Moore spaces follows from results in REED [1974a]. Hence, with the above result, there remains but one question. Let us call a 1-starcompact space simply starcompact. Is each starcompact Moore space compact?

Eric van Douwen observed that the Tikhonov plank is starcompact but not countably compact. Much later, the author showed that under  $\mathfrak{b} = \omega_1$ , a first countable space due to van Douwen and Nyikos given in VAN DOUWEN [1984] is also such an example. Finally Bill Roscoe and the author were able to show that under  $\mathfrak{b} = \mathfrak{c}$ , each starcompact Moore space is indeed compact. In fact, it follows from the proof of this result that if  $X$  is a starcompact Moore space which is not compact, then  $w(X)$  (the weight of  $X$ ) is of countable cofinality and  $\mathfrak{b} < w(X) < \mathfrak{c}$ .

The author acknowledges that the starcompact (and possibly the star-Lindelöf) properties have been studied by other authors under different terminology. In particular, upon a literature search it appears that the con-

cept of strong-starcompactness was introduced by Fleischman in his doctoral thesis FLEISCHMAN [1970]. In [1986] SARKHEL introduced the concept of starcompactness, and in [1984] MATVEEV defined  $k$ -pseudocompactness in a vein similar to the definitions of Fleischman. It is clear that the equivalence of strongly 1-starcompact and countably compact and the equivalence of strongly 3-starcompact and pseudocompact were known by some of these authors. In addition, Scott Williams informed the author that he had independently obtained certain of our lemmas about the weight of starcompact Moore spaces in unpublished work.

Finally, Ian Tree, has completed the study of the above properties in Hausdorff spaces, and made several new connections between these properties and well known topological structures (TREE [1989]). A complete presentation of the results mentioned here will appear in VAN DOUWEN, REED, ROSCOE and TREE [19 $\infty$ ].

**Problem 3.2.** *Does there exist in **ZFC** a Moore space with caliber  $(\omega_1, \omega)$  whose square does not have caliber  $(\omega_1, \omega)$ ?* **306. ?**

A topological space has *caliber*  $\omega_1$  (respectively, *caliber*  $(\omega_1, \omega)$ ) if every point-countable (respectively, point-finite) family of non-empty open sets is countable, and has Property K if every uncountable family of non-empty open sets has an uncountable linked (= pairwise non-disjoint) subfamily. Clearly, separability implies caliber  $\omega_1$ , caliber  $\omega_1$  implies caliber  $(\omega_1, \omega)$ , and caliber  $(\omega_1, \omega)$  implies the **ccc**. At a seminar in Oxford in 1987, Steve Watson showed that, although the product of two separable spaces is always separable, and it is consistent and independent that the product of two **ccc** spaces has **ccc**, there is an example in **ZFC** of two spaces with caliber  $(\omega_1, \omega)$  such that their product lacks caliber  $(\omega_1, \omega)$  (WATSON and ZHOU [1989]). The example given was not first-countable, and the author raised the question as to whether a first-countable example, or even a Moore space example, could be found. In particular, since caliber  $\omega_1$  is preserved by arbitrary products, it was unclear whether caliber  $(\omega_1, \omega)$  was in fact equivalent to caliber  $\omega_1$  or the **ccc** for Moore spaces.

In REED [1974c], the author noted that his Moore space machine over a first countable  $T_3$ -space produced a separable (respectively, **ccc** or DCCC) Moore space if and only if the original space had the same property. Dave McIntyre, a D.Phil. student at Oxford, has shown that the same situation is true for the properties given above. Hence, it is sufficient to study only first countable  $T_3$ -spaces. All the implications between the above calibers and chain conditions have now been established for Moore spaces. Most of these implications are due to MCINTYRE [1988]. In particular, using techniques from VAN DOUWEN and REED [19 $\infty$ ], McIntyre has shown that a Moore space over a Souslin line has caliber  $(\omega_1, \omega)$  but its square does not.



An example of a Moore space in **ZFC** with caliber  $(\omega_1, \omega)$  but not caliber  $\omega_1$  is given in REED and MCINTYRE [19 $\infty$ ]. McIntyre has however now shown that the square of this space does have caliber  $(\omega_1, \omega)$ . In fact, he defines a space  $X$  to have property  $K_\omega$  if, for every family  $\{U_\alpha : \alpha \in \omega_1\}$  of non-empty open sets, there is an uncountable subset  $\Lambda$  of  $\omega_1$  such that for each  $\Gamma \subseteq \Lambda$  with cardinality  $|\Gamma| = \omega$ ,  $\bigcap_{\alpha \in \Gamma} U_\alpha \neq \emptyset$ . He then establishes that if  $X$  has property  $K_\omega$  and  $Y$  has caliber  $(\omega_1, \omega)$ , then  $X \times Y$  has caliber  $(\omega_1, \omega)$ . The above Moore space example is shown to have property  $K_\omega$ .

Hence, the original question remains open in **ZFC**.

#### 4. The collectionwise Hausdorff property

Although somewhat a red herring enroute to a solution of the normal Moore space conjecture, this property has proved interestingly difficult to investigate and important in its set-theoretic implications.

- ? 307. **Problem 4.1.** *Does there exist a  $\sigma$ -discrete, collectionwise Hausdorff non-normal Moore space in **ZFC**?*

The question of whether there exists a CWH non-normal Moore space was first raised by R. L. Moore to his students. The existence of such a space was announced by John Worrell in 1964, but never published in the literature (although it apparently did appear in his thesis at the University of Texas). The author presented a simple first countable  $T_3$ -space example at a January, 1975 AMS meeting (REED [1975b]). Moore space examples were subsequently given under **MA** by ALSTER and POL [1975], and finally in **ZFC** by WAGE in [1976]. The above question involving  $\sigma$ -discreteness reduces the problem to its essence; it was often raised by the author in the early 1970's. In VAN DOUWEN and WAGE [1979], it was given a positive answer under  $\mathfrak{p} = \mathfrak{c}$ . The **ZFC** case remains open.

- ? 308. **Problem 4.2.** *Does there exist a collectionwise Hausdorff Moore space in **ZFC** which is not collectionwise normal w.r.t. compact sets?*

A variety of such spaces were given by the author in REED [1983] under **CH**, **MA**, and other assumptions. However the existence of such a space in **ZFC** remains open.

#### 5. Embeddings and subspaces

- ? 309. **Problem 5.1.** *Can each separable, complete Moore space be embedded in a DFCC Moore space?*

The DFCC is equivalent to Moore-closed in Moore spaces, and, as mentioned earlier, is equivalent to pseudocompact for completely regular Moore spaces. Each DFCC Moore space is complete and separable (GREEN [1974] and REED [1974a]). This question is the remaining open question from the list in REED [1974a].

**Problem 5.2.** *Can each locally compact, separable Moore space be densely embedded in a pseudocompact Moore space?* **310. ?**

Each locally compact, separable Moore space can be embedded in a locally compact, pseudocompact Moore space, and each such space which is zero-dimensional at each point of a countable dense subset can be densely embedded REED [1976]. There exists a pseudocompact Moore space which contains a copy of all metric spaces of weight  $\leq \mathfrak{c}$  (VAN DOUWEN and REED [19 $\infty$ ]).

This question was also raised in STEPHENSON [1987], where much useful information can be found. A related question from STEPHENSON [1987] is the following: Can each locally DFCC, separable Moore space be densely embedded in a DFCC Moore space?

**Problem 5.3.** *Does there exist in **ZFC** a first countable, perfect  $T_3$ -space with no dense Moore subset?* **311. ?**

From REED [1972b], this is equivalent to asking for such a space with no dense  $\sigma$ -discrete subset. Any first countable L-space is a counterexample. It is also not known in general in **ZFC** if there exist perfect, linearly ordered  $T_3$ -spaces or even simply perfect  $T_3$ -spaces with no dense  $\sigma$ -discrete subsets. These latter two questions have been independently raised by Dave Lutzer and Bob Stephenson, respectively.

In REED [1975a], the author gives an example in **ZFC** of a paracompact first countable  $T_3$ -space which contains no dense Moore subset.

**Problem 5.4.** *Under **MA** +  $\neg\mathbf{CH}$ , does there exist a Moore space which is not the union of fewer than  $\mathfrak{c}$  metrizable subsets?* **312. ?**

This question is from VAN DOUWEN, LUTZER, PELANT and REED [1980], where it was shown that

- (1) each Moore space is the union of  $\mathfrak{c}$  closed metrizable subsets,
- (2) there is a separable Moore space which is the union of  $\omega_1$  closed metrizable subsets but which is not the union of fewer than  $\omega_1$  metrizable subsets, and
- (3) under **MA** +  $\neg\mathbf{CH}$ , there exists a Moore space which is not the union of fewer than  $\mathfrak{c}$  closed metrizable subsets.

## 6. The point-countable base problem for Moore spaces

In COLLINS, REED and ROSCOE [19 $\infty$ , this volume], the authors discuss an open problem about the characterization of  $T_1$ -spaces having a point-countable base. As indicated, it is known that the property in question in that paper suffices in the class of Moore spaces. There is however an older problem in Moore spaces which is similar in many respects.

A base  $\mathcal{B}$  for a space  $X$  is said to be *uniform* (respectively, *weakly uniform*) if for each  $p \in X$  and each infinite subcollection  $\mathcal{H}$  of  $\mathcal{B}$ , each member of which contains  $p$ ,  $\mathcal{H}$  is a local base for  $p$  (respectively,  $\cap \mathcal{H} = \{p\}$ ). A  $T_3$ -space has a uniform base if and only if it is a metacompact Moore space (ALEKSANDROV [1960] and HEATH [1964]). Weakly uniform bases were defined by HEATH and LINDGREN in [1976].

In DAVIS, REED and WAGE [1976], the authors gave an example in **ZFC** of a non-metacompact Moore space with a weakly uniform base. They also showed that a Moore space having a weakly uniform base and with density  $\leq \omega_1$  (in fact, with no more than  $\omega_1$  isolated points) has a point-countable base. However, under **MA** +  $\omega_2 < 2^\omega$ , they constructed a normal Moore space with a weakly uniform base but without a point-countable base. The following remains an open and interesting question:

- ? 313. **Problem 6.1.** *Is it consistent with **ZFC** that each Moore space with a weakly uniform base has a point countable base?*

A more general issue concerns the existence in **ZFC** of *any* Moore space with no point-countable base yet in which each subspace of cardinality  $\leq \omega_1$  has a point-countable base. It is possible that under a supercompact reflection, no such Moore space exists (see TALL [1988]).

## 7. Metrization

We close with two metrization questions. One is quite natural; the other is a bit technical.

- ? 314. **Problem 7.1.** *Is each locally compact, locally connected pseudocompact Moore space metrizable?*

In particular, one might ask if each pseudocompact Moore manifold (i.e., locally Euclidian) is metrizable. The existence of locally compact, pseudocompact non-compact Moore spaces such as the space constructed by ZIPPIN in [1934] or the space  $\Psi$  in GILLMAN and JERISON [1976] is a major feature of Moore space pathology. A locally compact, locally connected Moore space containing  $\Psi$  as a subset is given in COOK [1970]. It is easy to see that a pseudocompact Moore space is metrizable if it is pseudonormal or if it has a regular

$G_\delta$ -diagonal. A related question concerns the existence of a locally compact, locally connected pseudonormal Moore space which is not metrizable.

**Problem 7.2.** *Is it consistent with **ZFC** that each (star-refining)-paracompact Moore space is metrizable?* **315. ?**

We now know that it is independent of and consistent with **ZFC** that countably paracompact separable Moore spaces are metrizable (WAGE, FLEISSNER and REED [1976]). The intended question here is to find the right “metrization” property which is implied by (countably paracompact + separability) and which can be proved both independent of and consistent with **ZFC** for Moore spaces in general, without the use of large cardinals.

In an attempt to find such a property, the author defined the concept of (star-refining)-paracompact (sr-paracompact) spaces in REED [1980]. [Note the original name was starcompact, however it is changed here to avoid confusion with the starcompact definitions in section 3.] A space  $X$  is said to be *sr-paracompact* provided if  $\mathcal{U}$  is a collection of open sets covering  $X$ , then there exists a locally finite collection  $\mathcal{F}$  of open sets covering  $X$  and refining  $\{st(x, \mathcal{U}) : x \in X\}$ .

Observe that a countably paracompact, separable space is clearly sr-paracompact. Hence under **MA** +  $\neg$ **CH**, there exist sr-paracompact Moore spaces which are not normal (REED [1980]) and normal, sr-paracompact Moore spaces which are not metrizable (TALL [1969]). In addition, it is easily seen that metacompact, sr-paracompact spaces are paracompact. Thus, under **MA** +  $\neg$ **CH**, there also exist normal Moore spaces which are not sr-paracompact (e.g., normal, metacompact, nonmetrizable Moore spaces such as Heath’s “V-space” over a  $Q$ -set).

The author had earlier considered *sr-screenable* and *sr-strongly screenable* spaces with locally finite being replaced by  $\sigma$ -disjoint and  $\sigma$ -discrete, respectively (REED [1971b]). Note that any separable space is sr-strongly screenable, whereas the tangent disc space over the real line is clearly (use a second category argument) not sr-paracompact. Hence, there exist sr-strongly screenable Moore spaces which are not sr-paracompact. The author does not know if each sr-paracompact Moore space is sr-strongly screenable. However, in [1978] PRZYMUSIŃSKI has given an example of a Moore space with a locally finite open cover which can not be refined by a  $\sigma$ -discrete open cover. Clearly, for normal Moore spaces, all three star-refining properties are equivalent.

Finally, it is also an interesting question whether it is consistent without large cardinals (perhaps, under  $V = L$ ) that normal Moore spaces are sr-paracompact. As observed by MIRIAM BROD in [1987], each sr-screenable Moore space has a  $\sigma$ -disjoint separating open cover, hence each normal, sr-paracompact Moore space is submetrizable.

## 8. Recent solutions

Two questions which the author had planned to include in this article have recently been answered.

**8.1. THEOREM.** *It is consistent with **ZFC** that there exists a  $\Delta$ -set which is not a  $Q$ -set.*  $\square$

The concept of a  $\Delta$ -set was originally defined by the author in a lecture at the 1975 Memphis State Topology Conference as a characterization of those subsets of the real line over which a tangent disc space would be countably paracompact. He raised the question of whether it was consistent with **ZFC** to have a  $\Delta$ -set which was not a  $Q$ -set, thereby producing a countably paracompact, non-normal tangent disc space.

A  $\Delta$ -set is an uncountable subset  $D$  of the real line such that for each non-increasing sequence  $\{H_n\}$  of subsets of  $D$  with empty intersection, there exists a sequence  $\{V_n\}$  of  $G_\delta$ -sets w.r.t.  $D$  with empty intersection such that for each  $n$ ,  $H_n \subseteq V_n$ .

Eric van Douwen observed that the term ' $G_\delta$ -sets' in the definition of a  $\Delta$ -set could be simply replaced by 'open sets'. Furthermore, in [1977] TEODOR PRZYMUSIŃSKI later proved that there exists a  $\Delta$ -set if and only if there exists a countably paracompact non-metrizable Moore space.

In [1989] ROBIN KNIGHT has now shown the consistency with **ZFC** of the existence of a  $\Delta$ -set which is not a  $Q$ -set. The remaining question is whether this situation is also consistent with  $2^\omega < 2^{\omega_1}$ . If true, this would show that **CH** could not be replaced by  $2^\omega < 2^{\omega_1}$  in FLEISSNER's theorem from [1978] that under **CH**, each countably paracompact, separable Moore space is metrizable.

**8.2. THEOREM.** *There is an open-compact mapping from a Moore space to  $\omega_1$ .*  $\square$

A. V. Arkhangel'skiĭ asked the author if a countably compact  $T_3$ -space must be metrizable if it were the open-compact image of a  $T_3$  space with a  $G_\delta$ -diagonal. Recall that in [1973] NAGAMI proved that a compact  $T_3$ -space must be metrizable if it were the open-compact image of a Moore space, and Chaber later obtained the same result for open compact images of  $T_3$ -spaces with a  $G_\delta$ -diagonal.

The author has now answered Arkhangel'skiĭ's question in the negative.

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## CHAPTER 10

### **Some Conjectures**

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A problem in Set Theoretic Topology which concerns us all, is, which problems are important? There is no rational answer as the existence of this volume proves. “Simple to state and difficult to solve” is part of the answer, although one can easily think of counterexamples to either of these requirements. “Either the theorem or the proof must have wide applications” or “the question has been a stumbling block for many over many years” are other aspects of an answer. Unlike, say Algebraic Topology, Abstract-Space-General-Topology problems seem to me to be all over the map, often with few recognizable relationships to each other, and very few hints as to which problems are important. I have to work on a problem for a while to have any feeling for its worth.

One looks for familiar pathologies and familiar set theoretic substructures. The combinatorics may be finite and still very hard; one prefers for the combinatorics to be decidable in Zermelo-Fraenkel Set Theory together with the Axiom of Choice (**ZFC**) but one must be aware that this may not be the case. The problems I list here have mostly been listed in RUDIN [1988]; I have worked on all of them long enough to have some real respect for them; they have all stood the test of time and labor by various people; and I feel any solution to any of them would have applications at least to other problems exhibiting similar pathologies. All spaces are assumed to be Hausdorff and regular; actually most are assumed to be normal but I will always mention this fact since the theme of these conjectures will be, how much strength can you build into spaces by assuming normality?

**I.** I have spent most of my time for the last several years working on problems involving normal nonmetrizable manifolds.

A *manifold* here is just a locally Euclidean Hausdorff space as opposed to a metric one as in the usual definition. Our concern is usually metrizability so we also assume that the manifold is connected.

I have shown in RUDIN [1990] that  $\diamond^+$  implies there is a normal not collectionwise normal (but collectionwise Hausdorff) manifold. TALL [1988] and BALOGH [1980] have shown that it is consistent with **ZFC** that all normal manifolds are collectionwise normal (and thus collectionwise Hausdorff.) It would be very nice to prove:

**Conjecture 1.** *Every normal manifold is collectionwise Hausdorff.*

**316. ?**

I believe strongly that (1) is true, but I have no idea how to begin a proof. Proving (2) would give an example that would amaze me.

**Conjecture 2.** *There is a normal, nonmetrizable manifold with a countable, point separating, open cover.*

**317. ?**

That is, for every  $p \neq q$  in the manifold, there is a member  $U$  of the open family with  $p \in U$  and  $q \notin U$ . If one instead assumes Hausdorff type

separation, the resulting manifold is easily proved to be metrizable. I have shown in RUDIN [1989] that the Continuum Hypothesis implies (2); and the manifold constructed is also a Dowker space. BALOGH [1983] has proved that Martin's Axiom implies that a manifold satisfying (2) must have Lindelöf number  $= \mathfrak{c}$ . I have given an example of a normal separable, nonmetrizable manifold in RUDIN [1990]. I fear this conjecture is false, but I would like for it to be true.

**II.** The normal, not collectionwise normal pathology is a basic one. A proof of Dowker's set theory conjecture would not yield a manifold but would give an especially elegant example of a 1-dimensional, connected, simplicial complex, with the stars of all the vertices open, which is normal but not collectionwise normal.

? **318. Conjecture 3.** (DOWKER [1952]) *There is a set  $S$  and a filter  $\mathcal{F}$  on  $S$  such that*

- (a) *If  $F: S \rightarrow \mathcal{F}$ , then there is  $x \neq y$  in  $S$  with  $y \in F(x)$  and  $x \in F(y)$ , but*
- (b) *If  $X \subset S$ , there is an  $F: S \rightarrow \mathcal{F}$  such that, for all  $x \in X$  and  $y \in (S - X)$ , either  $y \notin F(x)$  or  $x \notin F(y)$ .*

DOWKER proves in [1952] that any example proving (3) must have  $|S| \geq \omega_2$ . I prove in RUDIN [1984] that one can choose  $S = \mathbb{R}$  and define a separate filter  $\mathcal{F}_x$  for each  $x \in S$ , and then, requiring that  $F(x) \in \mathcal{F}_x$  and  $F(y) \in \mathcal{F}_y$ , satisfy (a) and (b). This construction yields a simplicial complex with the desired properties and it enabled me to construct the normal, not collectionwise normal, manifold in RUDIN [1990]. However, after forty years we essentially know only one way to construct a normal, not collectionwise normal, space: use BING'S  $G$  from [1951]; my construction uses Bing's  $G$ . However to prove Dowker's conjecture, a problem of the same era as Bing's  $G$ , we need a different technique. A different technique would be very interesting; and it is this fact which makes me push for a solution to this rather esoteric sounding question.

**III.** Speaking of Dowker brings us to consider the Dowker space pathology.

A space is said to be a *Dowker space* if it is normal but not countably paracompact; equivalently (DOWKER [1951]), it is normal but its product with an ordinary closed unit interval (or with any nondiscrete compact metric space) is not normal. This pathology occurs in many classes of abstract spaces.

We know of the existence of only one such space (RUDIN [1971]). It has few nice properties; its cardinality and weight are  $(\aleph_\omega)^\omega$ . On the other hand, by making various special set theoretic assumptions, we can prove the existence of Dowker spaces having a remarkable variety of additional properties, many of them with all of their cardinal functions small. I have mentioned that

the Continuum Hypothesis implies the existence of a Dowker manifold with a countable point separating open cover (RUDIN [1989]); the character of this space is, of course, countable while its weight and cardinality are  $\omega_1$ .

**Conjecture 4.** *There is a Dowker space of cardinality  $\omega_1$ .*

**319. ?**

This particular conjecture is picked out of a hat. I would be equally happy to see a Dowker space of cardinality  $\mathfrak{c}$ ; or one could replace *cardinality* in the conjecture by some other cardinal function; or one could ask for a Dowker square of two countably paracompact spaces (BEŠLAGIČ [19 $\infty$ ] has recently constructed such a Dowker space assuming the Continuum Hypothesis.) What I want is *another* Dowker space. After 20 years it is time. In any problem where Dowker spaces are required, we can only give a consistency example unless our one peculiar example can be modified to give an answer. It seems a ridiculous state of affairs. Of course part of the problem may be that we also have few models for set theory in which we know that some class of Dowker spaces does *not* exist.

A Dowker space problem on which I, personally, have spent too much time is:

**Conjecture 5.** *Every normal space with a  $\sigma$ -disjoint base is paracompact.*

**320. ?**

This would be a beautiful theorem and I think it is true. If there is a counterexample, it must be a Dowker space (NAGAMI [1955]). In RUDIN [1983] I give an example of a screenable Dowker space constructed assuming  $\diamond^{++}$ ; the construction is messy and falls short of having a  $\sigma$ -disjoint base; it gives me reason to hope that (5) is correct.

**IV.** I consider conjecture (5) to be a generalized metric problem: after all, a  $\sigma$ -locally finite base guarantees that a (regular) space is metrizable. There are many problems in Set Theoretic Topology which ask about pathologies that can perhaps occur in nonmetrizable spaces which imitate metrizability in some way. Another old problem of this type in which I have only recently become interested is:

**Conjecture 6.**  *$M_3$  spaces are  $M_1$ .*

**321. ?**

In [1961] CEDER defined  $M_1$ ,  $M_2$ , and  $M_3$  spaces and proved that  $M_1 \Rightarrow M_2 \Rightarrow M_3$ . GRUENHAGE [1976] and JUNNILA [1978] proved that  $M_3 = M_2$ . But whether  $M_3 \Rightarrow M_1$  or not is still unknown; and it would be a very nice theorem if true.

A *pseudobase* for a space  $X$  is a family  $\mathcal{B}$  of subsets of  $X$  such that, for every  $x \in X$  and neighborhood  $U$  of  $x$ , there is a  $B \in \mathcal{B}$  with  $x \in (\text{interior } B) \subset B \subset U$ . A space is  $M_3$  provided it has a  $\sigma$ -closure preserving closed pseudobase, and

$M_1$  provided it has a  $\sigma$ -closure preserving open base. GRUENHAGE [1980] has given an excellent survey of the partial results on this problem. Conjecture (6) seems much less set theoretic to me than (5) and I would, therefore, expect a real theorem or counterexample in this case (while I fear an undecidability result for (5)).

Another class of generalized metric spaces for which there is an oddly difficult conjecture is what I call Collins spaces. One should be aware that Mike Reed and Bill Roscoe are at least as involved with these spaces as Peter Collins; but my contact has always been through Collins and so this is my name for them.

A *Collins space* is one in which each point  $x$  has a special countable open base  $\mathcal{B}_x$  with the property that, if  $U$  is a neighborhood of a point  $y$  there is a neighborhood  $V$  of  $y$  such that, for all  $x \in V$  there is a  $B \in \mathcal{B}_x$  with  $y \in B \subset U$ . A rather easy theorem from COLLINS, REED, ROSCOE and RUDIN [1985] shows that a space is metrizable if and only if  $\mathcal{B}_x$ 's can be chosen for each  $x$  in such a way that the terms of each  $\mathcal{B}_x$  form a nested, decreasing, sequence.

? 322. **Conjecture 7.** *Every Collins space has a point-countable base.*

Trivially every space with a point-countable base is a Collins space; and every Collins space with a dense subset of cardinality  $\leq \omega_1$  has a point-countable base. One can prove a number of other very easy theorems like: every open cover of a Collins space has a point-countable refinement. However the actual conjecture is hard to pin down. In truth I feel there must be a counterexample to (7).

In her thesis of [1981], CARYN NAVY constructed a variety of very pretty normal, para-Lindelöf but not paracompact spaces. All of Navy's spaces were countably paracompact and not collectionwise normal; one has to wonder if this was an accident. The techniques used by Navy led to counterexamples for several important metrization problems in which noncollectionwise normality was the required pathology, see RUDIN [1983] and FLEISSNER [1982].

? 323. **Conjecture 8.** *Every normal, nonparacompact para-Lindelöf space fails to be collectionwise normal.*

Normal para-Lindelöf seems like a nice property to me and much weaker than paracompactness. No one has really explored the difference and, having been useful once, it seems to me it might be useful again. This is perhaps the most esoterically vague of my conjectures, but it is a pet idea.

**V.** Normality in products (and the related shrinking of open covers) is perhaps more important to me than to others for it has been a life-long theme in my mathematics.

An open cover  $\mathcal{V} = \{V_\alpha \mid \alpha < \kappa\}$  is a (closed) shrinking of an open cover  $\{U_\alpha \mid \alpha < \kappa\}$  if  $\overline{V_\alpha} \subset U_\alpha$  for all  $\alpha$ .

A space is normal if every finite open cover has a shrinking. A space is Dowker if  $\omega$  is the minimal cardinality of an open cover without a shrinking. We define a space to be a  $\kappa$ -Dowker space (RUDIN [1985]) if  $\kappa$  is the minimal cardinality of an open cover which has no shrinking.

If  $\kappa$  is the minimal cardinality of a nondiscrete subset of a space  $X$ , then the product of  $X$  with any  $\kappa$ -Dowker space is not normal.

We know exactly one (real)  $\kappa$ -Dowker space for each infinite cardinal  $\kappa$ , see RUDIN [1985]. Under the assumption  $\diamond^{++}$  we know exactly one normal space all of whose monotone open covers have a shrinking having an open cover with no shrinking, see BEŠLAGIČ and RUDIN [1985].

Just as one example of a Dowker space does not suffice, one example of a  $\kappa$ -Dowker space for uncountable  $\kappa$  does not suffice to give us much insight into the possible pathologies and theorems in this area. In the case of uncountable  $\kappa$  we even have a serious shortage of consistency results. This is a plea, not for the proof of one conjecture, but for a cottage industry into our understanding of normality in products.

I present a somewhat tongue-in-cheek proof that normality in products is important:

(T): TAMANO's theorem [1960] that  $X$  is paracompact if and only if  $X \times Y$  is normal for all compact  $Y$ .

(D): DOWKER's theorem [1951] that  $X$  is normal and countably paracompact if and only if  $X \times Y$  is normal for all compact metric  $Y$ .

**Morita's Conjectures** (MORITA [1975]):

**M(a)**:  $X$  is discrete if and only if  $X \times Y$  is normal for all normal  $Y$ .

**M(b)**:  $X$  is metric if and only if  $X \times Y$  is normal for all  $Y$  such that  $Y \times M$  is normal for all metric  $M$ .

**M(c)**:  $X$  is metric and  $\sigma$ -locally compact if and only if  $X \times Y$  is normal for all normal countably paracompact  $Y$ .

The existence of  $\kappa$ -Dowker spaces for all  $\kappa$ , gives us **M(a)**. In [1975] MORITA proves that **M(c)** follows from **M(b)**. In CHIBA, PRZYMUSIŃSKI and RUDIN [1986] it is shown that **M(b)** holds if and only if:

**Conjecture 9.** *There is an uncountable monotone open cover without a shrinking of a space  $X$  such that  $X \times M$  is normal for all metric  $M$ .* **324. ?**

In BEŠLAGIČ and RUDIN [1985] it is shown that  $\diamond$  implies (9).

If one wants a product to be normal, one needs a lot of structure on at least one of the spaces. Structural problems are still difficult even when one requires one factor to be metric or compact, for instance. Some elementary sounding problems which Bešlagič points out are difficult (and unsolved) include:

(A) *Given  $X \times Y$  normal,  $Y$  compact, and every open cover of  $X$  has a* **325. ?**



*shrinking. Does every open cover of  $X \times Y$  have a shrinking?*

- ? **326.** (B) *Given every open cover of  $X$  has a shrinking, and  $Y$  is a normal, perfect preimage of  $X$ . Does every open cover of  $Y$  have a shrinking?*
- ? **327.** (C) *Given  $X$  is normal and  $Y$  is a collectionwise normal perfect image of  $X$ . Is  $X$  collectionwise normal?*

**VI.** The Lindelöf property, like normality, is a delicate property; spaces with strange open covers are hard to construct and they misbehave in products. A conjecture I worked on 25 years ago is:

- ? **328.** **13. The Linearly-Lindelöf conjecture** (HOWES [1970] and MIŠČENKO [1962]): *There is a normal, non-Lindelöf space, every monotone open cover of which has a countable subcover.*

Interestingly enough there are still no significant partial results.

Another beautiful “Lindelöf” problem is:

- ? **329.** **14. Michael’s Conjecture** (MICHAEL [1971]): *There is a Michael space.*

A *Michael space* is a Lindelöf space whose product with the irrationals is not Lindelöf (or equivalently not normal). The space itself being regular and Lindelöf is, of course, normal.

In [1971] MICHAEL gave an example, assuming the Continuum Hypothesis, of a Michael space. More recently ALSTER [19 $\infty$ ] has constructed an example from Martin’s Axiom. A new paper by LAWRENCE [19 $\infty$ b] shows, among other things, that there is a concentrated-Michael-space if and only if  $\mathfrak{b} = \omega_1$ . A space  $X$  is *concentrated* on a subset  $A$  if; for every open  $U \supset A$ ,  $X - U$  is countable. A *concentrated-Michael-space* is one concentrated on a closed subset  $A$  for which  $A \times (\text{irrationals})$  is normal. Michael’s example is concentrated; but since  $\omega_1 < \mathfrak{b}$  is consistent with Martin’s Axiom, Alster’s example is not concentrated. The number  $\mathfrak{b}$  is the minimal cardinality of an  $\leq^*$  unbounded family in  ${}^\omega\omega$ . If the Continuum Hypothesis is true  $\mathfrak{b} = \omega_1$ ; under Martin’s Axiom,  $\mathfrak{b} = \mathfrak{c}$ .

The most important unsolved Lindelöf problem is surely:

- ? **330.** **Conjecture 15.** *There is an  $L$ -space.*

An  $L$ -space is a hereditarily Lindelöf (regular) space which is not separable. The best reading on the subject is TODORČEVIĆ’S new book [1989]; a complete biography on the problem would take a small book. After much effort by many, Todorčević has shown that the existence of  $S$ -spaces (hereditarily

separable non-Lindelöf) is undecidable in **ZFC**. It has long been known to be consistent with **ZFC** that there be both  $S$  and  $L$  spaces. Both pathologies frequently occur in abstract space problems.

**VII.** Box products are by their nature pathological, but they have been and hard enough to please anyone. What one would like to know is which box products are normal and which are paracompact. I recommend VAN DOUWEN [1980] and WILLIAMS [1984] for general information on the subject and earlier references.

Early on it was proved by Kunen and me that the Continuum Hypothesis implies that every box product of countably many locally compact, separable, metric spaces (or countably many compact ordinals) is paracompact. Kunen and van Douwen showed that for some countably many compact spaces of character  $> \omega_1$ , the box product need not be normal. Then van Douwen showed that no nontrivial box product of metric spaces is normal if one factor is the irrationals. LAWRENCE [19 $\infty$ a] more recently showed that  $\mathfrak{b} = \mathfrak{d}$  or  $\mathfrak{c} = \mathfrak{d}$  implies that every countable box product of countable metric spaces is paracompact. (The same is true for locally compact metric spaces.)  $\mathfrak{d}$  is the minimal cardinality of a  $\leq^*$  dominant (cofinal) family in  ${}^\omega\omega$ . The basic problems remain:

**Conjecture 16.** *Every box product of  $\omega_1$  copies of  $(\omega + 1)$  is normal (or paracompact).* **331. ?**

**Conjecture 17.** *Every box product of  $\omega$  copies of  $(\omega + 1)$  is normal (or paracompact).* **332. ?**

One peculiarity is that normality and paracompactness in box products seem hard to separate. The other is that we know almost nothing, even of a consistency nature, about uncountable box products.

We have only touched the surface in this area.

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## CHAPTER 11

### Small uncountable cardinals and topology

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## 1. Definitions and set-theoretic problems

Let  $\omega$  denote the set of natural numbers,  $[\omega]^\omega$  the set of all infinite subsets of  $\omega$  and  ${}^\omega\omega$  the set of all functions from  $\omega$  into  $\omega$ . We give a brief discussion of some problems which involve cardinal numbers defined from various properties on these and related sets such as the real and the irrational numbers. For background on small cardinals, we refer to the article by Eric van Douwen, VAN DOUWEN [1984]. In that article, van Douwen defined eight small cardinals, and studied six of them in detail. We take these eight cardinals as our starting point, and begin with some preliminary definitions.

Two countable, infinite sets are *almost disjoint* provided their intersection is finite. A family of pairwise almost disjoint subsets of a set  $X$  is *maximal* provided it is not properly contained in another pairwise almost disjoint family of subsets of  $X$ . Let  $\mathcal{P}(X)$  denote the power set of  $X$ ,  $[X]^\omega$  the set of all countably infinite subsets of  $X$ , and  $[X]^{<\omega}$  the set of all finite subsets of  $X$ .

For  $A, B$  in  $[\omega]^\omega$ , we say  $A$  is *almost included* in  $B$  (denoted  $A \subset^* B$ ) provided  $A - B$  is finite.

For a family  $\mathcal{F} \subset [\omega]^\omega$ , we say that  $\mathcal{F}$  has the *strong finite intersection property* provided every finite subfamily has an infinite intersection, and an infinite set  $A$  is called a *pseudointersection* of  $\mathcal{F}$  provided  $A \subset^* F$  for all  $F \in \mathcal{F}$ .

A family  $\mathcal{T} \subset [\omega]^\omega$  is a (decreasing) *tower* provided there exist an ordinal  $\alpha$  and a bijection  $f: \alpha \rightarrow \mathcal{T}$  such that  $\beta < \gamma < \alpha$  implies that  $f(\gamma) \subset^* f(\beta)$ , and no infinite set  $A$  is a pseudointersection of  $\mathcal{T}$ .

A family  $\mathcal{U} \subset [\omega]^\omega$  *generates an ultrafilter* (or is an *ultrafilter base*) provided every finite intersection of elements of  $\mathcal{U}$  contains an element of  $\mathcal{U}$ , and for every  $A \in [\omega]^\omega$  either there exists  $U \in \mathcal{U}$  such that  $U \subset A$ , or there exists  $U \in \mathcal{U}$  such that  $U \subset \omega - A$ . An ultrafilter base  $\mathcal{U}$  is called *free* (or *non-principal*) provided  $\bigcap \mathcal{U} = \emptyset$ .

A family  $\mathcal{I} \subset [\omega]^\omega$  is an *independent family* provided for every  $A, B \in [T]^{<\omega}$ , if  $A \neq \emptyset$  and  $A \cap B = \emptyset$  then  $\bigcap A - \bigcup B \neq \emptyset$ .

A family  $\mathcal{S} \subset [\omega]^\omega$  is a *splitting family* provided for every  $A \in [\omega]^\omega$  there exists  $S \in \mathcal{S}$  such that  $|A \cap S| = |A - S| = \omega$ .

We define the *mod finite order*  $\leq^*$  (a reflexive transitive order) on the set  ${}^\omega\omega$  as follows: for  $f, g \in {}^\omega\omega$  we say  $f \leq^* g$  provided there exists  $N \in \omega$  such that for all  $n \geq N$ ,  $f(n) \leq g(n)$ .

A set  $X \subset {}^\omega\omega$  is *dominating* (in the mod finite order) provided for every  $f \in {}^\omega\omega$  there exists  $g \in X$  such that  $f \leq^* g$ , and  $X$  is *bounded* (in the mod finite order) if there exists  $g \in {}^\omega\omega$  such that  $f \leq^* g$  for all  $f \in X$ .

We denote the *cardinality of the continuum* by  $\mathfrak{c}$ . Since German type is often used for this cardinal, van Douwen and I used this same convention in the mnemonic notation for the following cardinals which have been studied under a variety of names (see HECHLER [1972], VAN DOUWEN [1984, p.123], and VAUGHAN [1979b]). When German type is not available, we use ordinary

letters for these cardinals (including the cardinality of the continuum). The mnemonic is derived from the key concept in the definition of a cardinal. The cardinal  $\mathfrak{b}$  is called the (un)b $\mathfrak{b}$ ounding number,  $\mathfrak{d}$  is the  $\mathfrak{d}$ ominating number,  $\mathfrak{s}$  is the  $\mathfrak{s}$ plitting number,  $\mathfrak{p}$  is derived from the notion of  $\mathfrak{P}$ -points,  $\mathfrak{t}$  is the tower number,  $\mathfrak{i}$  is the independent family number, and  $\mathfrak{u}$  the ultrafilter character number. The cardinal  $\mathfrak{a}$  can be called the  $\mathfrak{a}$ lmost disjointness number (unfortunately, mnemonic does not necessarily imply euphonic).

$\mathfrak{a} = \min\{|A| : A \subset [\omega]^\omega \text{ is an infinite, maximal almost disjoint family in } \omega\}.$

$\mathfrak{b} = \min\{|B| : B \subset {}^\omega\omega \text{ is unbounded in the mod finite order}\}.$

$\mathfrak{d} = \min\{|D| : D \subset {}^\omega\omega \text{ is dominating in the mod finite order}\}.$

$\mathfrak{s} = \min\{|S| : S \subset [\omega]^\omega \text{ is a splitting family on } \omega\}.$

$\mathfrak{p} = \min\{|P| : P \subset [\omega]^\omega \text{ has the strong finite intersection property but no } X \in [\omega]^\omega \text{ is a pseudointersection for } P\}.$

$\mathfrak{t} = \min\{|T| : T \subset [\omega]^\omega \text{ is a tower on } \omega\}.$

$\mathfrak{i} = \min\{|I| : I \subset [\omega]^\omega \text{ is a maximal independent family on } \omega\}.$

$\mathfrak{u} = \min\{|U| : U \subset [\omega]^\omega \text{ is a base for an ultrafilter on } \omega\}.$

Diagram 1 below, the shape of which is based on a similar diagram of BLASS [1989], is intended to display the basic relations among these cardinals. A line connecting two cardinals indicates that the cardinal lower on the diagram is less than or equal to the cardinal higher on the diagram (in **ZFC**). It would be nice if the line also indicated that it is consistent that the two cardinals are different. For example, Rothberger proved  $\mathfrak{t} \leq \mathfrak{b}$ , and there is a model where  $\mathfrak{t} < \mathfrak{b}$  (see VAN DOUWEN [1984, 3.1 and 5.3]). This aspect of the diagram, however, is not completely settled (see 1.1).

The proofs of the results implied by Diagram 1, or references to them, can be found in VAN DOUWEN [1984], except for the recent result of SHELAH [1990], who proved in **ZFC** that  $\mathfrak{d} \leq \mathfrak{i}$  (he also mentions a model in which the inequality “ $\mathfrak{d} < \mathfrak{i}$ ” holds). These results of Shelah are included with his permission as an appendix to this paper.

? **333. Problem 1.1.** *Are any of the following inequalities consistent with **ZFC**?*

- (a)  $\mathfrak{p} < \mathfrak{t}$
- (b)  $\mathfrak{d} < \mathfrak{a}$
- (c)  $\mathfrak{i} < \mathfrak{a}$
- (d)  $\mathfrak{u} < \mathfrak{a}$
- (e)  $\mathfrak{i} < \mathfrak{u}$

The first two of these inequalities are of interest because they concern four of the six cardinals studied by VAN DOUWEN in [1984]. We believe that (a) is the most interesting. Concerning (a): Rothberger proved that  $\mathfrak{p} > \omega_1$  if and

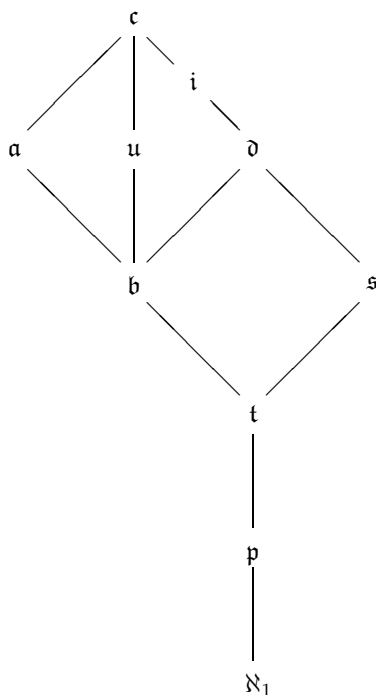


Diagram 1.

only if  $\mathfrak{t} > \omega_1$  (FREMLIN [1984, 14D]), and Szymański proved that  $\mathfrak{p}$  is regular (VAN DOUWEN [1984, 3.1(e)]) and that  $\mathfrak{t}$  cannot be real-valued measurable (SZYMAŃSKI [1988]). Given 8 cardinals there are  $64 - 8 = 56$  questions of the form “is  $\kappa < \lambda$ ”. For the above 8 cardinals, the questions in 1.1 are the only ones of this form which remain open (see VAN DOUWEN [1984], BLASS and SHELAH [1987, 1989], SHELAH [1984, 1990]).

There are many small cardinals which have been studied. Some of them are probably of more interest than some of those defined above. One interesting cardinal that has been discovered in several contexts is the cardinal  $\mathfrak{h}$ , called the *distributivity number* (the reason for the letter “ $\mathfrak{h}$ ” is given below). A family  $D \subset [\omega]^\omega$  is called a *dense* family provided for every  $X \in [\omega]^\omega$  there exists  $Y \in D$  such that  $Y \subset^* X$ , and  $D$  is called an *open* family provided for every  $Y \in D$  and every  $X \in [\omega]^\omega$ , if  $X \subset^* Y$  then  $X \in D$ . The ordered set  $([\omega]^\omega, \subset^*)$  is called  $\kappa$ -*distributive* if every set of less than  $\kappa$  dense open families has non-empty intersection. The distributivity number is defined by

$$\mathfrak{h} = \min\{\mathcal{D} : \mathcal{D} \text{ is a set of dense open families in } [\omega]^\omega \text{ with } \bigcap \mathcal{D} = \emptyset\}.$$



Now  $\mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b}$ , and  $\mathfrak{h} \leq \mathfrak{s}$ ; see BALCAR, PELANT and SIMON [1980]. The letter  $\mathfrak{h}$  comes from the word “height” in the interesting result, proved by Balcar, Pelant and Simon, that in **ZFC** one can prove that there exists a tree  $\pi$ -base for  $\omega^*$ , and further

$$\mathfrak{h} = \min\{\kappa : \text{there exists a tree } \pi\text{-base for } \omega^* \text{ of height } \kappa\}$$

where, as usual,  $\omega^* = \beta\omega - \omega$  and a family  $\mathcal{B}$  of non-empty open sets is called a  $\pi$ -base for a space  $X$  provided every nonempty open set contains a member of  $\mathcal{B}$ . A *tree*  $\pi$ -base  $T$  is a  $\pi$ -base which is a tree when considered as a partially ordered set under reverse inclusion (i.e., for every  $t \in T$  the set  $\{s \in T : t \subset s\}$  is well-ordered by “ $\supset$ ”). The height of an element  $t \in T$  is the ordinal  $\alpha$  such that  $\{s \in T : t \subset s \text{ and } s \neq t\}$  is of order type  $\alpha$ , and the *height* of a tree  $T$  is the smallest ordinal  $\alpha$  such that no element of  $T$  has height  $\alpha$ .

For any topological space, the *Novák number* (BALCAR, PELANT and SIMON [1980]) (resp. *weak Novák number* (VAN MILL and WILLIAMS [1983])) of  $X$ , denoted  $n(X)$  (resp.  $wn(X)$ ), is the smallest number of nowhere dense subsets of  $X$  needed to cover  $X$  (resp. to cover a dense subset of  $X$ ).

We consider here only the case  $X = \omega^*$  and write  $\mathfrak{n} = n(\omega^*)$ .

It is easy to see that  $\omega_2 \leq \mathfrak{t}^+ \leq n(\omega^*) \leq 2^{\mathfrak{c}}$  (HECHLER [1978, 4.11]). The equality  $wn(\omega^*) = \mathfrak{h}$  was proved in NYIKOS, PELANT and SIMON [19 $\infty$ ], and gives one way to see that  $\mathfrak{h} \leq \mathfrak{n}$ . In BALCAR, PELANT and SIMON [1980] it is proved that  $\mathfrak{h} \leq \min\{\mathfrak{b}, \text{cf}(\mathfrak{c})\}$ . In Cohen’s original models of not-**CH** one has  $\mathfrak{h} = \mathfrak{b} = \aleph_1 < \aleph_2 \leq \mathfrak{n}$ . Blass pointed out to us that to get a model where  $\mathfrak{n} < \mathfrak{b}$ , start with a model of **MA** in which  $\mathfrak{c}$  is at least  $\aleph_3$ , and add  $\aleph_1$  random reals (giving  $\mathfrak{h} = \aleph_1$  and  $\mathfrak{b} \geq \aleph_3$ ), and apply BALCAR, PELANT and SIMON [1980, 3.5(i)] which says that if  $\mathfrak{h} < \mathfrak{c}$  then  $\mathfrak{n} \leq \mathfrak{h}^+$ . In [1984] SHELAH gave a model where  $\mathfrak{h} < \mathfrak{s} = \mathfrak{b}$ . Dow has determined the value of  $\mathfrak{h}$  in a number of models. For example, he has a model where  $\mathfrak{h} = \mathfrak{s} < \mathfrak{b}$  (Dow [1989]).

A family  $\mathcal{G} \subset [\omega]^\omega$  is said to be *groupwise dense* (BLASS [1989]) provided

- (a)  $\mathcal{G}$  is an open family, and
- (b) for every family  $\Pi$  of infinitely many pairwise disjoint finite subsets of  $\omega$ , the union of some (necessarily infinite) subfamily of  $\Pi$  is in  $\mathcal{G}$ .

Clearly every groupwise dense family is a dense open family. Define

$$\mathfrak{g} = \min\{|\mathcal{G}| : \mathcal{G} \text{ is a set of groupwise dense families in } [\omega]^\omega \text{ with } \bigcap \mathcal{G} = \emptyset\}.$$

Obviously  $\mathfrak{h} \leq \mathfrak{g}$  and it is known that  $\mathfrak{g} \leq \mathfrak{d}$  (BLASS [1989]). There is a model of Blass and Shelah where  $\mathfrak{u} < \mathfrak{g}$  and  $\mathfrak{h} < \mathfrak{g}$  (BLASS and LAFLAMME [1989]). Blass has proved that if  $\mathfrak{u} < \mathfrak{g}$  then  $\mathfrak{b} = \mathfrak{u}$  and  $\mathfrak{g} = \mathfrak{d} = \mathfrak{c}$  (BLASS [19 $\infty$ ]).

The next number requires no further definitions:

$$\mathfrak{a}_s = \min\{|A| : A \text{ is a maximal family of almost disjoint subsets of } \omega \times \omega \text{ that are graphs of functions from subsets of } \omega \text{ to } \omega\}$$

Balcar and Simon proved that  $\mathfrak{s} \leq \mathfrak{a}_s$ , and  $\mathfrak{a} \leq \mathfrak{a}_s \leq \mathfrak{c}$ . Shelah has a model where  $\mathfrak{a} < \mathfrak{a}_s$ , and another where  $\mathfrak{s} < \mathfrak{b} \leq \mathfrak{a}$  (SHELAH [1984]).

Let  $(P, \leq)$  be a partially ordered set (poset). A set  $D \subset P$  is said to be *dense* provided for every  $p \in P$  there exists  $d \in D$  such that  $d \leq p$ . A set  $G \subset P$  is a *filter* provided (a) for every  $g \in G$  and every  $p \in P$ , if  $g \leq p$  then  $p \in G$ , and (b) for every  $g$  and  $g' \in G$  there exists  $r \in G$  such that  $r \leq g$  and  $r \leq g'$ . The set  $(P, \leq)$  satisfies the **ccc** provided every antichain is countable (i.e., if  $A \subset P$ , and  $A$  is uncountable, then there exist distinct  $a$  and  $a' \in A$  and  $r \in P$  such that  $r \leq a$  and  $r \leq a'$ ). Let “**MA**( $\kappa$ ) for **ccc** posets” (or “**MA**( $\kappa$ )” for short) be the statement: for every **ccc** partially ordered set and every family  $\mathcal{D}$  of no more than  $\kappa$  dense subsets of  $P$ , there exists a generic filter  $G$  for  $\mathcal{D}$  (i.e.,  $G$  is a filter and  $G \cap D \neq \emptyset$  for all  $D \in \mathcal{D}$ ). Define

$$\mathfrak{m} = \min\{\kappa : \mathbf{MA}(\kappa) \text{ for } \mathbf{ccc} \text{ posets fails}\}$$

Of course, Martin’s Axiom is the statement “ $\mathfrak{m} = \mathfrak{c}$ ”.

FREMLIN [1984] has given a proper class of definitions  $\mathfrak{m}_\Phi$ , where  $\Phi$  is a class of partially ordered sets, similar to the definition of  $\mathfrak{m}$ . We mention two of these here. A poset  $P$  is called  *$\sigma$ -centered* if there exists a partition  $\{P_i : i \in \omega\}$  of  $P$  such that each  $P_i$  is *centered* (i.e., if  $p, q \in P_i$  then there exists  $r \in P_i$  such that  $r \leq p$ , and  $r \leq q$ ). Define

$$\mathfrak{m}_{\sigma\text{-centered}} = \min\{\kappa : \mathbf{MA}(\kappa) \text{ for } \sigma\text{-centered posets fails}\},$$

$$\mathfrak{m}_{\text{countable}} = \min\{\kappa : \mathbf{MA}(\kappa) \text{ for countable posets fails}\}.$$

Bell [1981] proved that  $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$ , thus  $\mathfrak{m} \leq \mathfrak{p} \leq \mathfrak{m}_{\text{countable}}$ .

Let  $L(\kappa)$  be the statement: If  $P$  is a **ccc** partially ordered set of cardinality  $\leq \kappa$ , then  $P$  is  $\sigma$ -centered (FREMLIN [1984, 41L]). Define

$$\mathfrak{l} = \min\{\kappa : L(\kappa) \text{ is false}\}.$$

In [1987] TODORČEVIĆ and VELIČKOVIĆ have proved that  $\mathfrak{m} = \mathfrak{l}$ , and as a corollary, the result of FREMLIN [1984, 41C(d)]:  $\text{cf}(\mathfrak{m}) > \omega$  (also see KUNEN [1988] and Problem 1.3).

A family  $\mathcal{B}$  is called a  *$\pi$ -base* for a free ultrafilter  $u$  on  $\omega$  provided for every  $U \in u$  there exists  $B \in \mathcal{B}$  such that  $B \subset U$ . Define

$$\pi u = \min\{|\mathcal{B}| : \mathcal{B} \subset [\omega]^\omega \text{ is a } \pi\text{-base for a free ultrafilter on } \omega\}.$$

The *refinement number* is defined by

$$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subset [\omega]^\omega \text{ for every } X \in [\omega]^\omega \text{ there exists } R \in \mathcal{R} \text{ such that } R \subset^* X \text{ or } R \subset^* \omega - X\}.$$

PRICE [1982] was the first to (implicitly) consider this cardinal, and it was discovered independently by VOJTÁŠ [19 $\infty$ ], J. Cichoń, and BEŠLAGIĆ and VAN DOUWEN [19 $\infty$ ]. The last two looked at  $\mathfrak{r}$  from the following point of view: A set  $R \in [\omega]^\omega$  is said to *reap* a family  $\mathcal{F} \subset [\omega]^\omega$  provided for every  $F \in \mathcal{F}$ ,  $|F \cap R| = |F - R| = \omega$ . Thus,  $\mathfrak{r}$  is the smallest cardinality of a family  $\mathcal{F}$  such that no set  $R \in [\omega]^\omega$  reaps  $\mathcal{F}$ .

Balcar (unpublished) has shown that  $\mathfrak{r} = \pi u$ . Clearly no set reaps the Boolean algebra generated by a maximal independent family; so  $\pi u = \mathfrak{r} \leq \mathfrak{i}$ , and clearly  $\mathfrak{r} = \pi u \leq u$ . GOLDSTERN and SHELAH [19 $\infty$ ] have a model where  $\mathfrak{r} < u$  (thus this is also a model where  $\pi u < u$ ). Also see JUST [19 $\infty$ ]. In [1982] PRICE noted a model where  $\mathfrak{r} = \mathfrak{c}$  (for another such model, see BEŠLAGIĆ and VAN DOUWEN [19 $\infty$ ]). In this context, a plausible definition of a small cardinal is the smallest cardinality of a family  $\mathcal{F}$  such that every infinite set  $A \subset \omega$  contains a member of  $\mathcal{F}$ . ROTHBERGER [1948] pointed out, however, that since there exist almost disjoint families of cardinality  $\mathfrak{c}$ , every such family  $\mathcal{F}$  has cardinality  $\mathfrak{c}$ .

For sets  $A, B$  in  $[\omega]^\omega$  we say that  $A$  *splits*  $B$  provided the partition of  $B$ ,  $\{B \cap A, B - A\}$ , consists of two infinite sets (i.e.,  $|B \cap A| = |B - A| = \omega$ ). Using this term, the cardinal  $\mathfrak{s}$  is the minimal cardinality of a family  $\mathcal{S}$  of subsets of  $\omega$  such that every infinite subset of  $\omega$  is split by some member of  $\mathcal{S}$ , and  $\mathfrak{r}$  is the minimal cardinality of a family  $\mathcal{R}$  of subsets of  $\omega$  such that no infinite subset of  $\omega$  splits every member of  $\mathcal{R}$ . We now think of characteristic functions of subsets of  $\omega$ . For  $f \in 2^\omega$ , and  $A \in [\omega]^\omega$ , we say  $\lim_A f = i$  provided  $i \in 2$  and the subsequence  $f|A$  converges to  $i$  in the discrete space  $\{0, 1\}$  (thus,  $f^{-1}(1)$  splits  $A$  iff  $f^{-1}(0)$  splits  $A$  iff  $\lim_A f$  does not exist). The cardinals  $\mathfrak{r}$  and  $\mathfrak{s}$  can be formulated in terms of sequences of zeroes and ones as follows:  $\mathfrak{s}$  is the minimal cardinality of a family  $\mathcal{S} \subset 2^\omega$  such that for every  $A \in [\omega]^\omega$  there exists  $f$  in  $\mathcal{S}$  such that  $\lim_A f$  does not exist. Also,  $\mathfrak{r}$  is the minimal cardinality of a family  $\mathcal{R}$  of subsets of  $\omega$  such that for every  $f \in 2^\omega$  there exists  $A$  in  $\mathcal{R}$  such that  $\lim_A f$  exists. This leads to the following two cardinals defined by VOJTÁŠ [1988, 19 $\infty$ ]. Let  $l^\infty$  denote the set of all bounded real valued sequences. Here, of course, for  $f \in l^\infty$  and  $A \in [\omega]^\omega$ , we define  $\lim_A f = x$  provided the sequence  $f|A$  converges to  $x$  in the usual topology on  $\mathbb{R}$ .

$$\mathfrak{s}_\sigma = \min\{ |S| \subset l^\infty : (\forall A \in [\omega]^\omega)(\exists f \in S) \lim_A f \text{ does not exist} \},$$

$$\mathfrak{r}_\sigma = \min\{ |R| \subset [\omega]^\omega : (\forall f \in l^\infty)(\exists A \in R) \lim_A f \text{ exists} \}.$$

It is easily seen that  $\mathfrak{s} = \mathfrak{s}_\sigma$ , and  $\mathfrak{r}_\sigma \geq \mathfrak{r}$ , but it is not known if  $\mathfrak{r} = \mathfrak{r}_\sigma$ .

Some other cardinals which have been considered are the Ramsey number (which also is denoted by  $\mathfrak{r}$ )  $\mathfrak{R}_c$ ,  $\mathfrak{u}_p$ ,  $\mathfrak{q}$  and  $\mathfrak{q}_0$ . The *Ramsey number* is the smallest cardinality  $\kappa$  of a family of functions  $\pi_\alpha: [\omega]^2 \rightarrow 2$  such that for every  $X \in [\omega]^\omega$  there exists  $\alpha < \kappa$  such that for every  $i \in \omega$ ,  $|\pi_\alpha(X - i)| = 2$  (IHODA [1988]). Blass has recently proved (unpublished) that the

Ramsey number equals  $\min\{\mathfrak{b}, \mathfrak{s}\}$ . The cardinal  $\mathfrak{R}_c$  is defined to be the smallest cardinality of a family  $F \subset {}^\omega\omega$  such that for every  $A \in [\omega]^\omega$  there exists  $f \in F$  such that  $f(A) = \omega$ . In Cohen's original models of not-**CH**  $\mathfrak{R}_c = \aleph_1$  (HECHLER [1973]), and Nyikos recently has proved (unpublished)  $\mathfrak{s} \leq \mathfrak{R}_c \leq \mathfrak{d}$ . The cardinal  $\mathfrak{u}_p$  is defined as the smallest cardinality of a base for a  $P$ -point in  $\omega^*$  if there exists a  $P$ -point, and is defined to be  $\mathfrak{c}$  if there do not exist any  $P$ -points (there are models of set theory in which  $P$ -points do not exist (WIMMERS [1982])). It is easy to see that  $\mathfrak{r}_\sigma \leq \mathfrak{u}_p$  (VOJTÁŠ [19∞]). The cardinal  $\mathfrak{q}$  is defined as the smallest cardinal such that no set of reals of this size or larger is a  $\mathfrak{Q}$ -set (GRUENHAGE and NYIKOS [19∞]) (a set  $X \subset \mathbb{R}$  is a  $Q$ -set provided every subset of  $X$  is a  $G_\delta$ -set in the subspace topology of  $X$ ). In [1948] ROTHBERGER proved that  $\mathfrak{p} \leq \mathfrak{q}$ , and it is consistent that  $\mathfrak{p} < \mathfrak{q}$  (FLEISSNER and MILLER [1980]). The cardinal  $\mathfrak{q}_0$  is defined the supremum of the set of cardinals  $\kappa$  such that every subset of  $\mathbb{R}$  of cardinality strictly less than  $\kappa$  is a  $Q$ -set (GRUENHAGE and NYIKOS [19∞]).

We now define some cardinals related to measure and category. Let  $\mathcal{I}$  be an ideal of subsets of a set. Define

$$\begin{aligned} \text{add}(\mathcal{I}) &= \min\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}, \\ \text{cov}(\mathcal{I}) &= \min\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \bigcup \mathcal{J} = \mathbb{R}\}, \\ \text{non}(\mathcal{I}) &= \min\{|Y| : Y \subset \mathbb{R} \text{ and } Y \notin \mathcal{I}\}, \text{ and} \\ \text{cf}(\mathcal{I}) &= \min\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \mathcal{I} = \bigcup \{\mathcal{P}(E) : E \in \mathcal{J}\}\}. \end{aligned}$$

These cardinals have been studied mainly for the set  $\mathbb{R}$  of real numbers, and the ideals of meager (= first category) sets and Lebesgue null sets. There seems to be no standard notation for these two important ideals. We will denote the ideal of meager sets by  $\mathbb{K}$  and the ideal of Lebesgue null sets by  $\mathbb{L}$ .

The cardinal  $\text{cov}(\mathbb{K})$  has been considered under several names (MILLER [1981, 1982b]), and it is known that  $\text{cov}(\mathbb{K}) = \mathfrak{m}_{\text{countable}}$  (the key idea is in GRIGORIEFF [1975], and an explicit proof is in FREMLIN and SHELAH [1979]). BARTOSZYNSKI [1987] proved that  $\text{cov}(\mathbb{K})$  is the least cardinal of any  $F \subset {}^\omega\omega$  such that for every  $g \in {}^\omega\omega$  there exists  $f \in F$  such that  $f(n) \neq g(n)$  for all  $n \in \omega$ . The ideal  $\mathcal{F}$  of nowhere dense sets of  $\mathbb{R}$  has also been considered; for example, FREMLIN [19∞b, 3B(b), 1J(b)] has proved  $\text{cf}(\mathcal{F}) = \text{cf}(\mathbb{K})$ . The relations among these cardinals can be displayed in the following diagram (called Cichoń's diagram, see FREMLIN [1983/84]). We have redrawn Cichoń's diagram to follow the conventions of Diagram 1 (in addition, the dotted enclosures indicate the following two results:  $\text{add}(\mathbb{K}) = \min\{\mathfrak{b}, \text{cov}(\mathbb{K})\}$ , and  $\text{cf}(\mathbb{K}) = \max\{\mathfrak{d}, \text{non}(\mathbb{K})\}$ ). In the case  $\mathfrak{c} = \omega_2$ , a number of people have combined to give models for all cases which the diagram allows for assignment of the values  $\omega_1$ , and  $\omega_2$ ; see BARTOSZYNSKI, JUDAH and SHELAH [19∞]. Thus, the shape of Cichoń's diagram is settled.

Not much is known about the relations among the cardinals in Cichoń's diagram and the other cardinals above. The following diagram of Vojtáš indicates

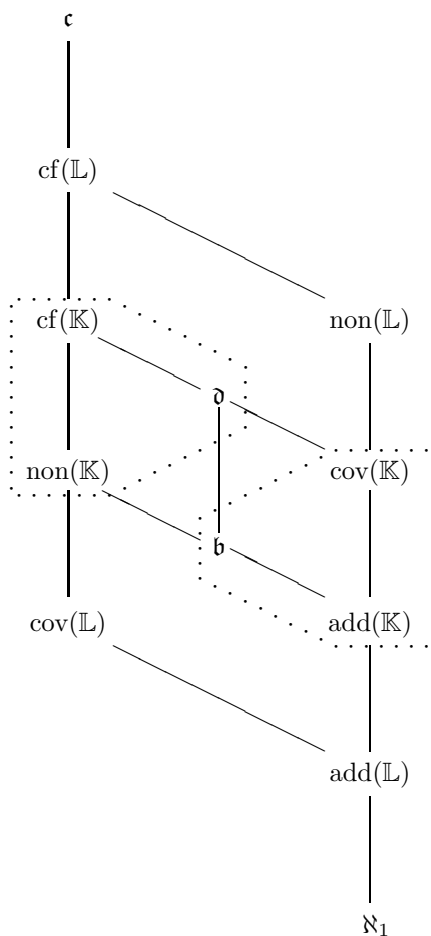


Diagram 2.

some of the known results. Among them are  $\text{cov}(\mathbb{K}) \leq \mathfrak{r}$  (VOJTÁŠ [19 $\infty$ ]),  $\mathfrak{t} \leq \text{add}(\mathbb{K})$  (PIOTROWSKI and SZYMAŃSKI [1987]),  $\mathfrak{b} \leq \text{non}(\mathbb{K})$  (ROTHBERGER [1941]). Also,  $\mathfrak{s} \leq \text{non}(\mathbb{L})$  and  $\mathfrak{s} \leq \text{non}(\mathbb{K})$  are attributed to J. Brzuchowski in CICHON [1981]. Note that  $\text{cov}(\mathbb{K}) \leq \mathfrak{d}$  follows at once from Bartoszyński's characterization of  $\text{cov}(\mathbb{K})$  mentioned in the preceeding paragraph.

Given the above cardinals, it is natural to consider their exponentiations (i.e.,  $2^\kappa$ ,  $2^{2^\kappa}$ , and possibly further exponentiations), and their cofinalities (if not regular in **ZFC**). It is natural to ask how these cardinals are related to each other. Furthermore, there are occasions when one wants to look at

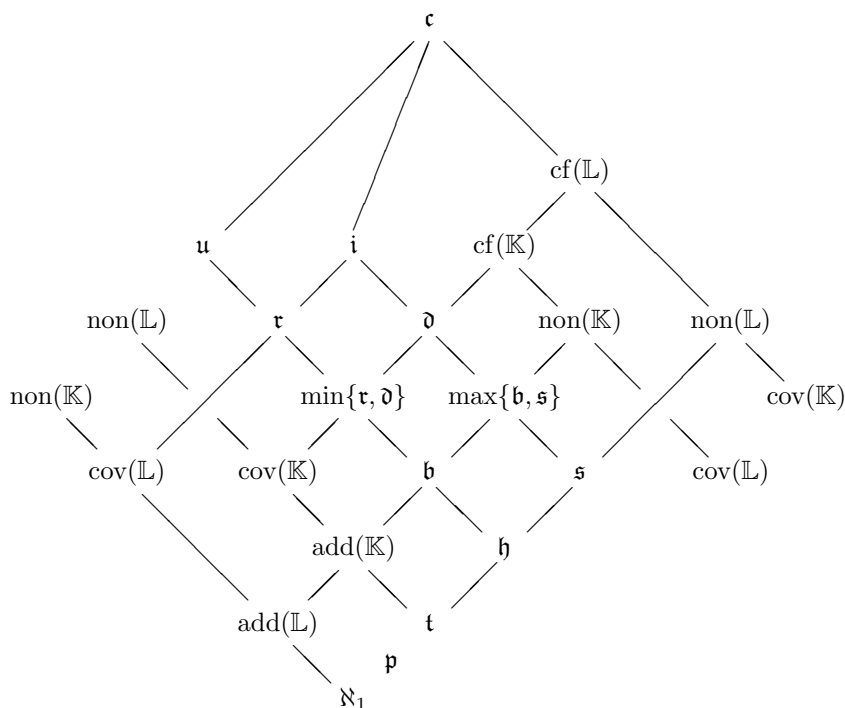


Diagram 3.

three or more of these cardinals at the same time. Indeed there are so many “obvious” open question about these cardinals that it is not possible to list them all here (an auxiliary problem is to find the most interesting among these questions). We mention several specific questions.

**Problem 1.2.** *Can  $\mathfrak{a}$  and  $\mathfrak{s}$  be singular?*

**334. ?**

**Problem 1.3.** *Can  $\mathfrak{m}$  be a singular cardinal of cofinality greater than  $\omega_1$ ?*

**335. ?**

See KUNEN [1988].

**Problem 1.4.** (VOJTÁŠ [1988]) *Is  $\mathfrak{r} = \mathfrak{r}_\sigma$ ?*

**336. ?**

In a letter of October 1989, W. Just notes that if  $\mathfrak{r} < \mathfrak{r}_\sigma$ , then either  $\mathfrak{r} < \mathfrak{u}$ , or  $\text{cf}([\mathfrak{u}]^{\aleph_0}) > \mathfrak{u}$ , and in the latter case  $\mathfrak{u} \geq \aleph_\omega$  and there is an inner model where the covering lemma fails.

? **337. Problem 1.5.** *Can  $\text{cf}(\text{cov}(\mathbb{K})) = \omega$ ?*

None of the other have countable cofinality; see BARTOSZYNSKI [1988], BARTOSZYNSKI and IHODA [19 $\infty$ ], BARTOSZYNSKI, IHODA and SHELAH [19 $\infty$ ], BARTOSZYNSKI and JUDAH [19 $\infty$ ], FREMLIN [1983/84], IHODA and SHELAH [19 $\infty$ ], and MILLER [1982a].

? **338. Problem 1.6.** *Can  $\text{cf}(\text{cov}(\mathbb{K})) < \text{cf}(\text{add}(\mathbb{K}))$ ?*

The same question is open for measure zero sets; see BARTOSZYNSKI and IHODA [19 $\infty$ ] and BARTOSZYNSKI, IHODA and SHELAH [19 $\infty$ ].

? **339. Problem 1.7.** *Is  $\mathfrak{t} \leq \text{add}(\mathbb{K})$ ?*

PIOTROWSKI and SZYMAŃSKI [1987] proved that  $\mathfrak{t} \leq \text{add}(\mathbb{K})$ .

## 2. Problems in topology

With many problems in topology, it is not immediately obvious whether small cardinals are involved or not. An example of this is the following rather old problem, raised in [1963] by MICHAEL:

? **340. Problem 2.1.** *Is there a Lindelöf space whose product with the space of irrational numbers is not normal?*

Consistent examples are known: any Lindelöf subspace of the Michael line has non-normal product with the irrationals, but such spaces exist iff  $\mathfrak{b} = \omega_1$  (VAN DOUWEN [1984, 10.2]). ALSTER [19 $\infty$ ] has proved that under **MA**, there is a space that shows the answer is “yes”. In Alster’s result, however, W. Fleissner has pointed out that the part of **MA** that Alster used can be stated in the notation of small cardinals as “ $\mathfrak{b} \leq \text{cov}(\mathbb{K})$ ”. Fleissner also noted that a model considered by MILLER [1982b] satisfies “ $\text{cov}(\mathbb{K}) < \mathfrak{b}$ ”. LAWRENCE [19 $\infty$ b] has proved that if  $X$  is Lindelöf and has non-normal product with the irrationals, then both the weight and cardinality of  $X$  are at least  $\min\{\mathfrak{b}, \aleph_\omega\}$ . Thus, the possibility is raised that the answer to Michael’s question might be equivalent to some statement involving small cardinals.

Another old question, raised in 1966, is the Scarborough-Stone problem (SCARBOROUGH and STONE [1966]). A space is called *sequentially compact* (resp. *countably compact*) provided every sequence in the space has a convergent subsequence (resp. cluster point). The problem asks:

? **341. Problem 2.2.** *Is every product of sequentially compact spaces countably compact?*

It was recently solved in the negative by Nyikos for the class of  $T_2$ -spaces, but for the classes of  $T_3$ -spaces or  $T_{3\frac{1}{2}}$ -spaces, the problem has been solved (in the negative) so far only by assuming some extra axiom such as  $\mathfrak{b} = \mathfrak{c}$  (VAN DOUWEN [1980a, 13.1]). It has also been solved in the negative in some models where  $\mathfrak{b} < \mathfrak{c}$  (NYIKOS and VAUGHAN [1987]). Also see VAUGHAN [1984].

A related problem raised by COMFORT [1977] comes from the theorem of Ginsburg and Saks that states: if  $X^{2^{\mathfrak{c}}}$  is countably compact then  $X^\alpha$  is countably compact for all  $\alpha$  (as was noted by Comfort, the proof of Ginsburg and Saks yields that if  $\{X_i : i \in I\}$  is a family of spaces such that for all  $J \subset I$  with  $|J| \leq 2^{\mathfrak{c}}$ , we have  $\prod\{X_i : i \in J\}$  is countably compact, then  $\prod\{X_i : i \in I\}$  is countably compact). The problem asks:

**Problem 2.3.** *Can  $2^{\mathfrak{c}}$  be replaced by a smaller cardinal in this result, i.e.,* **342. ?** *is there a cardinal  $\kappa < 2^{\mathfrak{c}}$  such that for every space  $X$ , if  $X^\kappa$  is countably compact then every power of  $X$  is countable compact?*

Examples of Z. Frolík show that the answer is in the negative under the assumption of the generalized continuum hypothesis, and examples of V. Saks (assuming  $\mathbf{MA} + \neg\mathbf{CH}$  and weaker statements) show the same thing (cf. VAUGHAN [1984]).

**Problem 2.4.** *Is every product of  $\mathfrak{h}$  sequentially compact spaces countably* **343. ?** *compact?*

See NYIKOS, PELANT and SIMON [19 $\infty$ ].

**Problem 2.5.** (NYIKOS [19 $\infty$ b]) *Does there exist a compact space which* **344. ?** *can be mapped continuously onto  $[0, 1]^{\mathfrak{s}}$  and has the following property: there exists a countable dense subset  $D$  such that every sequence in  $D$  has a subsequence that converges to some point in the space?*

**Problem 2.6.** *Is there a compact  $T_2$ -space  $X$  with no non-trivial convergent* **345. ?** *sequences and  $|X| < 2^{\mathfrak{s}}$ ?*

Nyikos pointed out to me that a construction of FEDORCHUK [1977] can be adapted to show that there exists such an  $X$  of cardinality  $2^{\mathfrak{s}}$ .

**Problem 2.7.** *Are there two countably compact topological groups whose* **346. ?** *product is not countably compact?*

Under  $\mathbf{MA}$  (i.e.,  $\mathfrak{m} = \mathfrak{c}$ ) the answer was given in the affirmative by VAN DOUWEN [1980b], and using a different technique HART and VAN MILL [19 $\infty$ ] proved that if  $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$ , then there exists a countably compact group  $H$  such that  $H \times H$  is not countably compact. The problem is still open in  $\mathbf{ZFC}$ .



As far I know, the following variation of the Scarborough-Stone problem is also open:

? **347. Problem 2.8.** *Is every product of sequentially compact, topological groups countably compact?*

A space is called *Fréchet* (or *Fréchet-Urysohn*) if every point in the closure of a set is the limit of a convergent sequence in the set.

? **348. Problem 2.9.** *Is there a countable Fréchet topological group that is not metrizable?*

Such groups have been constructed assuming  $\omega_1 < \mathfrak{p}$ , or  $\mathfrak{p} = \mathfrak{b}$  (NYIKOS [19 $\infty$ a]). A  $\Sigma$ -product of uncountably many copies of  $\{0, 1\}$  is a countably compact, non-compact Fréchet topological group, hence not metrizable (see ENGELKING [1989, 3.10.D]).

? **349. Problem 2.10.** *Is there a separable, first countable, countably compact, non-normal  $T_2$ -space? Is there one, which is also almost compact (a space  $X$  is called almost compact provided  $|\beta X - X| = 1$ )?*

If “separable” is not required, then such examples exist which are  $\omega$ -bounded (VAUGHAN [1979a, 1988]).

? **350. Problem 2.11.** *Is there a separable, first countable, countably compact, non-compact  $T_2$ -space?*

This problem is discussed by Nyikos in this book.

Let  $\mu_{cc}$  (resp.  $\mu_{sc}$ ) denote the cardinality of the smallest separable Hausdorff space with no isolated points which is countably (resp. sequentially) compact.

? **351. Problem 2.12.** *Is  $\mu_{cc} = \mathfrak{p}$ ? Is  $\mu_{sc} = \mathfrak{a}$ ? is it consistent that  $\mu_{cc} < \mu_{sc}$ ?*

See BEŠLAGIĆ, VAN DOUWEN, MERRILL and WATSON [1987].

? **352. Problem 2.13.** *Is the box product of countably many copies of the rational numbers paracompact (normal)?*

LAWRENCE [19 $\infty$ a] proved that the answer is in the affirmative under either assumption  $\mathfrak{b} = \mathfrak{d}$ , or  $\mathfrak{d} = \mathfrak{c}$ .

? **353. Problem 2.14.** *Is the box product of countably many copies of the convergent sequence  $\omega + 1$  normal?*

See VAN DOUWEN [1980a] and WILLIAMS [1984].

Many of the concepts considered in this paper can be extended to higher cardinals. To illustrate this, we list two such questions.

**Problem 2.15.** *Does there exist a first countable, initially  $\aleph_1$ -compact  $T_2$ -space which is not compact?* **354. ?**

A space is called initially  $\aleph_1$ -compact provided every open cover of cardinality  $\leq \aleph_1$  has a finite subcover. This question was raised by Dow, who showed that the answer is in the negative under **CH** and several other conditions (DOW [1985]). Fremlin showed the answer is in the negative under **PFA** (see BALOGH, DOW, FREMLIN and NYIKOS [1988]).

**Problem 2.16.** (COMFORT [1988]) *Consider  $\omega_1\omega$  as a product of  $\aleph_1$  countable, discrete spaces with the product topology. What is the smallest number of compact sets needed to cover  $\omega_1\omega$ ? In particular,  $(*)$  can  $\omega_1\omega$  be covered by fewer than  $2^{\aleph_1}$  compact sets?* **355. ?**

This is equivalent to asking if there exists a dominating (= cofinal) family  $\mathcal{F} \subset \omega_1\omega$  with  $|\mathcal{F}| < 2^{\aleph_1}$  with respect to the product order:  $f \leq g$  iff  $f(\alpha) \leq g(\alpha)$  for all  $\alpha \in \omega_1$  (TALL [1989]). It is the same question if we work with respect to the mod countable order (COMFORT [1977]), and therefore it is known that the problem involves large cardinals: JECH and PRIKRY [1984] have proved that if  $\mathfrak{c}$  is real-valued measurable, then the answer to  $(*)$  is “no”, and if the answer is “yes”, and  $2^{\aleph_1}$  has a certain property, then there are models with large cardinals.

### 3. Questions raised by van Douwen in his Handbook article

In his article VAN DOUWEN [1984], van Douwen raised ten questions related to small cardinals. For the convenience of the reader we will state all of them here, and use his enumeration of these problems. The ones that have been solved so far are VAN DOUWEN [1984, 6.6, 6.10, part of 6.11, 8.11, and 8.14].

**Problem 6.6.** *Is there a compact space of cardinality  $2^t$  which is not sequentially compact?*

**SOLUTION:** Alan Dow has answered the above question in the positive by noting that if  $X$  is compact Hausdorff and not sequentially compact, then  $\mathfrak{n} \leq |X|$ , and by constructing a model (a variation on model V in BALCAR, PELANT and SIMON [1980]) where  $2^t < \mathfrak{n}$ .

Van Douwen's question can be revived by asking: how can the cardinal, which is defined as the smallest cardinality of a compact, non-sequentially compact space, be expressed as a set-theoretically defined cardinal?

? **356. Problem 6.7.** *For compact  $X$ , in **ZFC** does “every countable compact subspace of  $X$  is closed” imply “ $X$  is a sequential space”?*

It is known that the answer is “yes” if  $\mathfrak{c} < 2^t$  (VAN DOUWEN [1984, 6.4]).  
Let

$$\mu = \min\{\kappa : \text{some product of } \kappa \text{ sequentially compact spaces is not sequentially compact}\}.$$

**Problem 6.10.** *Can the cardinal  $\mu$  be expressed as a set-theoretically defined cardinal?*

SOLUTION:  $\mu = \mathfrak{h}$  (NYIKOS, PELANT and SIMON [19 $\infty$ ] and FRIČ and VOJTÁŠ [1985] independently).

? **357. Problem 6.11.** (Restatement) *Is every product of sequentially compact spaces countably compact (i.e., is  $\{\kappa : \text{some product of } \kappa \text{ sequentially compact spaces is not countably compact}\}$  non-empty)?*

This is the Scarborough-Stone problem 2.2. If this set is non-empty, then let

$$\mu_1 = \min\{\kappa : \text{some product of } \kappa \text{ sequentially compact spaces is not countably compact}\}.$$

Can the cardinal  $\mu_1$  be expressed as a set-theoretically defined cardinal? It is known that  $\mu_1 \geq \mathfrak{n}$  (NYIKOS, PELANT and SIMON [19 $\infty$ ] and FRIČ and VOJTÁŠ [1985]).

Partial solution to the Scarborough-Stone problem: NYIKOS [1988] has shown that in **ZFC** there exists a family of Hausdorff (non-regular) sequentially compact spaces whose product is not countably compact. This answers the first part of the above question. For regular (or  $T_{3\frac{1}{2}}$ -spaces) all of VAN DOUWEN [1984, 6.11] is still open in **ZFC**. Some consistency results are discussed in 2.2.

**Problem 8.11.** *If  $X$  is a separable metric space and (a)  $X$  is analytic or (b) absolutely Borel, then is  $\text{cf}(\mathcal{K}(X)) = k(X) = \mathfrak{d}$ ?*

SOLUTION: By VAN DOUWEN [1984, 8.10] this question is clearly intended for  $X$  that are not  $\sigma$ -compact, and for them,  $\mathfrak{d} \leq k(X) \leq \text{cf}(\mathcal{K}(X))$ . Thus, the

question reduces to: is  $\text{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$ ? Here,  $\text{cf}(\mathcal{K}(X))$  denotes the smallest cardinality of a family  $\mathcal{L}$  of compact subsets of  $X$  such that for every compact set  $K \subset X$ , there exists  $L \in \mathcal{L}$  with  $K \subset L$ . A space is called *analytic* if it is the continuous image of the space of irrational numbers, and *absolutely Borel* if it is a Borel set in any of its metrizable compactifications. The answer to (b) is in the affirmative, but the answer to (a) is independent of the axioms of **ZFC**.

Concerning (a): BECKER [19 $\infty$ ] has constructed a model in which there is an analytic space  $X \subset 2^\omega$  with  $\text{cf}(\mathcal{K}(X)) > \mathfrak{d}$ . On the other hand, under **CH**,  $\text{cf}(\mathcal{K}(X)) = \mathfrak{d} = \omega_1$ .

Concerning (b): VAN ENGELEN [19 $\infty$ ] proved that if  $X$  is co-analytic (absolutely Borel sets are both analytic and co-analytic), then  $\text{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$ . The same follows from Fremlin's theory of Tukey's ordering (FREMLIN [19 $\infty$ a, 4, 15, 16]). Also see FREMLIN [19 $\infty$ b].

**Problem 8.14.** *Let  $S$  be a subset of a separable metric space  $X$  and assume that  $S$  is absolutely Borel. Is it true that if  $S \cap \text{cl}_X(X - S)$  is noncompact, then  $\chi(S, X) = \mathfrak{d}$ ?*

SOLUTION: Van Engelen and Becker have observed (independently) that the answer is “yes”. It follows from van Engelen's result “ $\text{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$ ” and the method of proof of VAN DOUWEN [1984, 8.10(c), 8.13(c)].

**Problem 8.17.** *Is there a (preferably metrizable) not locally compact space  $X$  with  $\text{Exp}_{\mathbb{R}}(X) < \text{Exp}_{\omega} < \infty$ ?* **358. ?**

Here  $\infty$  is defined to be larger than any cardinal, and

$\text{Exp}_{\mathbb{R}}(X) = \min\{\kappa : X \text{ embeds as a closed subspace in } {}^\kappa\mathbb{R}\}$ , and

$\text{Exp}_{\omega}(X) = \min\{\kappa : X \text{ embeds as a closed subspace in } {}^\kappa\omega\}$ .

Let

$\mathfrak{a}_p = \min\{|X| : X \text{ is first countable and pseudocompact but not countably compact}\}$ .

It is trivial that  $\mathfrak{b} \leq \mathfrak{a}_p \leq \mathfrak{a}$ .

**Problem 12.5.** *What is  $\mathfrak{a}_p$ ?*

**359. ?**

**Problem 12.6.** *Is there in **ZFC** a first countable (preferably separable and locally compact) pseudocompact space that is not countably compact and that has no uncountable closed discrete subset?* **360. ?**

**Problem 13.4.** *Is the following true in **ZFC**: Each first countable space of* **361. ?**

cardinality at most  $\mathfrak{c}$  is a quasi-perfect image of some locally compact space.

It is true under “ $\mathfrak{b} = \mathfrak{c}$ ” (VAN DOUWEN [1984, 13.4]). Is the condition “of cardinality at most  $\mathfrak{c}$ ” essential?

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# Appendix

## Remarks on some cardinal invariants of the continuum

by

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**A.1. THEOREM.**  $\mathbf{ZFC} \vdash \mathfrak{d} \leq \mathfrak{i}$ .

**A.2. NOTATION.** Let  $\mathcal{A} \subseteq [\omega]^\omega$  and  $\text{Fn}(\mathcal{A}) = \{f : f \text{ finite, } \text{dom}(f) \subseteq \mathcal{A}, \text{rng}(f) \subseteq \{0, 1\}\}$ . For the following  $f, g$  and  $h$  will always range over  $\text{Fn}(\mathcal{A})$ .

For  $f \in \text{Fn}(\mathcal{A})$ , let

$$X_f = \bigcap_{a \in \text{dom}(f)} a^{f(a)},$$

where  $a^1 = a$ ,  $a^0 = \omega - a$ .

From now on let  $\mathcal{A}$  be independent i.e.,  $X_f$  is infinite for all  $f$ .

Let  $I = I_{\mathcal{A}} = \{A \subseteq \omega : \forall f \exists g \supseteq f \text{ } (X_g \cap A \text{ is finite})\}$ .

Clearly,  $I$  is an ideal containing all finite sets and  $X_f \notin I_{\mathcal{A}}$  for all  $f$ .

**A.3. LEMMA** (Assuming  $|\mathcal{A}| < \mathfrak{d}$ ). *Let  $E \in I_{\mathcal{A}}$  and assume that  $f \in \text{Fn}(\mathcal{A})$  and  $A_0, A_1, \dots, A_n, \dots \in \mathcal{A}$  ( $n \in \omega$ ) are such that  $\text{dom}(f) \subseteq \mathcal{A}' =^{def} \mathcal{A} - \{A_0, \dots\}$ , then there is a set  $E'$  such that:*

( $\alpha$ )  $E' \in I_{\mathcal{A}}$ .

( $\beta$ )  $E' \cap E = \emptyset$

( $\gamma$ )  $\forall g \in \text{Fn}(\mathcal{A}')$ : If  $g \supseteq f$  then  $X_g \cap E'$  is infinite.

PROOF. For any  $H: \omega \rightarrow \omega$  let

$$E'_H = \bigcup_n \left[ (A_n - \bigcup_{i < n} A_i) \cap H(n) \right] - E.$$

Then clearly  $E'_H \in I_{\mathcal{A}}$  and  $E'_H \cap E = \emptyset$ , so any  $E'_H$  satisfies ( $\alpha$ ) and ( $\beta$ ). We have to find a suitable  $H$  such that ( $\gamma$ ) is satisfied.

Note that if  $g \supseteq f$  and  $\text{dom}(g) \subset \mathcal{A}'$  then  $X_g \cap (A_n - \bigcup_{i < n} A_i) - E \notin I_{\mathcal{A}}$ , so in particular it is infinite. (Since  $(A_n - \bigcup_{i < n} A_i)$  is of the form  $X_h$  for some  $h$  with  $\text{dom}(h) \cap \text{dom}(g) = \emptyset$ , it is not in  $I_{\mathcal{A}}$ .)

For each  $g \in \text{Fn}(\mathcal{A}')$  extending  $f$ , let

$$H_g(n) = \min(X_g \cap (A_n - \bigcup_{i < n} A_i) - E).$$

Clearly  $H_g$  is a 1-to-1 function, and if  $H_g(n) < H(n)$ , then  $H_g(n) \in X_g \cap E'_H$ . Hence if for infinitely many  $n$ ,

$$H_g(n) < H(n)$$

then

$$X_g \cap E'_H \text{ is infinite.}$$

Since  $|\mathcal{A}| < \mathfrak{d}$ , we can find  $H$  such that for all  $g \in \text{Fn}(\mathcal{A})$  with  $g \supseteq f$  there are infinitely many  $n$  for which  $H_g(n) < H(n)$ .

Then  $E' = E'_H$  satisfies the requirements of the lemma.  $\square$

PROOF OF THE THEOREM: Assume  $\mathcal{A}$  is an independent family of size  $< \mathfrak{d}$ . We will show that  $\mathcal{A}$  is not maximal.

Let  $N \prec \langle H(\lambda), \in \rangle$  for sufficiently large  $\lambda$  with  $N$  countable and  $\mathcal{A} \in N$ . Let  $\{f_n : n \in \omega\}$  list  $\text{Fn}(\mathcal{A}) \cap N$ , such that each element of  $\text{Fn}(\mathcal{A}) \cap N$  appears with even and with odd index. By induction choose  $E_n \in N$  such that:

(A)  $E_n \in I_{\mathcal{A}}$

(B)  $E_n \cap (\bigcup_{l < n} E_l) = \emptyset$

(C) If  $f_n \subseteq g \in \text{Fn}(\mathcal{A})$  and  $\text{dom}(g) \cap N = \text{dom}(f_n)$  then  $X_g \cap E_n$  is infinite

We can do this by the previous lemma, letting  $E = \bigcup_{l < n} E_l$ ,  $f = f_n$  and  $\{A_0, A_1, \dots\} \in N$  be some family disjoint from  $\text{dom}(f_n)$ . (we can have  $E_n \in N$  by elementarity of  $N$ ).

Now let  $Y = \bigcup_n E_{2n}$ . Then  $\mathcal{A} \cup \{Y\}$  is independent: Let  $g \in \text{Fn}(\mathcal{A})$ . Find  $n$  such that  $f_{2n} = g \cap N$ . Then  $X_g \cap Y$  contains  $X_g \cap E_{2n}$  which is infinite. If  $g \cap N = f_{2k+1}$  for some  $k$  then  $X_g \cap (\omega - Y)$  contains  $X_g \cap E_{2k+1}$  which is also infinite. This finishes the proof of the theorem.  $\square$

**A.4. REMARK.**  $\mathfrak{d} < \mathfrak{i}$  is consistent: e.g., take a model of **CH** and add  $\aleph_2$  many random reals with countable support. Then the old reals still form a dominating family. But an independent family of size  $\omega_1$  must be in an intermediate model, so it cannot be maximal, since the next random real will be independent from it. We can understand this argument more generally: if the set of reals is not the union of fewer than  $\lambda$  sets of measure zero, then any independent family of subsets of  $\omega$  has cardinality at least  $\lambda$ . So if  $P$  is the forcing of the measure algebra of dimension  $\lambda > \aleph_0$  then in  $V^P$  one has  $\mathfrak{i} \geq \lambda$ , whereas  $\mathfrak{d}$  is not changed by forcing with  $P$ .

# PART II

## GENERAL TOPOLOGY

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## CHAPTER 12

### **A Survey of the Class MOBI**

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In his landmark paper “Mappings and Spaces” [1966] A. V. ARKHANGEL’SKIĬ introduced the class MOBI (Metric Open Bicomact Images) as the intersection of all classes of topological spaces satisfying

- (i) Every metric space belongs to the class, and
- (ii) the image of any space in the class under an open-compact map is also in the class.

Recall that an open compact map is a continuous function such that the images of open sets are open and the inverse image of points are compact.

In BENNETT [1971] an equivalent definition of the class MOBI was obtained, namely,

**1. DEFINITION.** A space  $Y$  is in MOBI if there is a metric space  $M$ , a finite sequence  $(f_1, \dots, f_k)$  of open-compact maps and a finite sequence  $(X_1, \dots, X_k)$  of spaces such that

$$M \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{k-1}} X_{k-1} \xrightarrow{f_k} X_k = Y.$$

The space  $Y$  is said to be in the  $k$ -th generation of MOBI.

Note that in defining the class MOBI no separation axioms are asserted. In this paper only spaces that are at least  $T_1$  will be considered. Since bicomact was used in ARKHANGEL’SKIĬ [1966] to denote compact and Hausdorff, we will assume that compactness implies  $T_2$ .

In 1961 S. Hanai characterized the first generation of MOBI.

**2. THEOREM (HANAI [1961]).** *A space  $X$  is an open-compact image of a metric space if and only if  $X$  is a metacompact developable space.*

Thus a space  $Y$  is in the second generation of MOBI if and only if  $Y$  is the open-compact image of a metacompact developable space.

In [1966, 5.7] ARKHANGEL’SKIĬ observed that all spaces in MOBI have a point-countable base and asked many questions concerning this class. Most of these questions were answered negatively by the following examples:

**3. EXAMPLE (WICKE and WORRELL [1967, Example 3] (see BENNETT and BERNEY [1971])).** There exists a Hausdorff non-regular space  $Y$  in MOBI that is not developable. Moreover,  $Y$  is not (countably) metacompact.

**4. EXAMPLE (BENNETT [1971] (see WICKE and WORRELL [1967, Example 2])).** There exists a regular space  $Y$  in MOBI that is Lindelöf and hereditarily paracompact. The space  $Y$  is not developable (in fact, neither perfect nor a  $p$ -space). Moreover,  $Y$  can be modified to a linearly ordered topological space (see BENNETT and BERNEY [1971, 3.3]).

The pathological spaces in MOBI described in the above examples are in the second generation. The space  $Y$  in Example 3 is an open-compact image of a regular metacompact developable space, while in the second example a

Hausdorff non-regular space was used in the first generation. The following theorem shows that this space cannot be regular.

**5. THEOREM (WICKE and WORRELL [1967]).** *Let  $f$  be an open-compact map from a regular space  $X$  onto a space  $Y$ . If  $X$  has a base of countable order then so does  $Y$ .*

Since metacompact developable (developable) spaces are precisely metacompact (submetacompact) spaces with bases of countable order (WICKE and WORRELL [1965]), Theorem 5 implies that if a space  $Y$  in the second generation of MOBI is not developable, then either it is not (sub)metacompact (as in Example 3), or it is an image of a non-regular space in the first generation (as in Example 4).

In view of the above, it is natural to consider subclasses  $\text{MOBI}_i$  ( $i = 1, 2, 3, 4$ ) of the class MOBI obtained by adding to Definition 1 the assumption that all the terms of the sequence  $(X_1, \dots, X_k)$  are  $T_i$ -spaces.

In CHABER [1976], several examples of spaces in  $\text{MOBI}_3$  were constructed. In particular, the following improvement of Example 3 was given:

**6. EXAMPLE (CHABER [1976]).** There exists a regular space in  $\text{MOBI}_3$  that is not developable (in fact, neither perfect nor a  $p$ -space), hence not submetacompact.

Finally, examples of spaces without a very weak covering property (weak submetacompactness) were constructed, first in  $\text{MOBI}_2$  (CHABER [1988a]) and, later, in  $\text{MOBI}_3$  (BENNETT and CHABER [1988d]). In general, it seems to be helpful to look at the solution of a problem for  $\text{MOBI}_2$  while solving the corresponding problem on  $\text{MOBI}_3$ . In CHABER [1988a, 1988b] an example of a space in  $\text{MOBI}_2$  without a very weak separation property was constructed. It is not known whether such an example can be found in  $\text{MOBI}_3$ .

Again, all these pathological spaces were found in the second generation of MOBI. This suggests

? **362. Problem 1.** *Is each space in  $\text{MOBI}_i$  ( $i = 1, 2, 3, 4$ ) in the second generation? Does there exist a  $k$  such that each space in  $\text{MOBI}_i$  is in the  $k$ -th generation?*

The first part of Problem 1 was asked by Eric van Douwen, the second is equivalent to the following:

? **363. Problem 2.** (NAGAMI [1973]) *Are the classes  $\text{MOBI}_i$  ( $i = 1, 2, 3, 4$ ) countably productive?*

Another question concerning the general properties of the classes  $\text{MOBI}_i$  (the last unsolved problem from ARKHANGEL'SKIĬ [1966] for  $i < 4$ ) is

**Problem 3.** (ARKHANGEL'SKIĬ [1966]) *Are the classes  $\text{MOBI}_i$  ( $i = 1, 2, 3, 4$ ) invariant under perfect maps?* **364. ?**

The general problem for MOBI is

**Problem 4.** *Find characterizations of the classes  $\text{MOBI}_i$  ( $i = 1, 2, 3, 4$ ).* **365. ?**

In CHABER [1988b] a class of spaces related to  $\text{MOBI}_1$  was characterized (since this class can be described by Definition 1 if compactness is not assumed to imply  $T_2$ , the appropriate name for this class seems to be MOCI) as the class of all  $T_1$ -spaces with point-countable bases (all spaces in MOCI are in the second generation and MOCI is invariant under perfect mappings).

This characterization, the earlier discussion and the fact that all spaces in  $\text{MOBI}_4$  are perfect suggest a more specific version of Problem 4.

**Problem 5.** **366. ?**

- (1) *Are all spaces with point-countable bases in  $\text{MOBI}_1$ ?*
- (2) *Are all Hausdorff spaces with point-countable bases in  $\text{MOBI}_2$ ?*
- (3) *Are all regular spaces with point-countable bases of countable order in  $\text{MOBI}_3$ ?*
- (4) *Are all perfectly normal spaces with point-countable bases of countable order in  $\text{MOBI}_4$ ?*

Note that the properties listed in Problem 5 are known to be satisfied by all spaces in the corresponding class  $\text{MOBI}_i$ . Examples 4 and 6 show that the classes  $\text{MOBI}_i$  become strictly smaller with the increase of  $i \geq 2$ . We do not know the answer to the following:

**Problem 6.** *Are all Hausdorff spaces from  $\text{MOBI}_1$  in  $\text{MOBI}_2$ ?* **367. ?**

Problems 1–6 seem to be difficult and closely related to Problem 4. The lack of progress in investigating the general questions concerning MOBI led to the study of subclasses of this class.

There are two natural methods of defining natural subclasses of the classes  $\text{MOBI}_i$ . One can modify Definition 1 by putting additional restrictions either on the initial metric space  $M$  or on the open-compact mappings  $f_1, \dots, f_k$ .

Let  $P$  be a topological property. The subclass of  $\text{MOBI}_i$  generated by assuming that the metric space  $M$  in Definition 1 satisfies  $P$  (and the spaces  $X_1, \dots, X_k$  are  $T_i$ ) will be denoted by  $\text{MOBI}_i(P)$ .

Clearly, all spaces in  $\text{MOBI}_i(\text{discrete})$  are discrete and all spaces in  $\text{MOBI}_i(\text{locally separable})$  are locally separable (hence, metrizable if  $i \geq 3$ ). On the other hand, almost all the examples of pathological spaces in MOBI start with a  $\sigma$ -discrete metric space (Example 4 starts with a  $\sigma$ -locally separable metric space).



From JUNNILA [1978], it follows that the first generation of  $\text{MOBI}_i$  ( $\sigma$ -discrete) is the class of  $\sigma$ -discrete metacompact developable  $T_i$ -spaces. The examples from CHABER [1988a] and BENNETT and CHABER [19 $\infty$ d] show that the second generation of  $\text{MOBI}_i$  ( $\sigma$ -discrete) ( $i = 1, 2, 3$ ) contains spaces that are not  $\sigma$ -discrete (not even  $\sigma$ -locally separable). A property equivalent to  $\sigma$ -discreteness in the class of metric spaces and invariant under open-compact mappings is the property of having a closure-preserving closed cover by countable sets (JUNNILA [1978]). These results were used in CHABER [1988a] and BENNETT and CHABER [19 $\infty$ c] to prove

**7. THEOREM.** *For a  $T_i$ -space  $Y$ , where  $i = 1, 2$  (CHABER [1988a]) or  $i = 3$  (BENNETT and CHABER [19 $\infty$ c]), the following conditions are equivalent:*

- (i)  *$Y$  is the second generation of  $\text{MOBI}_i$  ( $\sigma$ -discrete),*
- (ii)  *$Y$  is in  $\text{MOBI}_i$  ( $\sigma$ -discrete),*
- (iii)  *$Y$  is in  $\text{MOBI}_i$  and has a closure-preserving closed cover by countable sets,*
- (iv)  *$Y$  has a point-countable base (a point-countable base of countable order for  $i = 3$ ) and a closure-preserving closed cover by countable sets.*

Since the classes of spaces satisfying (iv) are invariant under perfect mappings, Theorem 7 gives partial solutions to Problems 1–6 for  $i \leq 3$ .

One should emphasize that the proof of Theorem 7 for  $i = 3$  is much more difficult than the proof for  $i < 3$ . In fact, the proof from BENNETT and CHABER [19 $\infty$ c] is the first construction of an open-compact mapping using the existence of the base of countable order and, therefore, giving an evidence that the answer to Problem 5(3) could be ‘yes’. In spite of the apparent similarity, the gap between  $i = 2$  and  $i = 3$  was so big that the authors needed an intermediate step (the inductive construction from BENNETT and CHABER [19 $\infty$ b] giving a characterization of  $\text{MOBI}_3(\text{scattered}) = \text{MOBI}_i(\sigma\text{-discrete and complete})$ ) in order to fill it. Thus, it is not surprising that  $i = 4$  is not considered in Theorem 7. We shall discuss the  $\text{MOBI}_4$  later.

A natural way to generalize Theorem 7 is to replace  $\sigma$ -discreteness with  $\sigma$ -local separability. A property equivalent to  $\sigma$ -local separability in the class of metric spaces and invariant under open-compact mappings is the property of having a closure-preserving closed cover by separable sets.

We do not know whether Theorem 7 holds if  $\sigma$ -locally separable replaces  $\sigma$ -discrete and separable sets replace countable in the closure-preserving closed cover. The most interesting part of this question is

**? 368. Problem 7.** *Is  $\text{MOBI}_3(\sigma\text{-locally separable})$  equal to its second ( $k$ -th) generation?*

Since the property of having a closure-preserving closed cover by separable sets is preserved, in both directions, by open-compact mappings between

spaces with point-countable bases, the negative solution of Problem 7 would imply the negative solution of Problem 1 (for  $i < 3$  one should investigate the number of generations needed to get  $\text{MOBI}_i(\text{separable})$ ).

The class  $\text{MOBI}_4$  is axiom sensitive since the first generation space is a normal metacompact Moore space. Thus, if all normal Moore spaces are metrizable, then  $\text{MOBI}_4$  is just the class of all metric spaces. However, we do not know what  $\text{MOBI}_4$  is in the models in which there are normal nonmetrizable Moore spaces. In fact, we do not know the answer to the following:

**Problem 8.** *Is each space in  $\text{MOBI}_4$  metacompact?*

**369. ?**

**Problem 9.** *Is metacompactness preserved by open-compact mappings between (perfectly) normal spaces? What are the covering properties of an open-compact image of a (perfectly) normal metacompact space?*

**370. ?**

If it could be shown that the open-compact image of a perfectly normal metacompact space was weakly submetacompact then the image space would be submetacompact. This would imply that any space in the second generation of  $\text{MOBI}_4$  would be developable. It is known that the open compact image of a perfectly normal boundedly metacompact space is submetacompact (BENNETT and CHABER [19 $\infty$ a]).

Almost all the results concerning  $\text{MOBI}_3$  were obtained by restricting the area of investigation to the class  $\text{MOBI}_3(\text{scattered})$  (see CHABER [1976] and BENNETT and CHABER [19 $\infty$ d, 19 $\infty$ b, 19 $\infty$ c]). The first examples of non-metacompact spaces in  $\text{MOBI}_3$  have been found in the class  $\text{MOBI}_3(\text{scattered of height 2})$  (see Example 6 and CHABER [1976]). On the other hand, in BENNETT and CHABER [19 $\infty$ b] it was shown that an open-compact image of a normal metacompact space is metacompact if the domain is a scattered space of height 2. Hence, all spaces in  $\text{MOBI}_4(\text{scattered of height 2})$  are metacompact and the first generation gives all of  $\text{MOBI}_4(\text{scattered of height 2})$ . The reasoning from BENNETT and CHABER [19 $\infty$ b] does not rule out the existence of open-compact mappings defined on normal scattered spaces of height 3 and not preserving metacompactness.

**Problem 10.** *Is every space in  $\text{MOBI}_4(\text{scattered})$  metacompact? What is the structure of  $\text{MOBI}_4(\text{scattered})$ ?*

**371. ?**

Let us turn now to the second method of generating the subclasses of the class MOBI.

In all our examples, as well as in the examples from KOFNER [1982, Example 2] and CHABER [1988a, 4.5], the first mapping is finite-to-one while the infinite fibres of the second mapping are convergent sequences. In Theorem 7 the situation is similar but the fibres of the second mapping can be arbitrary compact scattered metric spaces (convergent sequences suffice if  $i = 1, 2$ ).

This suggests considering subclasses  $\text{MOBI}_{i,S}$  ( $\text{MOBI}_{i,I}$ ) of  $\text{MOBI}_i$  generated by assuming that all the fires of the open-compact mappings  $f_1, \dots, f_k$  in Definition 1 are scattered or, equivalently, countable (have isolated points).

Clearly,  $\text{MOBI}_i(\sigma\text{-discrete}) \subseteq \text{MOBI}_{i,S} \subseteq \text{MOBI}_{i,I} \subseteq \text{MOBI}_i$ .

It is known (see CHABER [1982] and CHOBAN [1978]), that the first generation in  $\text{MOBI}_{i,I}$  is the class of metacompact developable  $T_i$ -spaces having a countable cover by closed metrizable subspaces (= open finite-to-one  $T_i$ -images of metric spaces). Thus the first generations of  $\text{MOBI}_{i,S}$  and  $\text{MOBI}_{i,I}$  are equal and strictly smaller than the first generation of  $\text{MOBI}_i$  (unless  $i = 4$  and there is no 'reasonable' nonmetrizable normal Moore spaces).

? **372. Problem 11.** *Is the second generation of  $\text{MOBI}_{i,S}$  equal to the second generation of  $\text{MOBI}_{i,I}$ ? Find characterizations of (the second generations of) these classes.*

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## CHAPTER 13

### Problems in Perfect Ordered Spaces

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## 1. Introduction

A *linearly ordered topological space* is a triple  $(X, \mathcal{T}, <)$  where  $<$  is a linear ordering of the set  $X$  and  $\mathcal{T}$  is the usual open interval topology defined by  $<$ . A *generalized ordered space*, also called a suborderable space, is a topological space which can be embedded in a linearly ordered space. Introductory material on generalized ordered spaces can be found in LUTZER [1971, 1980].

A topological space is *perfect* if each of its closed subsets is a  $G_\delta$ -set. It is not true that generalized ordered spaces must be perfect: consider the familiar ordinal space  $[0, \omega_1)$ . In fact, the hypothesis that a generalized ordered space  $X$  is perfect is quite strong: a theorem in LUTZER [1971] shows that a perfect generalized ordered space must be paracompact.

The structure of perfect generalized ordered spaces is not well understood and the purpose of this paper is to remind readers of a few problems in the theory of perfect generalized ordered spaces. These problems have already appeared in the literature, and there is substantial overlap between this article and BENNETT and LUTZER [1977] and [1984b], and LUTZER [1983], but there has been little progress to date.

## 2. Perfect subspaces vs. perfect superspaces

Generalized ordered spaces are, by definition, subspaces of linearly ordered spaces. For many topological properties  $P$ , one can prove that a generalized ordered space with property  $P$  can be embedded in a linearly ordered space which also has property  $P$ . Examples of properties for which this is known to be true include separability, metrizability, and paracompactness. Indeed, for the last two of these three topological properties  $P$ , a generalized ordered space with  $P$  embeds as a *closed* subspace of a linearly ordered space which also has property  $P$ . The situation in which  $P = \text{“perfect”}$  is unclear, and we have:

**Question 1.** *Is it true that a perfect generalized ordered space can be embedded in a perfect linearly ordered space?* **373. ?**

To put Question 1 in context, it may be worth mentioning that there is no hope that a perfect generalized ordered space can be embedded as a closed subset in a perfect linearly ordered space: the familiar Sorgenfrey line  $S$  provides the required example, see LUTZER [1971]. However,  $S$  does not provide the example necessary to answer Question 1 negatively, since  $S$  is a dense subspace of the perfect linearly ordered space  $T = [0, 1] \times \{0, 1\}$  with the lexicographic ordering.

### 3. Perfect ordered spaces and $\sigma$ -discrete dense sets

If a space  $X$  has a dense subset  $D$  which can be written as a countable union of closed, discrete subsets, then  $D$  is said to be a  $\sigma$ -discrete dense subset of  $X$ . In generalized ordered spaces one can prove:

**3.1. PROPOSITION.** *If a generalized ordered space  $X$  has a  $\sigma$ -discrete dense subset, then  $X$  is perfect.*

PROOF. Let  $D = \bigcup \{D(n) : n \geq 1\}$  be a  $\sigma$ -discrete dense subset of  $X$ . It is easy to see that every singleton subset of  $X$  is a  $G_\delta$ -subset of  $X$  and hence that every order-convex subset of  $X$  is a  $F_\sigma$ -subset of  $X$ . Now consider any open set  $G$  and write  $G = \bigcup \{G(i) : i \in I\}$  where the sets  $G(i)$  are the maximal order-convex subsets of  $G$ . Express each  $G(i)$  as  $G(i) = \bigcup \{F(i, m) : m \geq 1\}$  where each  $F(i, m)$  is a closed convex subset of  $X$ . Let  $H(m, n) = \bigcup \{F(i, m) : i \in I \text{ and } F(i, m) \cap D(n) \neq \emptyset\}$ . Each  $H(m, n)$  is closed in  $X$  and  $G = \bigcup \{H(m, n) : m, n \geq 1\}$  so that  $G$  is an  $F_\sigma$ -subset of  $X$ .  $\square$

The converse of 3.1 is not provable in **ZFC** as the following consistency result shows. Recall that a *Suslin line* is a non-separable linearly ordered topological space in which each collection of pairwise disjoint open sets is countable. It is known that the existence of a Suslin line is consistent with, and independent of, the usual axioms of **ZFC**.

**3.2. PROPOSITION.** *If there is a Suslin line  $S$ , then  $S$  is perfect but has no  $\sigma$ -discrete dense subset.*  $\square$

Given that the converse of 3.1 is not provable in **ZFC**, it is reasonable to ask whether it is false in **ZFC**, and that is the second question about perfect ordered spaces. It was originally posed by M. Maurice and J. van Wouwe.

? **374. Question 2.** *Assuming only **ZFC**, is there an example of a perfect ordered space which does not have a  $\sigma$ -discrete dense subset?*

The question posed by Maurice and van Wouwe is closely related to an older problem posed by R. W. Heath.

? **375. Question 3.** *In **ZFC**, is there an example of a perfect linearly ordered space which has a point-countable base and yet which is not metrizable?*

It is consistent with **ZFC** that such an example exists, as can be seen from the following result established in BENNETT [1968] and PONOMAREV [1967].

**3.3. PROPOSITION.** *If there is a Suslin line, then there is a Suslin line having a point-countable base.*  $\square$

The relation between Questions 2 and 3 may be seen from a result established in BENNETT and LUTZER [1984a]:

**3.4. PROPOSITION.** *Let  $X$  be a generalized ordered space having a  $\sigma$ -discrete dense set and a point-countable base. Then  $X$  is metrizable.*  $\square$

Therefore, any space which answers Question 3 will also answer Question 2.

#### 4. How to recognize perfect generalized ordered spaces

To prove that an arbitrary topological space is perfect, it is not necessary to examine every closed set. It is easy to see that it will be enough to show that every nowhere dense closed set is a  $G_\delta$ -set, i.e., it is enough to consider the small closed sets. In generalized ordered spaces, another approach is possible: it turns out to be enough to consider only the large closed sets. To say that  $F$  is a *regular closed set* means that  $F = \text{cl}(\text{int}(F))$ . In BENNETT and LUTZER [1983] the following result is proved:

**4.1. PROPOSITION.** *A generalized ordered space  $X$  is perfectly normal if and only if each regularly closed subset of  $X$  is a  $G_\delta$ -set.*  $\square$

#### 5. A metrization problem for compact ordered spaces

The solution of at least one problem in generalized ordered space theory will hinge on being able to recognize whether a certain type of generalized ordered space must be perfect. Recall that a collection  $\mathcal{C}$  of subsets of  $X$  is *minimal* if  $\bigcup \mathcal{D} \neq \bigcup \mathcal{C}$  whenever  $\mathcal{D}$  is a proper subcollection of  $\mathcal{C}$ . A collection which can be expressed as the union of countably many minimal collections is said to be  $\sigma$ -*minimal*. The study of spaces which have  $\sigma$ -minimal bases was initiated in AULL [1974]. In answer to one of the questions in AULL [1974], BENNETT and BERNEY [1977] proved that the lexicographic square  $L$  (i.e., the space  $[0, 1] \times [0, 1]$  with the lexicographic ordering and the usual open interval topology) is a compact, first countable linearly ordered topological space which has a  $\sigma$ -minimal base. A key point about the structure of  $L$ —and this will be important later—is that  $L$  has two parts. The first is the open metrizable subspace  $[0, 1] \times ]0, 1[$  and the second is the pathological subspace  $[0, 1] \times \{0, 1\}$  which cannot have a  $\sigma$ -minimal base for its topology.

Examination of  $L$  and other examples suggests that the existence of pathological subspaces may be the key to understanding generalized ordered spaces which have  $\sigma$ -minimal bases. This leads to the following question first posed in BENNETT and LUTZER [1977].

**Question 4.** *Suppose that  $X$  is a compact linearly ordered topological space and that every subspace of  $X$  has a  $\sigma$ -minimal base for its relative topology. Must  $X$  be metrizable?* **376. ?**

It is known that a space  $X$  as described in Question 4 must be first countable, hereditarily paracompact, and such that for every subspace  $Y$  of  $X$ ,



there is a dense subspace  $M$  of  $Y$  which is metrizable. Further, there is a kind of structure theorem for such spaces which has proved useful in the study of other hereditary properties and metrizability in ordered spaces. The theorem suggests that, like the lexicographic square  $L$ , such spaces have two, quite different, parts.

**5.1. PROPOSITION.** *Suppose that  $X$  is a first countable paracompact generalized ordered space. Then there are subspaces  $G$  and  $H$  of  $X$  satisfying:*

- (a)  $G$  is open and metrizable;
- (b)  $H = X - G$ ;
- (c)  $H$  is dense in itself and can be written as  $H = D \cup E$  where  $D \cap E = \emptyset$  and where  $[d_1, d_2] \cap D$  is not compact whenever  $d_1 < d_2$  are points of  $D$  (respectively  $[e_1, e_2] \cap E$  is not compact whenever  $e_1 < e_2$  are points of  $E$ ). □

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## CHAPTER 14

### **The Point-Countable Base Problem**

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# Introduction

It is the purpose of this article to show how the problem arose, to place it in the context of the most natural of structuring mechanisms, to indicate some powerful metrisation theorems which are available on the addition of a single condition, and to review some recent work which gives some partial answers and suggests lines of enquiry which should be further pursued. Despite the attention shown to the problem over the last five years, we would not wish to claim that the results presented here go more than a small distance towards a solution. We believe that some new ideas are required and would encourage our colleagues to provide them.

Since the review paper COLLINS [19 $\infty$ ] was given at the Baku Topology Symposium in 1987, new insights have encouraged us to vary the presentation and to include hitherto unpublished material.

All spaces will have the  $T_1$  separation axiom and  $\mathbb{N}$  will denote the set of natural numbers.

## 1. Origins

The structuring mechanism which spawned the problem arose in the search for a simple, yet natural, condition which would produce a countable basis in a separable space. The model was, naturally enough, a standard elementary proof for a metric space with countable dense subset  $A$ .

If, in this context,  $x$  belongs to open  $U$  and  $a$  is an element of  $A \cap S_\epsilon(x)$ , where the open ball  $S_{3\epsilon}(x) \subseteq U$ , then  $x \in S_r(y) \subseteq U$  for any  $y \in S_\epsilon(x)$  and any rational  $r$  such that  $\epsilon < r < 2\epsilon$ . The picture is given in Figure 1.

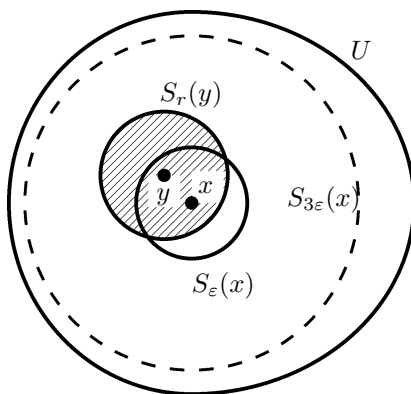


Figure 1: The proof that a separable metric space is second countable

Then  $\{S_r(a) : a \in A \wedge r \in \mathbb{Q}^+\}$ , where  $\mathbb{Q}^+$  is the set of positive rational numbers, is a countable basis.

An essential feature of the above proof, which must be borne clearly in mind in constructing generalisations, is the need, not only for  $S_r(y)$  to be small enough to be within  $U$ , but also large enough to ‘pick up’  $x$ . We shall return to this point later.

The first generalisation to be investigated (COLLINS and ROSCOE [1984]), which allows an immediate parody of the above proof, runs as follows. For each  $x$  in a space  $X$ , let

$$\mathbf{W}(x) = \{W(n, x) : n \in \mathbb{N}\}$$

be a countable family of subsets of  $X$ , each containing  $x$ .  $\mathcal{W} = \{\mathbf{W}(x) : x \in X\}$  is said to *satisfy* (A)<sup>1</sup> if it satisfies

- (A) if  $x \in U$  and  $U$  is open, then there exist a positive integer  $s = s(x, U)$  and an open set  $V = V(x, U)$  containing  $x$  such that  $x \in W(s, y) \subseteq U$  whenever  $y \in V$ .

The picture is the same as Figure 1, once one sets  $V = S_\epsilon(x)$  and  $W(s, y) = S_r(y)$ . Second countability follows from separability when each  $W(n, x)$  is open, or indeed is a neighbourhood of  $x$  ( $\mathcal{W}$  satisfies ‘*open* (A)’, or ‘*neighbourhood* (A)’). In fact, one can go further if  $\mathcal{W}$  satisfies ‘*neighbourhood decreasing* (A)’, that is, if  $W(n+1, x) \subseteq W(n, x)$  holds for each  $x$  and  $n$  in addition to the  $W(n, x)$  being neighbourhoods of  $x$ .

**1. THEOREM** (COLLINS and ROSCOE [1984]). *In order that  $X$  be metrisable it is necessary and sufficient that  $X$  has  $\mathcal{W}$  satisfying neighbourhood decreasing (A).*  $\square$

In COLLINS and ROSCOE [1984], it is shown that *eventually decreasing neighbourhood* (A) will not suffice for metrisability. Theorem 1 is proved in one page, relying on no other results, and a number of classical metrisation theorems, such as those of Nagata-Smirnov and of Moore-Arkhangel’skiĭ-Stone, are quickly deduced.

We should like to stress how natural condition (A) is by restating Theorem 1 to provide a set-theoretic model for metric spaces.

**2. THEOREM** (COLLINS and ROSCOE [1984]). *Suppose that for each  $x$  in a set  $X$  there is a decreasing sequence  $\mathbf{W}(x) = \{W(n, x) : n \in \mathbb{N}\}$  of subsets of  $X$ , each containing  $x$ , such that*

- (1) *given  $x$  and  $y$ ,  $x \neq y$ , there exists a positive integer  $m$  with  $y \notin W(m, x)$ ,*
- (2) *given  $x$  in  $X$  and a positive integer  $n$ , there exist positive integers  $r = r(n, x)$  and  $s = s(n, x)$  such that  $y \in W(r, x)$  implies that  $x \in W(s, y) \subseteq W(n, x)$ .*

---

<sup>1</sup>The names (A), (F) and (G) used for conditions in this paper are taken from COLLINS and ROSCOE [1984].

Then there is a metric for  $X$  such that, for each  $x$  in  $X$ ,  $\mathbf{W}(x)$  is a basis for the neighbourhood system of  $x$  in the metric topology.  $\square$

Condition (2) is just (A) re-stated in terms of the  $W(n, x)$ 's and obviously strengthens the usual neighbourhood axioms for a topological space. Condition (1) ensures appropriate separation.

In the proof of second countability of a separable metric space given in the last section, the same  $r$  sufficed for each  $y$  in  $S_\epsilon(x)$ . This is reflected in condition (A) where  $s = s(x, U)$  does not depend on  $y \in V$ . It is natural to ask what happens if  $s$  also depends on  $y$ . We say (COLLINS and ROSCOE [1984]) that  $\mathcal{W}$  satisfies (G) if it satisfies

- (G) if  $x \in U$  and  $U$  is open, then there exists an open  $V = V(x, U)$  containing  $x$  such that  $x \in W(s, y) \subseteq U$  for some  $s = s(x, y, U) \in \mathbb{N}$  whenever  $y \in V$ .

The picture is much the same as before (Figure 2).

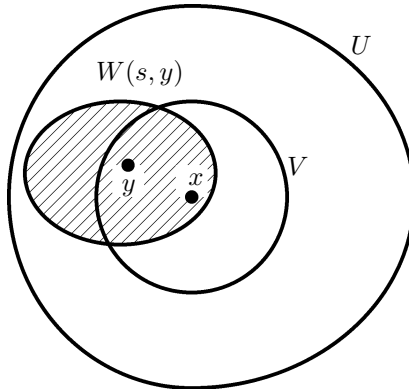


Figure 2: Condition (G)

Again, with analogous definitions, if  $\mathcal{W}$  satisfies open (G), then separability implies second countability (Lemma 3 of COLLINS and ROSCOE [1984]). How finely conditions (A) and (G) are balanced on the point of what is possible in metrisation theory may be judged from the following results.

**3. THEOREM** (COLLINS and ROSCOE [1984]). *There is a space  $X$  (the ‘bow-tie’ space of L. F. MCAULEY [1955]) which has  $\mathcal{W}$  satisfying neighbourhood decreasing (G) but which is not metrisable.*  $\square$

**4. THEOREM** (COLLINS, REED, ROSCOE and RUDIN [1985]). *In order that  $X$  be metrisable it is necessary and sufficient that  $X$  has  $\mathcal{W}$  satisfying open decreasing (G).*  $\square$

It should be noted that Theorems 1 and 4 are *not* inter-dependent.

The value of considering generalisations of open decreasing (G) is exemplified by the next result.

**5. THEOREM** (BALOGH [1985], COLLINS, REED, ROSCOE and RUDIN [1985]). *A space is stratifiable (and hence a Nagata space if first countable) if and only if it has  $\mathcal{W}$  satisfying decreasing (G) and has countable pseudocharacter.*  $\square$

Comparison of Theorems 4 and 5 prompts the following open question.

**? 377. Problem 1.** (COLLINS and ROSCOE [1984]) *Which spaces are characterised as having  $\mathcal{W}$  satisfying neighbourhood decreasing (G)?*

It is known (see COLLINS, REED, ROSCOE and RUDIN [1985]) that there are stratifiable (indeed, Nagata) spaces which do not have neighbourhood decreasing (G).

It is another generalisation of open decreasing (G) which provides the title of this article and the next section.

## 2. The point-countable base problem

Whilst investigating the structuring mechanism described in the last section, the authors made a number of conjectures, many of which have now been answered by theorems or counterexamples. Of those that remain, the point-countable base problem is the most intriguing, both because of the number of partial solutions that have been discovered and because of the effort that has been expended on it.

A basis for a space  $X$  is *point-countable* if every element of  $X$  is contained in at most a countable number of elements of the basis. It may quickly be deduced that, if  $X$  has such a basis  $\mathcal{B}$ , then  $X$  has  $\mathcal{W}$  satisfying open (G) (by defining  $\mathbf{W}(x) = \{B \in \mathcal{B} : x \in B\}$ ). The converse remains an open question.

**? 378. Problem 2.** (The Point-Countable Base Problem (COLLINS, REED, ROSCOE and RUDIN [1985])) *If  $X$  has  $\mathcal{W}$  satisfying open (G), need  $X$  have a point-countable basis?*

Note that it is not possible usefully to reduce ‘open non-decreasing  $\mathbf{W}(x)$ ’ to ‘open decreasing  $\mathbf{V}(x)$ ’ (which one might hope to do, so as to apply results of the last section) by the formula

$$V(n, x) = \bigcap_{i=1}^n W(i, x)$$

since, even if  $\mathcal{W}$  satisfies (G),  $\mathcal{V}$  may not, as the  $V(s, x)$  may not ‘pick up’  $x$ . (See our comment in Section 2.)

That a space be meta-Lindelöf (a condition clearly implied by the existence of a point-countable basis) does not require *open* (G), as the following result shows. This result is not only useful, but exemplifies a common style of proof found when (G) and related conditions are used.

**6. LEMMA** (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space  $X$  has  $\mathcal{W}$  satisfying (G), then  $X$  is hereditarily meta-Lindelöf (i.e., each open cover has a point-countable open refinement).*

PROOF. Suppose  $\mathcal{G} = \{U_\alpha : \alpha \in \gamma\}$  is an open cover of  $X$ , enumerated using some ordinal  $\gamma$ . For each  $\alpha \in \gamma$ , define the set

$$S_\alpha = \bigcup \{V(x, U_\alpha) : x \in U_\alpha \setminus \bigcup_{\beta \in \alpha} U_\beta\}$$

where  $V(x, U)$  is given by (G). By construction,  $\mathcal{S} = \{S_\alpha : \alpha \in \gamma\}$  is an open cover of  $X$ . We claim that  $\mathcal{S}$  is point-countable. If  $y$  belongs to  $S_\alpha$ , then there is  $x_\alpha \in U_\alpha \setminus \bigcup_{\kappa \in \alpha} U_\kappa$  such that  $y \in V(x_\alpha, U_\alpha)$ , and hence there is  $W_\alpha \in \mathbf{W}(y)$  such that

$$x_\alpha \in W_\alpha \subseteq U_\alpha.$$

There can only be countably many such  $\alpha$ , for otherwise there would be two ordinals  $\alpha, \beta$  (with  $\alpha \in \beta$ , say) such that  $W_\alpha = W_\beta$ . But then  $x_\beta \in W_\beta \subseteq U_\alpha$ , giving a contradiction. So,  $\mathcal{S}$  is point-countable as claimed.

The fact that  $X$  is *hereditarily* meta-Lindelöf follows simply from the observation that any subspace  $Y$  trivially has  $\mathcal{W}'$  satisfying (G).  $\square$

A large number of partial answers have been found to the point-countable base problem. Many of them turn out to be consequences of a simple Lemma (which was actually discovered after many of its consequences). To state it, we need the concept of a pointed open cover. A *pointed open cover* for a space  $X$  with topology  $\mathcal{T}$  is a subset  $\mathcal{P}$  of  $X \times \mathcal{T}$  such that  $\{U : \exists x \in X (x, U) \in \mathcal{P}\}$  is a cover for  $X$ .  $\mathcal{P}$  is said to be *point-countable* if  $\{(x, U) \in \mathcal{P} : y \in U\}$  is countable for all  $y$ , and *dense* if

$$y \in \overline{\{x : (x, U) \in \mathcal{P} \wedge y \in U\}}$$

for all  $y$ . Note that we have not insisted that  $(x, U) \in \mathcal{P}$  implies  $x \in U$ .<sup>2</sup>

**7. LEMMA** (MOODY, REED, ROSCOE and COLLINS [19∞]). *If the space  $X$  has  $\mathcal{W}$  satisfying open (G), then  $X$  has a point-countable base if and only if  $X$  has a dense, point-countable, pointed open cover.*

---

<sup>2</sup>Indeed, it is often more natural to generate these pointed open covers in such a way that  $x \notin U$  for some  $(x, U)$ . However it is easy to show that if a space has  $\mathcal{W}$  satisfying open (G) and a dense, point-countable pointed open cover  $\mathcal{P}$ , then there is another one,  $\mathcal{P}'$ , where all  $(x, U)$  have  $x \in U$ .

PROOF. First suppose that  $X$  has a point-countable base  $\mathcal{B}$ . For each non-empty element  $U$  of  $\mathcal{B}$ , pick an  $x_U \in U$ , and define  $\mathcal{P} = \{(x_U, U) : U \in \mathcal{B} \setminus \{\emptyset\}\}$ . It can be easily verified that  $\mathcal{P}$  is a dense, point-countable, pointed open cover. Conversely, define

$$\mathcal{B} = \{U \cap W : \exists x (x, U) \in \mathcal{P} \wedge W \in \mathbf{W}(x)\}.$$

Clearly,  $\mathcal{B}$  is a point-countable collection of open sets. To see that  $\mathcal{B}$  is a base, consider any  $x \in X$  and any open set  $O$  containing  $x$ . Since  $\mathcal{P}$  is dense, there must exist a  $(y, U) \in \mathcal{P}$  such that  $x \in U$  and  $y \in V(x, O)$ . Pick a  $W \in \mathbf{W}(y)$  for which  $x \in W \subseteq O$ . Then  $x \in U \cap W \subseteq O$  and  $U \cap W \in \mathcal{B}$ , so that  $\mathcal{B}$  is a base as required.  $\square$

Notice how this proof follows the one that a separable space with  $\mathcal{W}$  satisfying open (G) is second-countable. In fact, possessing a dense, point-countable pointed open cover is a natural generalisation of separability: there are countably many points available to have each point as a limit, only now, *which* points are available varies from place to place. (Each point is available in an open set.) Every separable space  $X$  with countable dense subset  $D$  has such a pointed open cover  $\{(x, X) : x \in D\}$ .

Given these two lemmas, it is possible to establish a number of results rather easily. We now give sketch proofs of the three such theorems, in each case showing how the dense, point-countable, pointed open covers are constructed.

**8. THEOREM (MOODY, REED, ROSCOE and COLLINS [19 $\infty$ ]).** *If the space  $X$  has  $\mathcal{W}$  satisfying open (G) and has density  $\leq \aleph_1$ , then it has a point-countable basis.*

PROOF. If  $X$  is separable then we already know it is second countable, so we may assume it has a dense subset  $D = \{x_\alpha : \alpha \in \omega_1\}$ . It is then easy to show that the pointed open cover

$$\mathcal{P} = \{(x_\alpha, X \setminus \overline{\{x_\beta : \beta \in \alpha\}}) : \alpha \in \omega_1\}.$$

is point-countable and dense.  $\square$

Considerable work has been done to show that, under the assumption that large cardinals exist, if certain topological properties are true of all subsets with cardinal  $\leq \aleph_1$  of a given space, then they are true of the space (see F. D. Tall's questions on reflection, this volume). In this vein, it has been conjectured that, if every  $\leq \aleph_1$  subset of a first countable regular space  $X$  has a point-countable base, then  $X$  has a point-countable base. If this could be proved, then, of course, Theorem 8 would answer the point-countable base problem in the affirmative for regular spaces, on the assumption of large cardinals.



**9. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19∞]). *If  $X$  is semi-stratifiable and has  $\mathcal{W}$  satisfying open  $(G)$ , then it has a point-countable basis.*

PROOF. We shall use the characterisation of semi-stratifiable given by G. D. CREEDE in [1970], which states that a space  $X$  with topology  $\mathcal{T}$  is semi-stratifiable if and only if there exists a function  $g$  from  $\mathbb{N} \times X$  to  $\mathcal{T}$  such that

- (i)  $\{x\} = \bigcap \{g(n, x) : n \in \mathbb{N}\}$ , and
- (ii) if  $y \in X$  and  $(x_n)$  is a sequence of points in  $X$  such that  $y \in g(n, x_n)$  for all  $n$ , then  $(x_n)$  converges to  $y$ .

Let  $g$  be such a function. By Lemma 6 we may let  $\mathcal{U}_n$  be a point-countable open refinement of  $\{g(n, x) : x \in X\}$ . For each  $U \in \mathcal{U}_n$ , choose  $x_U$  such that  $U \subseteq g(n, x_U)$ . Define  $\mathcal{P} = \{(x_U, U) : U \in \mathcal{U}_n \wedge n \in \mathbb{N}\}$ . By construction,  $\mathcal{P}$  is a point-countable, pointed open cover. It also follows easily from (ii) above that  $\mathcal{P}$  is dense.  $\square$

**10. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19∞]). *If  $X$  has  $\mathcal{W}$  satisfying open  $(G)$  and is the locally countable sum of spaces which have point-countable bases (i.e.,  $X = \bigcup \{X_\lambda : \lambda \in \Lambda\}$ , where each subspace  $X_\lambda$  has a point-countable base and where there is a neighbourhood  $N(x)$  of each  $x$  which meets only countably many  $X_\lambda$ ), then  $X$  has a point-countable base.*

PROOF. In fact, we will show, without using the openness of  $\mathcal{W}$ , that if each  $X_\lambda$  has a dense, point-countable, pointed open cover  $\mathcal{P}_\lambda$  then so does  $X$ . If  $U$  is a set open in one of the  $X_\lambda$ , let  $U^X$  denote some set chosen to be open in  $X$  and such that  $U^X \cap X_\lambda = U$ . The dense, point-countable, pointed open cover of  $X$  is then given by

$$\{(x, \bigcup \{V(y, N(y) \cap U^X) : y \in U\}) : \lambda \in \Lambda, (x, U) \in \mathcal{P}_\lambda\}.$$

$\square$

G. Gruenhage has solved the point-countable base problem for GO-spaces.

**11. THEOREM** (GRUENHAGE [19∞]). *Every GO-space with  $\mathcal{W}$  satisfying open  $(G)$  has a point-countable base.*  $\square$

And P. J. NYIKOS [1986] and one of us (AWR) have established the following result (which does not use open  $(G)$ ).

**12. THEOREM.** *If  $X$  is a first countable, non-archimedean space which has  $\mathcal{W}$  satisfying  $(G)$ , then  $X$  has a point-countable base.*  $\square$

We have demonstrated that the answer to the point-countable base problem is 'yes' in a number of cases where there is extra structure for is available. Our next result demonstrates that any counter-example must be particularly unpleasant. It is a consequence of Lemma 6 and Theorem 10.

**13. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19 $\infty$ ]). *If the space  $X$  is a counterexample (i.e., has  $\mathcal{W}$  satisfying open (G) but no point-countable base) then there is a non-empty subspace  $X'$  of  $X$ , every non-empty open subset of which is also a counterexample.*  $\square$

Thus, if there is a counter-example, then it has a subspace which is a counterexample and none of whose open sets (viewed as subspaces)

- (1) have density  $\leq \aleph_1$ ,
- (2) are semi-stratifiable
- (3) are GO-spaces
- (4) are non-archimedean
- (5) or are locally countable sums of such spaces,

which excludes many common ways of constructing counter-examples.

It is worth remarking that the property of having  $\mathcal{W}$  satisfying open (G) shares a number of other properties with that of having a point-countable basis. For example, both are countably productive and both are hereditary. In [19 $\infty$ ] GRUENHAGE showed (i) that a submetacompact  $\beta$ -space with  $\mathcal{W}$  satisfying open (G) is developable (and hence has a point-countable base; this result actually generalises Theorem 7 above), and (ii) that a countably compact space with  $\mathcal{W}$  satisfying open (G) is metric.

The next result gives a little more insight into the problem.

**14. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19 $\infty$ ]). *If  $X$  is a space with  $\mathcal{W}$  satisfying (G), then  $X$  has a point-countable pointed open cover  $\mathcal{P}$  such that*

- (i)  $(x, U) \in \mathcal{P} \Rightarrow x \in U$ , and
- (ii)  $\{x : (x, U) \in \mathcal{P}\}$  is dense in  $X$ .

*Thus, any space with  $\mathcal{W}$  satisfying open (G) has a dense subspace with a point-countable base (the set in (ii)).*  $\square$

The same techniques used in the proof of Theorem 8 demonstrate that the point-countable base  $\mathcal{B}$  for the dense subset  $D$  can be lifted to a point-countable set  $\mathcal{B}'$  of subsets of  $X$  which are a basis for its topology at all points in  $D$ . However, there is no obvious way of making them into a basis for the whole of  $X$ .

The condition (G) may be strengthened to (G') as follows:

- $$(G') \quad \begin{array}{l} \text{if } x \in U \text{ and } U \text{ is open, then there exists an open} \\ V = V(x, U) \subseteq U \text{ containing } x \text{ such that } x \in W(s, y) \subseteq V \\ \text{for some } s = s(x, y, U) \in \mathbb{N} \text{ whenever } y \in V. \end{array}$$

The picture here has changed in that now  $W(s, y) \subseteq V$  rather than  $U$ . It is easy to see that the  $\mathcal{W}$  constructed earlier, for spaces with point-countable bases, satisfies (G'). In fact, it is possible to prove the following result.

**15. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19 $\infty$ ]).  *$X$  has  $\mathcal{W}$  satisfying  $(G')$  if, and only if, it has a dense, point-countable, pointed open cover.*  $\square$

Thus, if it also has  $\mathcal{W}'$  (not necessarily equal to  $\mathcal{W}$ ) satisfying open  $(G)$ , then it has a point-countable base. Unfortunately, the techniques used in the proof of Theorem 15 do not seem to generalise to the weaker condition  $(G)$ . However, that result does give rise to the following problem (an affirmative answer to which would solve the point-countable base problem):

**Problem 3.** *Does every space with  $\mathcal{W}$  satisfying  $(G)$  have a dense, point-countable, pointed open cover?* **379. ?**

Since almost all of our positive results about the point-countable base problem are consequences of Lemma 7 after constructing a dense, point-countable, pointed open cover, there is reason to believe that this problem may be best attacked via Problem 3. Direct analogues of all of Theorems 8–13 hold for Problem 3. (We have shown that Gruenhage's proof of Theorem 11 can be adapted to show that any GO-space with  $\mathcal{W}$  satisfying  $(G)$  has a dense, point-countable, pointed open cover.) The single caveat is in the case of Theorem 8, where the proof relies on first countability (implied by open  $(G)$  but not  $(G')$ ). However any first countable space with density  $\leq \aleph_1$  has a dense, point-countable, pointed open cover, as does *any* space with *cardinality*  $\leq \aleph_1$ .

We have already remarked that the property of having a point-countable, pointed open cover is rather like separability. And, like separability, it is not in general hereditary: a counterexample can be constructed by assuming  $\mathfrak{c} = \aleph_2$  and using the Sierpinski construction of a topology on the real line where a neighbourhood of a point  $x$  consists of all points within  $\epsilon > 0$  which are not less than  $x$  in an  $\omega_2$  well-order. This space does not have such a pointed open cover, but by adding the rational points of the plane in a suitable way the space becomes separable. However, if Problem 3 were to have a positive answer, then any space with  $\mathcal{W}$  satisfying  $(G)$  would have this property hereditarily. A simple modification to the proof of Theorem 1 of COLLINS, REED, ROSCOE and RUDIN [1985] shows that the property is hereditary if the space has  $\mathcal{W}$  satisfying  $(G)$  (i.e., if  $X$  has such a  $\mathcal{W}$  and a dense, point-countable pointed open cover, then so does every subspace). This is a small piece of positive evidence towards the conjecture.

### 3. Postscript: a general structuring mechanism

We have already seen that conditions (A) and (G) give a powerful structuring mechanism for topological spaces when we impose various conditions on the  $\mathbf{W}(x)$ . This mechanism can be further extended when we relax the condition that each  $\mathbf{W}(x)$  is countable. If  $\mathbf{W}(x)$  is, for each  $x$  in a space  $X$ , a set of

subsets of  $X$  containing  $x$ , we say  $\mathcal{W} = \{\mathbf{W}(x) : x \in X\}$  satisfies (F) when it satisfies

- (F) if  $x \in U$  and  $U$  is open, then there exists an open  $V = V(x, U)$  containing  $x$  such that  $x \in W \subseteq U$  for some  $W \in \mathbf{W}(y)$  whenever  $y \in V$ .

The picture here is the same as in Figure 2. Every topological space clearly has  $\mathcal{W}$  satisfying open (F), and metrisability is given when  $X$  has  $\mathcal{W}$  satisfying open decreasing (G), of which open (F) is a generalisation. Therefore, it should not surprise the reader that restrictions on  $\mathcal{W}$  satisfying (F) relate naturally to certain well-known generalised metric spaces.  $\mathcal{W}$  satisfies *chain* (F) if each  $\mathbf{W}(x)$  is a chain with respect to inclusion.

**16. THEOREM** (MOODY, REED, ROSCOE and COLLINS [19 $\infty$ ]). *If  $X$  has  $\mathcal{W}$  satisfying chain (F) and each  $\mathbf{W}(x) = \mathbf{W}_1(x) \cup \mathbf{W}_2(x)$ , where  $\mathbf{W}_1(x)$  consists of neighbourhoods of  $x$  and  $\mathbf{W}_2(x)$  is well-ordered by  $\supseteq$ , then  $X$  is paracompact.*  $\square$

Not all paracompact spaces have such  $\mathcal{W}$ .

? **380. Problem 4.** *Characterise the spaces which have  $\mathcal{W}$  satisfying chain (F), where the  $\mathbf{W}(x)$  are*

- (i) *all neighbourhoods,*
- (ii) *well-ordered by  $\supseteq$ , or*
- (iii) *as in the statement of Theorem 14.*

**17. THEOREM** (COLLINS and ROSCOE [1984]). *If  $X$  has  $\mathcal{W}$  satisfying chain (F), then  $X$  is monotonically normal* (in the sense of R. W. HEATH, D. J. LUTZER and P. L. ZENOR [1973]).  $\square$

It is possible to characterise the spaces that have chain (F): we define a space  $X$  to be *acyclically monotonically normal* if there is, for each  $x$  and open  $U$  such that  $x \in U$ , an open set  $V(x, U)$  such that

- (i)  $x \in U_1 \subseteq U_2 \Rightarrow V(x, U_1) \subseteq V(x, U_2)$
- (ii)  $x \neq y \Rightarrow V(x, X \setminus \{y\}) \cap V(y, X \setminus \{x\}) = \emptyset$
- (iii) If  $n \geq 2$ ,  $x_0, \dots, x_{n-1}$  are all distinct and  $x_n = x_0$ , then

$$\bigcap_{r=0}^{n-1} V(x_r, X \setminus \{x_{r+1}\}) = \emptyset.$$

Conditions (i) and (ii) are just the usual conditions for monotone normality, and (iii) is an extension of (ii) (notice that (ii) is just condition (iii) when  $n = 2$ ). The effect of (iii) is to ban certain types of cycles, hence the name.

**18. THEOREM.** *A space  $X$  is acyclically monotonically normal if and only if it has  $\mathcal{W}$  satisfying chain (F).*

However, we do not know if there are any monotonically normal spaces which do not have chain (F). This leads to our final problem.

**Problem 5.** *Is every monotonically normal space acyclically monotonically normal?* **381. ?**

(The definition of acyclic monotone normality, Theorem 15 and Problem 5 all first appeared in ROSCOE [1984] and were further discussed in MOODY, REED, ROSCOE and COLLINS [19∞].)

GO spaces and stratifiable spaces, the two best known classes of monotonically normal spaces, are both acyclically monotonically normal (MOODY, REED, ROSCOE and COLLINS [19∞]), as are elastic spaces (MOODY [1989]). It is known that no counter-example can be scattered. In his thesis [1989], P. J. MOODY did a considerable amount of work on this problem and proved that acyclic monotone normality has many of the same properties enjoyed by monotone normality. He showed that there is a close relationship between this problem and the problem of E. K. VAN DOUWEN [1975] of whether every monotonically normal space is  $K_0$ , since he observed that every acyclically monotonically normal space is  $K_0$ . He also showed that a counter-example exists to van Douwen's problem, and hence to ours, if there is what he terms a  $\lambda$ -Gower space (see J. VAN MILL [1982]) which is monotonically normal, for any infinite cardinal  $\lambda$ . However, it is not known if such a space exists.

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## CHAPTER 15

### Some Open Problems in Densely Homogeneous Spaces

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## 1. Introduction

This survey of open problems overlaps somewhat with a survey paper that appeared recently in the Annals of the New York Academy of Sciences (FITZPATRICK and ZHOU [1989]). Again we remark how these problems range in flavor from the geometric to the set-theoretic.

If  $X$  is a topological space, then  $\mathcal{H}(X)$  denotes the group of all autohomeomorphisms on  $X$ . The statement that  $X$  is *countable dense homogeneous* (CDH) means that

- (1)  $X$  is separable, and
- (2) if  $A$  and  $B$  are two countable dense sets in  $X$ , then there is an  $h \in \mathcal{H}(X)$  such that  $h(A) = B$ .

The statement that  $X$  is *densely homogeneous* (DH) means that

- (1)  $X$  has a  $\sigma$ -discrete dense subset, and
- (2) if  $A$  and  $B$  are two  $\sigma$ -discrete dense subsets of  $X$  then there is  $h \in \mathcal{H}(X)$  such that  $h(A) = B$ .

Here, by a  $\sigma$ -discrete set we mean the union of countably many sets, each with the relative topology, being a discrete space. The statement that  $X$  is *strongly locally homogeneous* (SLH) means that  $X$  has a basis of open sets  $\mathcal{U}$  such that if  $p$  and  $q$  are two point of  $U \in \mathcal{U}$  then there is an  $h \in \mathcal{H}(X)$  such that  $h(p) = q$  and such that  $h(x) = x$  for every  $x$  not in  $U$ .

## 2. Separation Axioms

We have previously in FITZPATRICK and ZHOU [1988] discussed countable dense homogeneity and dense homogeneity in the context of  $T_1$ -spaces. The  $T_1$ -hypothesis is redundant, that is, we have the following theorem:

**1. THEOREM** (FITZPATRICK, WHITE and ZHOU [1989]). *Every topological space that is CDH or DH is also a  $T_1$ -space.*

Note that if, instead of  $\sigma$ -discrete as defined above, we mean by  $\sigma$ -discrete the union of countably many sets, each with no limit points in the space, then there exists a space that is DH but not  $T_1$ , or even  $T_0$ . To see this, consider a sequence which converges to two different points.

One might wonder whether additional separation axioms might follow in the presence of CDH or DH. The answer is in the negative.

**2. EXAMPLE.** That a CDH space may fail to be  $T_2$  is evinced by an uncountable space with the co-finite topology.

**3. EXAMPLE.** That a CDH  $T_2$ -space may fail to be  $T_3$  is evinced by the space  $(\mathbb{R}^2, \Gamma')$  discussed in FITZPATRICK and ZHOU [1988].



**4. EXAMPLE.** That a CDH, completely regular Hausdorff space may fail to be normal is evidenced by Moore's manifold  $\Sigma_B$  (MOORE [1942]), proved in FITZPATRICK and ZHOU [1988] to be CDH.

We do not currently have examples to show that  $T_3$  does not imply  $T_{3\frac{1}{2}}$  or that  $T_4$  does not imply  $T_5$  or  $T_6$ , in the presence of CDH.

### 3. The Relationship between CDH and SLH

#### ? 382. Problem 1. *Is every connected, CDH, metric space SLH?*

The connection between CDH and SLH was established by BENNETT [1972] in his ground-breaking paper on CDH spaces. Therein, he showed that if  $X$  is locally compact, separable, metric, and SLH, then it is CDH. He did this in order to exhibit CDH connected spaces that are not manifolds; his test space was the Menger universal curve, shown by R. D. ANDERSON [1958] to be SLH.

The condition of local compactness was relaxed to that of completeness by FLETCHER and MCCOY in [1974] and by ANDERSON, CURTIS and VAN MILL in [1982]. In the nonseparable case it was proved by FITZPATRICK and LAUER in [1987] that every completely metrisable SLH space is DH.

In [1982], VAN MILL showed that connected SLH Baire spaces need not be CDH; actually, using transfinite induction, he established the existence of a connected, locally connected, Baire, SLH subspace of the plane that is not CDH, and he implicitly raised Question 1 above. Recently, W. L. SALTS-MAN [1989] has shown that the set of points in the plane whose coordinates are both rational or both irrational provides an effective example, with the same properties: connected, locally connected, Baire, SLH, not CDH. We have been told that E. K. van Douwen also knew of these properties of this set. In the nonmetric case STEPRĀNS and ZHOU [1988] have given an example of a separable manifold (therefore SLH) which is not CDH.

As noted in FITZPATRICK and ZHOU [1989], 0-dimensional spaces that are CDH or DH must be SLH.

There are only two known examples of connected CDH spaces that are not SLH: the example  $(\mathbb{R}^2, \Gamma')$  of FITZPATRICK and ZHOU [1988], and the "0-angle" space of WATSON and SIMON [19 $\infty$ ]. The latter has the advantage of being regular.

Recently, speculation as to a possible source for a metric counterexample has centered on a class of spaces introduced by F. B. JONES [1942]. There, using Hamel bases, he showed the existence of an additive, discontinuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that the graph of  $f$ , as a subspace of  $\mathbb{R}^2$ , is connected. Let us, in this discussion, call such a subspace of  $\mathbb{R}^2$  a *Jones group*, as all such are topological groups. As R. W. HEATH [1988] has observed, no Jones group can be SLH. The question is, can one be CDH? Might they all be CDH? Well,

they needn't all be CDH. Heath has shown in unpublished work that under the assumption of the Continuum Hypothesis, there is a Jones group that is not CDH. SALTSMAN [1989] has shown the same in **ZFC**. But whether there is one that *is* CDH is a vexing question.

One might think that, in the case of continua (= compact, connected metric spaces), we could get that CDH implies SLH. Even this is unknown. Recall that Bennett showed in BENNETT [1972], in this class, that SLH implies CDH. G. S. UNGAR [1978] showed that CDH continua must be  $n$ -homogeneous for all  $n$ , and, therefore, 2-homogeneous, that is, homogeneous with respect to pairs of distinct points. J. A. KENNEDY [1984] showed that a 2-homogeneous continuum  $X$  must be SLH, provided that  $X$  admits a nontrivial homeomorphism that is the identity on some nonempty open set. Whether every 2-homogeneous continuum must admit such an autohomeomorphism remains an open question.

**Problem 1'.** *Is every CDH continuum SLH?*

**383. ?**

#### 4. Open Subsets of CDH Spaces.

**Problem 2.** *If  $X$  is CDH and metric and  $U$  is open in  $X$ , must  $U$  be CDH?* **384. ?**

The question as to when open subsets of CDH spaces inherit the CDH property was raised by UNGAR [1978].

The first known results are

- (1) components of CDH spaces are CDH and are, if nontrivial, open (FITZPATRICK and LAUER [1987]), and
- (2) components of DH spaces are DH and are, if nontrivial, open (proved in the metric case in FITZPATRICK and ZHOU [1988] and in the general case in SALTSMAN [1989]).

So, some open subsets do inherit these properties. Other partial results are as follows FITZPATRICK and ZHOU [19∞].

- (3) If  $X$  is locally compact, metric, and CDH, and every dense open set in  $X$  is CDH, then so is every open subset.
- (4) If  $X$  is CDH and metric, and  $U$  is a locally compact set that is both open and closed in  $X$ , then  $U$  is CDH.
- (5) If  $X$  is CDH, compact metric, and  $\dim X \leq 1$ , and  $U$  is open in  $X$ , then  $U$  is CDH.
- (6) If  $X$  is CDH, complete metric, and  $\dim X = 0$ , and  $U$  is open in  $X$ , then  $U$  is CDH. We note that CDH can be replaced by DH in this result.

In the negative direction, the only known examples of CDH spaces with open, non-CDH subspaces, are those in FITZPATRICK and ZHOU [1988] and WATSON and SIMON [19∞] mentioned earlier as examples where CDH does

not imply SLH. They work as examples of both phenomena because they have open connected subsets that are not homogeneous, and it is known that connected CDH or SLH spaces are homogeneous, and SLH is inherited by open subsets. This leads us to ask the following.

- ? **385. Problem 2'.** *If  $X$  is connected, CDH, and metric, and  $U$  is an open, connected set in  $X$ , must  $U$  be homogeneous? If  $U$  is homogeneous, is it necessarily CDH?*

Again, this is apparently unknown even for continua.

## 5. Local Connectedness

- ? **386. Problem 3.** *If  $X$  is a CDH, connected, complete metric space, must  $X$  be locally connected?*

This is known to have an affirmative answer in case  $X$  is also locally compact (FITZPATRICK [1972]). But the only examples we know of CDH connected metric spaces are also locally connected. A possible example here is Jones group, which is, of course, far from being complete.

## 6. Cartesian Products

- ? **387. Problem 4.** *For which 0-dimensional subsets  $X$  of  $\mathbb{R}$  is  $X^\omega$  homogeneous? CDH?*

- ? **388. Problem 5.** *Is the  $\omega$ th power of the Niemytzki plane homogeneous?*

That CDH, SLH, and 2-homogeneity are not preserved under finite products, even for continua, was established by K. KUPERBERG, W. KUPERBERG and W. R. R. TRANSUE [1980]. On the other hand, it has long been known that homogeneity of CDH can be induced in infinite products. M. K. FORT [1962] proved that the product of countably infinitely many compact, connected, metric manifolds with boundary is CDH. Recently, Z. YANG [1989] has shown the same without the compactness hypothesis. D. B. MOROTOV [1985] proved that the  $\omega$ th power of any first countable, 0-dimensional, compact Hausdorff space must be homogeneous. Using Fort's techniques, it is not hard to see that the  $\omega$ th power of the long ray is homogeneous.

## 7. Completeness

**Problem 6.** *Does there exist a CDH metric space that is not completely metrisable?* **389. ?**

The answer is in the affirmative, assuming either the Continuum Hypothesis (abbreviated: **CH**) or Martin's Axiom (abbreviated: **MA**). That **CH** implies the existence of such a space was proved in FITZPATRICK and ZHOU [19 $\infty$ ]; there we obtained a connected, locally connected, Baire, homogeneous, SLH subset of the plane that is not completely metrisable. BALDWIN and BEAUDOIN [19??] have shown, assuming **MA**, that there is a CDH subset of the line that is not completely metrisable. G. Gruenhage has, assuming **CH**, obtained a CDH subspace of  $\mathbb{R}$  of universal measure 0.

**Problem 6'.** *Is there an absolute example of a CDH metric space of cardinality  $\omega_1$ ?* **390. ?**

## 8. Modifications of the Definitions.

B. Knaster called a space  $X$  *bihomogeneous* provided every two points in  $X$  can be interchanged by means of an autohomeomorphism on  $X$ . Recently, K. KUPERBERG [19 $\infty$ ] has given an example of a homogeneous continuum that is not bihomogeneous. Clearly, one could analogously define *countable dense bihomogeneity*, *dense bihomogeneity*, and *strong local bihomogeneity*, and investigate those properties in relation to the properties currently under consideration.

In 1985, at the New York Independence Day Conference on Limits, on seeing Moore's manifold  $\Sigma_B$  as an example of a CDH space that is not DH, E. K. van Douwen asked the following. Is Moore's manifold  $\Sigma_B$  homogeneous with respect to  $\sigma$ -discrete sets that are homeomorphic to one another? If it is, then is there an example of a CDH space that isn't DH in this weaker sense? Van Douwen's question has now been answered (FITZPATRICK, WHITE and ZHOU [19 $\infty$ ]). It may still be of interest to investigate this form of homogeneity and to determine conditions under which all  $\sigma$ -discrete dense sets are homeomorphic.

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## CHAPTER 16

### Large Homogeneous Compact Spaces

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## 1. The Problem

Throughout this paper, “space” always means “Hausdorff space”. Most of the spaces considered are compact. We use  $2$  to denote the 2-point space,  $\{0, 1\}$ , with the (necessarily) discrete topology, and  $[0, 1]$  to denote the unit interval with the usual topology. The *cellularity* of a space is the supremum of all sizes of families of disjoint open sets in the space; thus, a space is **ccc** iff its cellularity is  $\aleph_0$  or less. The *weight* of a space is the least cardinality of a basis, and the *character* of a point in a space is the least cardinality of a local base at that point.

A space,  $X$ , is called *homogeneous* iff for any  $x, y \in X$ , there is a homeomorphism,  $h$ , of  $X$  onto  $X$ , such that  $h(x) = y$ . For example,  $2^\kappa$  (with the product topology) is homogeneous for any cardinal  $\kappa$ . Thus, homogeneous compact spaces can be made arbitrarily large in the sense of the cardinal functions cardinality, weight, and character; however the spaces  $2^\kappa$  are small in the sense of cellularity, since they all are **ccc**. There are also non-**ccc** compact homogeneous spaces; for example, let  $X$  be the space  $2^\gamma$  for an ordinal  $\gamma$ , where now  $X$  has the order topology, using lexicographical order. By MAURICE [1964], if  $\gamma$  is countable and indecomposable ( $\forall \alpha < \gamma (\alpha + \gamma = \gamma)$ ), then  $X$  is homogeneous. If  $\gamma = \omega$ , then  $X$  is just the Cantor set, but if  $\gamma > \omega$  (for example,  $\gamma$  is the ordinal  $\omega^\omega$ ), then  $X$  has cellularity  $2^{\aleph_0}$ . However, since  $\gamma$  really must be countable here,  $2^{\aleph_0}$  is the largest cellularity obtainable by this method. Thus, a natural question, posed first by van Douwen, is:

**Problem.** *Is there a compact homogeneous space with cellularity greater than  $2^{\aleph_0}$ ?* 391. ?

Perhaps this question depends on the axioms of set theory, although as far as I know, the problem is open under any axioms. In the following, I survey some partial results, with the hope that it might be helpful in the solution. For a much more detailed survey, see ARKHANGEL'SKIĬ [1987].

If the answer is “yes”, you need a non-trivial technique for producing homogeneous spaces; unfortunately, I don't know of any that seem helpful here. There are a number of results on products. Trivially, the product of homogeneous spaces is homogeneous. It is true (KELLER [1931]), and non-trivial, that  $[0, 1]^\kappa$  is homogeneous for any infinite  $\kappa$ , but these all have the **ccc**. Likewise, Motorov showed that  $X^\kappa$  is homogeneous whenever  $\kappa$  is infinite and  $X$  is compact, first countable, and 0-dimensional, but the cellularity of such a product cannot exceed  $2^{\aleph_0}$ . It is easy to produce homogeneous Boolean algebras, but their Stone spaces are not in general homogeneous (see, e.g., below). Ordered spaces won't work; in fact, every homogeneous compact LOTS is first countable (and hence has cardinality (and hence cellularity) no more than  $2^{\aleph_0}$ ). For the proof, observe, by taking a nested sequence of intervals,



that every compact LOTS contains either a  $P$ -point or a point of countable character. Thus, if  $X$  is a homogeneous compact LOTS, then either every point is a  $P$ -point (so  $X$  is finite), or every point has character  $\aleph_0$ .

If the answer is “no”, you need a non-trivial technique for proving compact spaces nonhomogeneous. One such technique, due independently to M. E. Rudin and Z. Frolík, establishes that  $\mathbb{N}^*$  is not homogeneous. Here,  $\mathbb{N}$  denotes  $\omega$  with the discrete topology,  $\beta\mathbb{N}$  is its Čech compactification, and  $\mathbb{N}^*$  is the remainder,  $\beta\mathbb{N} \setminus \mathbb{N}$ . We identify each point  $p \in \mathbb{N}^*$  with a non-principal ultrafilter. If  $x_n$  ( $n \in \omega$ ) are points in any compact space,  $X$ , and  $p \in \mathbb{N}^*$ , we define the  $p$ -limit,  $\lim_p \langle x_n : n \in \omega \rangle$ , to be the (unique)  $y \in X$  such that for each neighborhood,  $U$ , of  $y$ ,  $\{n : x_n \in U\} \in p$  (viewing  $p$  as an ultrafilter). Then, in  $\mathbb{N}^*$ , they showed that for any such  $p$ ,  $p$  is not a  $p$ -limit of any discrete sequence of points. Thus, taking  $y$  to be a  $p$ -limit of a discrete sequence, no homeomorphism of  $\mathbb{N}^*$  can move  $p$  to  $y$ .

The property which distinguishes  $p$  from  $y$  in the above proof (being a  $p$ -limit of a discrete sequence) is a little complicated. Under the Continuum Hypothesis or Martin’s Axiom, a simpler proof would be to use W. Rudin’s theorem that there is a  $P$ -point in  $\mathbb{N}^*$ ; however, Shelah showed that one cannot prove outright in **ZFC** that there is such a  $P$ -point. However, there is always a weak  $P$ -point. In general, a point,  $x \in X$  is called a *weak  $P$ -point* iff  $x$  is not a limit of any countable subset of  $X$ . I proved in KUNEN [1978] that there is a weak  $P$ -point in  $\mathbb{N}^*$ , a result which has since been greatly improved by van Mill and others; see VAN MILL [1984]. Since every infinite compact space must also contain points which are not weak  $P$ -points, this establishes the nonhomogeneity of  $\mathbb{N}^*$  via a more quotable property.

The Rudin-Frolík proof is still of great importance, since their method applies to spaces which do not contain weak  $P$ -points. For example, one can use the study of  $p$ -limits to prove that no infinite compact  $F$ -space is homogeneous (or even stronger results—see Section 2). Here,  $X$  is called a *compact  $F$ -space* iff  $X$  is compact and in  $X$ , any two disjoint open  $F_\sigma$  sets have disjoint closures. For example,  $\mathbb{N}^*$  is a compact  $F$ -space. Not every compact  $F$ -space has weak  $P$ -points; for example, the absolute of  $[0, 1]$  (equivalently, the Stone space of the regular open algebra of  $[0, 1]$ ) is separable, and thus has no weak  $P$ -points.

The Stone space of any complete Boolean algebra is a compact  $F$ -space; this provides a large class of examples of homogeneous Boolean algebras whose Stone spaces are not homogeneous. Conversely, VAN DOUWEN [1981] has shown that there is a non-homogeneous Boolean algebra whose Stone space is homogeneous.

Returning to the problem of homogeneous compact spaces with large cellularity, it might be hoped that one might do this by taking products, in analogy to the results mentioned earlier for first countable spaces. If so, the factors must be chosen with some care as we show in Section 2, for example, no prod-

uct of infinite compact  $F$ -spaces is homogeneous. So, it now seems natural to consider the following classes of compact spaces,  $X$ , graded by successively weaker homogeneity properties.

**Class 1.**  $X$  is homogeneous.

**Class 2.** For some compact  $Y$ ,  $X \times Y$  is homogeneous.

**Class 3.**  $X$  is a retract of a compact homogeneous space.

**Class 4.**  $X$  is a continuous image of a compact homogeneous space.

Perhaps Class 4 contains all compact spaces. Of course, if it contains any compact space of cellularity greater than  $2^{\aleph_0}$ , then the answer to the Problem is “yes”.

Class 3 does not contain all compact spaces. If  $X$  is the closure in the plane of the graph of  $\sin(1/x)$ ,  $x \in (0, 1]$ , then Motorov has shown, by a connectedness argument, that  $X$  is not a retract of any compact homogeneous space; see ARKHANGEL'SKIĬ [1987] for a more general result along this line. Note that this  $X$  is a continuous image of the Cantor set, and hence in Class 4.

It is not clear whether Classes 2 and 3 are distinct, or whether they contain all compact 0-dimensional spaces. By Motorov's previously mentioned result, every compact, 0-dimensional, first countable space is in Class 2. It is also not clear whether Class 2 contains any infinite compact  $F$ -space; if so, then the proof in section 2 puts some restrictions about what the  $Y$  can be. Specifically,  $Y$  cannot be any compact LOTS or any compact metric space or any compact  $F$ -space, or any product of such spaces.

Finally, a simple sequence is a trivial example to show that Classes 1 and 2 are different, even for metric spaces.

## 2. Products

The results of this section show that any product of an infinite compact  $F$ -space with any collection of other compact  $F$ -spaces or “simple” compact spaces is not homogeneous. I am not sure what the best definition of “simple” is to get the strongest result (perhaps “simple” = “any”), but it certainly includes all sequentially compact spaces. A space,  $X$ , is called *sequentially compact* iff in  $X$ , every  $\omega$ -sequence has a convergent subsequence. So,  $2^\omega$  is sequentially compact, as is any compact metric space. Every compact LOTS is sequentially compact (just choose the subsequence to be increasing or decreasing). Many compact spaces fail to be sequentially compact; for example,  $\beta\mathbb{N}$ , or  $\mathbb{N}^*$ , or  $2^\kappa$  for any  $\kappa \geq 2^{\aleph_0}$ ; our Theorem below will, however, apply to  $2^\kappa$ , since it allows arbitrary products of sequentially compact spaces.

Martin's Axiom implies that every compact space of weight less than  $2^{\aleph_0}$  is sequentially compact. This is not provable in **ZFC**, although our Theorem will apply to such spaces anyway by the following extension, which we now discuss.

Call a space,  $X$ , *sequentially small* iff whenever  $A$  is an infinite subset

of  $X$ , there is an infinite  $B \subseteq A$  whose closure does not contain a copy of  $\beta\mathbb{N}$ . Obviously, every compact sequentially compact space and every space of weight less than  $2^{\aleph_0}$  is sequentially small. In fact, if all points of  $X$  have character less than  $2^{\aleph_0}$ , then  $X$  is sequentially small, since by a theorem of Pospíšil (see VAN MILL [1984]),  $\beta\mathbb{N}$  must contain a point of character  $2^{\aleph_0}$ . No infinite compact  $F$ -space can be sequentially small, since in such a space the closure of every infinite set contains a copy of  $\beta\mathbb{N}$ .

Then, our result is:

**1. THEOREM.** *Suppose  $X = \prod_{\alpha < \kappa} X_\alpha$ , where each  $X_\alpha$  is either an infinite compact  $F$ -space or contains a weak  $P$ -point or has a non-empty sequentially small open subset. Suppose further that at least one  $X_\alpha$  is an infinite compact  $F$ -space. Then  $X$  is not homogeneous.*

We begin with some preliminaries on products. If we have a product,  $X = \prod_{\alpha < \kappa} X_\alpha$ , we shall use subscripts for the coordinates, and superscripts for indices of sequences in  $X$ . Thus, we might consider a sequence,  $\langle x^n : n \in \omega \rangle$  from  $X$ ; each  $x^n$  would be a  $\kappa$ -sequence  $\langle x_\alpha^n : \alpha \in \kappa \rangle$ , where  $x_\alpha^n \in X_\alpha$ . If  $S \subseteq \kappa$ , we let  $\pi_S$  be the natural projection from  $X$  onto  $\prod_{\alpha \in S} X_\alpha$ .

Now, let us look more closely at sequences. In any compact space, call the sequence  $\langle d^n : n \in \omega \rangle$  *nicely separated* iff the  $d^n$  are all distinct and there are open neighborhoods,  $U^n$  ( $n \in \omega$ ) of the  $d_n$  such that for all  $A \subseteq \omega$ ,

$$\text{cl}\left(\bigcup_{n \in A} U^n\right) \cap \text{cl}\left(\bigcup_{n \notin A} U^n\right) = \emptyset.$$

We say that the  $U^n$  *nicely separate* the  $d^n$ . Nicely separated is the “opposite” of being convergent; it implies that the closure of  $\{d^n : n \in \omega\}$  is homeomorphic to  $\beta\mathbb{N}$ . If  $X$  is an  $F$ -space, every discrete  $\omega$ -sequence is nicely separated—just take the  $U^n$  to be disjoint open  $F_\sigma$  sets.

By the next lemma, in a product space, “nicely separated” depends only on countable subproducts.

**2. LEMMA.** *Suppose  $X = \prod_{\alpha < \kappa} X_\alpha$ , where each  $X_\alpha$  is compact, and suppose each  $d^n \in X$  ( $n \in \omega$ ). Then in the following, (a)  $\Rightarrow$  (b) and (b)  $\iff$  (c):*

- (a) *For some  $\alpha \in \kappa$ ,  $\langle d_\alpha^n : n \in \omega \rangle$  is nicely separated in  $X_\alpha$ .*
- (b) *For some countable  $S \subseteq \kappa$ ,  $\langle \pi_S(d^n) : n \in \omega \rangle$  is nicely separated in  $\prod_{\alpha \in S} X_\alpha$ .*
- (c)  *$\langle d^n : n \in \omega \rangle$  is nicely separated in  $X$ .*

**PROOF.** It is easy to see that (a)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (c); just pull back the nicely separating neighborhoods. To show (c)  $\Rightarrow$  (b), let the  $U^n$  nicely separate the  $d^n$ . We may, by shrinking them if necessary, assume that each  $U^n$  is an open  $F_\sigma$  set; say  $U^n = \bigcup_k F^{n,k}$ , where each  $F^{n,k}$  is closed. But then, by a standard compactness argument, each  $U^n$  is a cylinder over a countable set of co-ordinates. That is, each  $F^{n,k}$  is covered by finitely many basic subsets of

$U^n$ ; taking the union of the supports of these basic sets for all  $n, k$  produces a countable  $S \subseteq \kappa$  such that each  $U^n = \pi_S^{-1}\pi_S(U^n)$ . Then the  $\pi_S(d^n)$  are nicely separated by the  $\pi_S(U^n)$ .  $\square$

We remark that in Lemma 2, it is easy to produce examples where (b) does not imply (a), even when  $\kappa = 2$ .

Next, we look more closely at weak  $P$ -points in  $\mathbb{N}^*$ . As before, we identify  $\beta\mathbb{N}$  with the set of ultrafilters on  $\omega$  and  $\mathbb{N}^*$  with the set of non-principal ultrafilters. If  $p \in \beta\mathbb{N}$  and  $\phi: \omega \rightarrow \omega$ , we define  $\phi_*(p) \in \beta\mathbb{N}$  by:  $A \in \phi_*(p) \iff \phi^{-1}(A) \in p$ . If  $p, q \in \mathbb{N}^*$ , we define  $p \leq q$  iff there is a function,  $\phi: \omega \rightarrow \omega$  such that  $p = \phi_*(q)$ . It is easy to verify that  $\leq$  is transitive and reflexive. This partial order is due to M. E. Rudin and H. J. Keisler. We call  $p$  and  $q$  *Rudin-Keisler incomparable* iff  $p \not\leq q$  and  $q \not\leq p$ . Then we quote the following lemma from KUNEN [1978]:

**3. LEMMA.** *There are weak  $P$ -points,  $p$  and  $q$  in  $\mathbb{N}^*$  such that  $p$  and  $q$  are Rudin-Keisler incomparable.*  $\square$

The next lemma uses the method of M. E. Rudin and Z. Frolík to show that in a compact  $F$ -space, a  $p$ -limit of a *discrete* sequence cannot be a  $q$ -limit of *any* sequence, except in the trivial case of a constant sequence.

**4. LEMMA.** *Suppose  $p, q \in \mathbb{N}^*$  are weak  $P$ -points and are Rudin-Keisler incomparable. Let  $X$  be any compact  $F$ -space. In  $X$ , let  $\langle d^m : m \in \omega \rangle$  be a discrete sequence of distinct points, and  $\langle e^n : n \in \omega \rangle$  any sequence of points (possibly not distinct). Suppose that  $x = \lim_p \langle d^m : m \in \omega \rangle = \lim_q \langle e^n : n \in \omega \rangle$ . Then  $\{n : e^n = x\} \in q$ .*

PROOF. Let  $K$  be the closure of  $\{d_m : m \in \omega\}$ , and  $K^* = K \setminus \{d_m : m \in \omega\}$ . Choose open sets,  $U^m$  ( $m \in \omega$ ) so that each  $d^m \in U^m$ , and the  $U^m$  are disjoint from  $K^*$  and from each other. Let  $A = \{n : e^n \in K^*\}$ ,  $B = \{n : e^n \in \bigcup_{m \in \omega} U^m\}$ , and  $C = \omega \setminus (A \cup B)$ . Then one of  $A, B, C$  is in  $q$ , resulting in three cases; the second and third will lead to contradictions.

Case 1:  $A \in q$ : Define a map  $f$  on  $\mathbb{N}$  by  $f(n) = d^n$ . Then, by the properties of the Čech compactification,  $f$  extends to a map, which we also call  $f$ , from  $\beta\mathbb{N}$  onto  $K$ . Since  $X$  is an  $F$ -space,  $f$  is a homeomorphism, let  $g$  be its inverse. If  $n \in A$ , then  $g(e^n) \in \mathbb{N}^*$ , and  $g(x) = p = \lim_q \langle g(e^n) : n \in \omega \rangle$ . Since  $p$  is a weak  $P$ -point, we must have that  $\{n : e^n = x\} = \{n : g(e^n) = p\} \in q$ .

Case 2:  $B \in q$ : For  $n \in B$ , let  $\phi(n)$  be that  $m$  such that  $e^n \in U^m$ . Then  $p = \phi_*(q)$ , contradicting that  $p$  and  $q$  were Rudin-Keisler incomparable.

Case 3:  $C \in q$ : Observe that for  $n \in C$ ,  $e^n \notin K$ . By induction on  $n \in \omega$ , choose open  $F_\sigma$  sets,  $V^n$  and  $W^n$ , such that for each  $n$ ,

- (i)  $d^n \in V^n \subseteq cl(V^n) \subseteq U^n$ .
- (ii) If  $n \in C$ , then  $e^n \in W^n$ .
- (iii)  $cl(W^n)$  is disjoint from  $K$  and from  $V^i$  for all  $i \leq n$  (If  $n \notin C$  we can take  $W^n = \emptyset$ ).

(iv)  $V^n$  is disjoint from  $W^i$  for all  $i < n$ ; this is possible by (iii).

But then  $\bigcup_{n \in \omega} V^n$  and  $\bigcup_{n \in \omega} W^n$  are disjoint open  $F_\sigma$  sets and  $x$  is in the closure of both of them (by (ii)), contradicting that  $X$  is an  $F$ -space.  $\square$

Of course, it follows that no infinite compact  $F$ -space can be homogeneous, since a  $p$ -limit of a discrete sequence cannot be moved to the  $q$ -limit of any discrete sequence by a homeomorphism. The two distinct “types” of points we produce, “ $p$ -limit of discrete sequence” and “ $q$ -limit of discrete sequence”, are not very “quotable”, but the types produced in the proof of our main Theorem will be even less quotable. Lemma 4 will be false in a product of two compact  $F$ -spaces, since in such a product a  $p$ -limit of a discrete sequence along the  $x$ -axis can equal a  $q$ -limit of a discrete sequence along the  $y$ -axis. Thus, a more complex argument is needed to refute the homogeneity of such a product. First, we need to quote one more lemma, due to MALYKHIN [1979].

**5. LEMMA.** *If  $\beta\mathbb{N}$  is embeddable in  $\prod_{i \in \omega} X_i$ , then  $\beta\mathbb{N}$  is embeddable in at least one  $X_i$ .*  $\square$

PROOF. (Theorem 1) The hypotheses on the  $X_\alpha$  are not mutually exclusive, but partition  $\kappa$  arbitrarily into 3 subsets,  $R$ ,  $S$  and  $T$ , such that  $R \neq \emptyset$ , and such that each  $X_\alpha$  is an infinite compact  $F$ -space for  $\alpha \in R$ , contains a weak  $P$ -point for  $\alpha \in S$ , and contains a non-empty sequentially small open set for  $\alpha \in T$ .

Choose  $d^n \in X$  for  $n \in \omega$  as follows: For each  $\alpha \in R$  let  $\langle d_\alpha^n : n \in \omega \rangle$  be a discrete sequence in  $X_\alpha$ . For each  $\alpha \in S$  let the  $d_\alpha^n$  be all the same weak  $P$ -point in  $X_\alpha$ . For each  $\alpha \in T$  let  $U_\alpha$  be a non-empty open subset of  $X_\alpha$  whose closure is sequentially small, and let the  $d_\alpha^n$  be all the same element of  $U_\alpha$ .

Let  $p$  and  $q$  be as in Lemma 3. Let  $x = \lim_p \langle d^m : m \in \omega \rangle$  and  $y = \lim_q \langle d^m : m \in \omega \rangle$ . We assume that  $h$  is a homeomorphism of  $X$  with  $h(y) = x$ , and derive a contradiction.

Let  $e^n = h(d^n)$ . The  $d^n$  are nicely separated in  $X$  since  $R \neq \emptyset$ ; thus the  $e^n$  are also nicely separated in  $X$ . Applying Lemma 1, fix a countable  $J \subseteq \kappa$  such that  $\langle \pi_J(e^n) : n \in \omega \rangle$  is nicely separated in  $\prod_{\alpha \in J} X_\alpha$ .

Observe that for each  $\alpha$ ,  $x_\alpha = \lim_q \langle e_\alpha^m : m \in \omega \rangle = \lim_p \langle d_\alpha^m : m \in \omega \rangle$ . We consider the three kinds of spaces  $X_\alpha$  separately.

First, for each  $\alpha \in J \cap R$ , we may apply Lemma 4 to  $X_\alpha$  and choose an  $A_\alpha \in q$  so that  $e_\alpha^m = x_\alpha$  for all  $m \in A_\alpha$ . We may do likewise for each  $\alpha \in J \cap S$ , using the fact that  $x_\alpha$  is a weak  $P$ -point. Since  $J \cap (R \cup S)$  is countable, we may now choose an infinite  $B$  which is almost contained in each of these  $A_\alpha$ ; so, for these  $\alpha$ ,  $e_\alpha^m = x_\alpha$  for all but at most finitely many  $m \in B$ . So, for these  $\alpha$ ,  $\{e_\alpha^m : m \in B\}$  is finite. We are not claiming that  $B \in q$ .

Next, for those  $\alpha \in J \cap T$ , apply the definition of “sequentially small”  $\omega$  times and diagonalize to get an infinite  $D \subseteq B$  so that for each such  $\alpha$ ,  $cl(\{e_\alpha^m : m \in D\})$  does not embed  $\beta\mathbb{N}$ . (At each application, one first tries

to find a subsequence where the  $e_\alpha^m$  are distinct; if this is impossible, then  $\{e_\alpha^m : m \in D\}$  will be finite.)

For each  $\alpha \in J$ , let  $P_\alpha = cl(\{e_\alpha^m : m \in D\})$ . Then each  $P_\alpha$  does not embed  $\beta\mathbb{N}$ , so neither does  $\prod_{\alpha \in J} P_\alpha$  by Lemma 5. But this is a contradiction; the product contains the closure of  $\{\pi_J(e^n) : n \in D\}$ , which is homeomorphic to  $\beta\mathbb{N}$ , since the points are nicely separated.  $\square$

If one assumes the Continuum Hypothesis or Martin's Axiom, some of the minor steps along the way to the proof can be strengthened. Now, one can take  $p$  and  $q$  to be *selective*; i.e., minimal in the Rudin-Keisler order. In the proof of the Theorem, we can then indeed take  $B \in q$ . Also, in Lemma 4, we do not have to assume that the sequence  $\langle d^m : m \in \omega \rangle$  is discrete; it is enough to assume the  $d^m$  are distinct, since then there will be an  $E \in p$  such that  $\langle d^m : m \in E \rangle$  is discrete anyway. However, it is not clear whether the Theorem itself can be strengthened in any essential way.

The Theorem overlaps somewhat with results of VAN DOUWEN [1978], who established nonhomogeneity by a different method—considering cardinal functions rather than limit points. He showed, for example, that if  $X$  is compact and  $|X| > 2^{\pi(X)}$ , then no power of  $X$  is homogeneous. So, this gives a different proof that no power of  $\beta\mathbb{N}$  is homogeneous.

Another method, emphasizing limit points again, also may be used to establish nonhomogeneity of products in some cases. Call a *strict*  $F_{\omega_1}$  any union of the form  $\bigcup\{U_\xi : \xi < \omega_1\}$ , where each  $U_\xi$  is open and contains the closure of  $\bigcup\{U_\eta : \eta < \xi\}$ . Call  $x$  an  $L$ -point of  $X$  iff  $x$  is in the boundary of a strict  $F_{\omega_1}$ . Observe that if  $X$  is a product of spaces,  $X_\alpha$ , then  $x$  is an  $L$ -point of  $X$  iff some  $x_\alpha$  is an  $L$ -point of  $X_\alpha$ . Thus, if each  $X_\alpha$  contains a non- $L$ -point and some  $X_\alpha$  contains an  $L$ -point, then the product is not homogeneous. So, for example, the ordinal  $\omega_1 + 1$  cross any product of **ccc** spaces and spaces containing a point of countable tightness is nonhomogeneous.

Yet another nonhomogeneity result is due to DOW and VAN MILL [1980], who showed that no compact space can be covered by nowhere dense **ccc**  $P$ -sets. Thus, if  $X$  is compact and contains a non-isolated  $P$ -point, then  $X$  cross any compact **ccc** space is not homogeneous. This applies to  $X = \omega_1 + 1$ , although here, the “ $L$ -point” argument gives a stronger result. More interestingly, it applies when  $X$  is the LOTS,  $2^{\omega_1}$ , ordered lexicographically; here, every point is an  $L$ -point, so that the “ $L$ -point” argument says nothing.

It seems that the state of the art on homogeneity of products can be summarized by saying that there are a lot of special results, without any unifying theme emerging yet.

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## CHAPTER 17

### Some Problems

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#### *Contents*

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## 0. Introduction

This note collects some open questions on three separate topics. Sections 1 and 2 are concerned with different aspects of compact-covering maps, and Section 3 with continuous selections. All our questions deal with fairly basic, intuitive concepts, and the questions in Section 1 are of interest even for subsets of the plane.

### 1. Inductively perfect maps, compact-covering maps, and countable-compact-covering maps

Recall that a surjective map  $f: X \rightarrow Y$  (all *maps* are continuous) is *inductively perfect* if there is an  $X' \subset X$  such that  $f(X') = Y$  and  $f|X'$  is perfect. (If  $X$  is Hausdorff, this  $X'$  must be closed in  $X$ ). Since the inverse image under a perfect map of a compact set is always compact, every inductively perfect map  $f: X \rightarrow Y$  must be *compact-covering* (i.e., every compact  $K \subset Y$  is the image of some compact  $C \subset X$ ), and clearly every compact-covering  $f: X \rightarrow Y$  is *countable-compact-covering* (i.e., every *countable* compact  $K \subset Y$  is the image of some compact (not necessarily countable)  $C \subset X$ ).

The following two questions were raised in MICHAEL [1981b], where their background and motivation are explained. In essence, negative answers to these questions would provide negative answers to two rather natural questions about whether two theorems about open maps remain valid for the larger class of tri-quotient maps introduced in MICHAEL [1977]. Unlike these motivating questions, however, Questions 1.1 and 1.2 do not involve tri-quotient maps and are entirely about elementary, standard concepts.

**Question 1.1.** *Let  $f: X \rightarrow Y$  be a map from a separable metrizable space  $X$  onto a metrizable space  $Y$ , with each  $f^{-1}(y)$  compact.* **392. ?**

- (a) *If  $f$  is countable-compact-covering, must  $f$  be compact-covering?*
- (b) *If  $f$  is compact-covering, must  $f$  be inductively perfect?*

**Question 1.2.** *Let  $f: X \rightarrow Y$  be a map from a separable metrizable space  $X$  onto a countable metrizable space  $Y$ . If  $f$  is compact-covering, must  $f$  be inductively perfect?* **393. ?**

The following remarks may be helpful.

**1.3.** I don't know the answer to Question 1.1 even when  $Y \subset \mathbb{I}$  (or even  $Y = \mathbb{I}$  for 1.1(a)),  $X \subset Y \times \mathbb{I}$ , and  $f(y, t) = y$ .

**1.4.** I don't know the answer to Question 1.2 even when  $Y = \mathbb{Q}$  (rationals),  $X \subset Y \times \mathbb{I}$ , and  $f(y, t) = y$ .

**1.5.** The answer to both parts of Question 1.1 is "yes" if  $f$  is open, or, more generally, if  $f(U)$  is a  $G_\delta$  in  $Y$  for every open  $U$  in  $X$ . (Indeed, every open

map  $f: X \rightarrow Y$  from a metric space  $X$  onto a paracompact space  $Y$ , with every  $f^{-1}(y)$  complete in the given metric on  $X$ , must be inductively perfect (MICHAEL [1959b, Corollary 1.2]). If  $Y$  is metrizable, this remains true with “open map  $f: X \rightarrow Y$ ” weakened to “tri-quotient map  $f: X \rightarrow Y$  with  $f(U)$  a  $G_\delta$  in  $Y$  for every open  $U \subset X$ ” (OSTROVSKY [1986, Theorem 2])).

**1.6.** The answer to Question 1.2 is “yes” if  $f$  is open. (Indeed, every open map from a first-countable space  $X$  onto a countable regular space  $Y$  has a cross-section (MICHAEL [1981a, Theorem 1.1])).

**1.7.** The answer to both parts of Question 1.1 is “yes” if  $X$  is completely metrizable. (Indeed, every countable-compact-covering map  $f: X \rightarrow Y$  from a completely metrizable space  $X$  onto a paracompact space  $Y$ , with each  $f^{-1}(y)$  separable, is inductively perfect (MICHAEL [1977, Theorems 1.6 and 6.5(a) and Remark 5.3])).

**1.8.** The answer to Question 1.2 is “yes” if each  $f^{-1}(y)$  is completely metrizable (and thus the answer to Question 1.1(b) is “yes” if  $Y$  is countable). (Indeed, every compact-covering map  $f: X \rightarrow Y$  from a metrizable space  $X$  onto a countable regular space  $Y$ , with each  $f^{-1}(y)$  separable and completely metrizable, is inductively perfect (MICHAEL [1981b, 1.2(c) and Theorem 1.4])).

**1.9.** The answer to both parts of Question 1.1 becomes “no” if the sets  $f^{-1}(y)$  are not assumed to be compact (or at least complete), even if  $f$  is open. For 1.1(a) this follows from MICHAEL [1959b, Example 4.1], and for 1.1(b) from MICHAEL [1977, Example 9.7].

**1.10.** The answer to Question 1.1 becomes “no” if “countable-compact-covering” is weakened to “sequence-covering” (in the sense that every convergent sequence, including its limit, in  $Y$  is the image of some compact set (not necessarily a convergent sequence)  $C \subset X$ ). See MICHAEL [1979, (3)].

## 2. Quotient $s$ -maps and compact-covering maps

Recall that a map  $f: X \rightarrow Y$  is an  $s$ -map if every  $f^{-1}(y)$  is separable. Compact-covering maps were defined in Section 1.

**? 394. Question 2.1.** (MICHAEL and NAGAMI [1973, Problem 1.5]) *If a Hausdorff space  $Y$  is a quotient  $s$ -image of a metric space, must  $Y$  also be a compact-covering quotient  $s$ -image of a (possibly different) metric space?*

The following remarks will help to explain the origin of this question.

**2.2.** The answer to Question 2.1 is “yes” if “quotient” is strengthened to “open” in both hypothesis and conclusion (MICHAEL and NAGAMI [1973, Theorem 1.4]).

**2.3.** The answer to Question 2.1 is “yes” if “ $s$ -image of a metric space” is strengthened to “image of a separable metric space” in both hypothesis and conclusion (MICHAEL [1966, Theorem 11.4 and Corollary 11.5]).

**2.4.** The answer to Question 2.1 is “yes” if “compact-covering” is weakened to “sequence-covering” in the sense of (1.10) above (GRUENHAGE, MICHAEL and TANAKA [1984, Theorem 6.1 (b)  $\rightarrow$  (a)]).

**2.5.** For a positive answer to Question 2.1, it would suffice to conclude from the hypotheses that  $Y$  must be a compact-covering  $s$ -image of a metric space; indeed, since  $Y$  is Hausdorff and (being a quotient image of a metric space) a  $k$ -space, any compact-covering map onto  $Y$  must be quotient. (That  $Y$  must be a compact-covering *image* (rather than  $s$ -image) of a metric space follows from MICHAEL and NAGAMI [1973, Theorem 1.1] and FILLIPOV [1969, Corollary 3]).

### 3. Continuous selections

The questions in this section are all related to the following result.

**3.1. THEOREM** (MICHAEL [1956c, Theorem 1] and [1956a, Theorem 3.2’]). *Let  $X$  be paracompact,  $Y$  a Banach space,  $K \subset Y$  convex and closed, and  $\varphi: X \rightarrow \mathcal{F}_c(K)$  l.s.c. Then  $\varphi$  has a selection.*

In the above theorem,  $\mathcal{F}_c(K) = \{E \subset K: E \neq \emptyset, E \text{ closed in } K \text{ and convex}\}$ ,  $\varphi: X \rightarrow \mathcal{F}_c(K)$  is l.s.c. if  $\{x \in X: \varphi(x) \cap V \neq \emptyset\}$  is open in  $X$  for every open  $V$  in  $K$ , and  $f: X \rightarrow K$  is a *selection* for  $\varphi$  if  $f$  is continuous and  $f(x) \in \varphi(x)$  for every  $x \in X$ .

Theorem 3.1 remains true, with essentially the same proof, if  $Y$  is only a complete, metrizable, locally convex topological linear space. More generally, and with rather more effort, one can show that it suffices if  $Y$  is only a complete metric space with a suitably defined “convex structure”; see MICHAEL [1959a] and a recent improvement by D. W. CURTIS in [1985]. That is as far as our knowledge extends in this direction; without dimensional restrictions on  $X$ , no way has been found to significantly weaken these rigid convexity requirements on  $Y$ . (By contrast, if  $X$  is *finite-dimensional*, then there are purely topological conditions on the sets  $\varphi(x)$  which are not only sufficient but also necessary (MICHAEL [1956b, Theorem 1.2]). It is hoped that answers to the following questions may shed additional light on various aspects of this problem; my conjecture is that all three answers are negative.

**Question 3.2.** *Let  $X$  be paracompact,  $Y$  an infinite-dimensional Banach space, and  $\varphi: X \rightarrow \mathcal{F}_c(Y)$  l.s.c. with each  $\varphi(x)$  a linear subspace of deficiency one (or of finite deficiency) in  $Y$ . Must  $\varphi$  have a selection  $f$  such that  $f(x) \neq 0$  for every  $x \in X$ ?* **395. ?**

Before turning to our next question, observe that the assumption in Theorem 3.1 that  $K$  is closed in  $Y$  cannot simply be omitted, even if  $X = \mathbb{I}$  (MICHAEL [1956a, Example 6.2]).

- ? **396. Question 3.3.** (MICHAEL [1988b]) *Does Theorem 3.1 remain true if  $K$  is only assumed to be a convex  $G_\delta$  subset of  $Y$ ?*
- ? **397. Question 3.4.** *Does Theorem 3.1 remain true if  $Y$  is only assumed to be a complete, metrizable (but not necessarily locally convex) topological linear space?*

The following remarks may be helpful.

**3.5.** All three questions are open even if  $X$  is a compact metric space.

**3.6.** The answers to all three questions are “yes” if  $\dim X < \infty$ ; see MICHAEL [1956b, Theorem 1.2] for Questions 3.2 and 3.4, and SAINT-RAYMOND [1984] or MICHAEL [1988a, Theorem 1.3] for Question 3.3.

**3.7.** The answer to Question 3.2 becomes “no” if the sets  $\varphi(x)$  are only assumed to be infinite-dimensional, closed linear subspaces of  $Y$ ; this result, which answers MICHAEL [1988b, Question 2], follows from a recent example obtained independently by DRANISHNIKOV [1988] and by TORUŃCZYK and WEST [1989]. (A somewhat simpler example, where the sets  $\varphi(x)$  are only infinite-dimensional, closed convex subsets of  $Y$ , can be found in MICHAEL [1988a, Example 10.2]).

**3.8.** It follows from MICHAEL [1988a, Theorem 1.4] that the answer to Question 3.2 is “yes” if the function  $\psi: X \rightarrow \mathcal{F}_c(Y)$ , defined by  $\psi(x) = \varphi(x) \cap \{y \in Y: \|y\| \leq 1\}$ , is *continuous* with respect to the Hausdorff metric on  $\mathcal{F}_c(Y)$  (rather than merely l.s.c.). In fact, this remains true even if the sets  $\varphi(x)$  are only assumed to be infinite-dimensional, closed linear subspaces of  $Y$ .

**3.9.** It follows from (3.8) that, if  $B$  is an infinite-dimensional Banach space and if  $X = B^* \setminus \{0\}$  and  $Y = B \setminus \{0\}$  (both with the norm topology), then there exists a continuous  $f: X \rightarrow Y$  such that  $u(f(u)) = 0$  for every  $u \in X$  (MICHAEL [1988a, Theorem 1.5]). If the answer to Question 3.2 is “yes”, then such an  $f$  exists even when  $X$  carries the weak\* topology and  $Y$  the norm topology.

**3.10.** The usual proof of Theorem 3.1 (see MICHAEL [1956c] or [1956a]) depends on the existence of a metric on  $K$  (the one obtained from the norm) which is complete and which “relates well” to the natural convex structure on  $K$ . Under the weaker hypotheses of Question 3.3, however, there exist metrics on  $K$  satisfying either of these two requirements but apparently no

metric satisfying both of them. That helps to explain the difficulty in trying to obtain a positive answer to this question.

**3.11.** The answer to Question 3.3 is “yes” if  $(\text{conv } C)^- \subset K$  (closure in  $Y$ ) whenever  $C$  is a compact subset of some  $\varphi(x)$  (by MICHAEL [1959b, Theorem 1.1] and [1956a, Propositions 2.6 and 2.3 and Theorem 3.2’]). While this condition is not always satisfied under the hypotheses of Question 3.3, it is satisfied, for example, if  $K$  is the intersection of open convex subsets of  $Y$  or if  $\dim \varphi(x) < \infty$  for every  $x \in X$ .

**3.12.** Question 3.4 is related to the following old problem: Let  $Y$  be a metrizable topological linear space. Must  $Y$  (or even every convex subset of  $Y$ ) be an absolute retract? (Some discussions which relate to this problem can be found in KLEE [1960b, 1960a] and DUGUNDJI [1965]).

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## CHAPTER 18

### **Questions in Dimension Theory**

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The questions we shall consider, with two significant exceptions—Questions 4 and 5, are problems in general topology in the sense that their solutions would probably avoid either homological or deep geometrical methods. The aim of our note is very limited. We would like to recall some natural questions in dimension theory (all well-known to specialists and a few of them classical) which appeal by the simplicity of the statement and illustrate the diversity of the subject. We refer the reader to the articles by V. V. FEDORCHUK, V. V. FILIPPOV and B. A. PASYNKOV [1979], V. V. FEDORCHUK [1988], B. A. PASYNKOV [1985], R. ENGELKING and E. POL [1983] and the surveys by J. NAGATA [1967, 1983c, 1983b], where many other interesting problems and extensive bibliographies of the topic can be found. An excellent exposition of homological dimension theory, not discussed here, is given by A. I. DRANISHNIKOV [1988], this survey includes his fundamental recent results. Many problems in continua theory are concerned with dimension; we refer the reader to the article by W. LEWIS [1983] for that matter.

Our terminology follows NAGATA [1983a]. For a completely regular space  $X$ ,  $\dim X$  is the Lebesgue covering dimension  $\dim \beta X$  of the Čech-Stone compactification of the space  $X$ , i.e.,  $\dim X \leq n$  if each finite cover of  $X$  by functionally open sets has a finite refinement by functionally open sets such that each point belongs to at most  $n + 1$  of them.

The Menger-Urysohn inductive dimension  $\text{ind}$ , i.e., the small inductive dimension, is defined as follows:  $\text{ind } X = -1$  if and only if  $X = \emptyset$ ,  $\text{ind } X \leq n$  if each point can be separated, in  $X$ , from every closed set that does not contain it, by a closed set  $L$  with  $\text{ind } L \leq n - 1$ ;  $\text{ind } X$  is the minimal  $n$  with  $\text{ind } X \leq n$ , or, if no such  $n$  exists, we say that  $X$  is infinite dimensional (cf., sec. 14)

The large inductive dimension  $\text{Ind}$ , considered for normal spaces only, is defined similarly, only now one defines  $\text{Ind } X \leq n$  if and only if each pair of disjoint closed sets can be separated, in  $X$ , by a closed set  $L$  with  $\text{Ind } L \leq n - 1$ .

A separable metrizable space  $X$  is *weakly infinite-dimensional* if for each infinite sequence  $(A_1, B_1), (A_2, B_2), \dots$  of pairs of disjoint closed sets there are closed sets  $L_i$  separating  $A_i$  from  $B_i$  in  $X$  such that  $\bigcap_{i=1}^{\infty} L_i = \emptyset$ ; otherwise we shall call  $X$  *strongly infinite-dimensional*.

Separable metrizable spaces that are countable unions of zero-dimensional subspaces are called *countable-dimensional*.

Countable-dimensional spaces (in particular, all finite-dimensional spaces) are weakly infinite-dimensional and the Hilbert cube is strongly infinite-dimensional.

A compact metrizable will be called a compactum.

The real unit interval will be denoted by  $I$ , and  $I^{\infty}$  denotes the Hilbert cube, i.e., the countable infinite product of copies of  $I$ .

**1. How large can the gap between the inductive dimensions be for non-separable metrizable spaces?** **398. ?**



Two fundamental facts in dimension theory are that  $\dim X = \operatorname{ind} X = \operatorname{Ind} X$  for any separable metrizable space  $X$  and that  $\dim X = \operatorname{Ind} X$  for any metrizable  $X$ .

- ? 399. 1.1. QUESTION. Does there exist for each natural number  $n \geq 1$  a metrizable space  $X$  with  $\operatorname{ind} X = 0$  and  $\operatorname{Ind} X = n$ ?

Spaces of this kind were defined only for  $n = 1$ , the simplest example of such a (complete, of weight  $\aleph_1$ ) space was given by J. KULESZA [19 $\infty$ ]; all known spaces illustrating this phenomenon are related to P. ROY's construction from [1968]. No examples appeared to refute the impression that for any metrizable  $X$  with  $\operatorname{ind} X = 0$  one has  $\operatorname{Ind}(X \times X) = \operatorname{Ind} X$ , see MRÓWKA [1985]. It is also an open question if there are metrizable topological groups  $G$  with  $\operatorname{ind} G < \operatorname{Ind} G$ .

- ? 400. 2. How can the basic dimensions differ for compact spaces?

This question was raised by P. S. ALEKSANDROV in [1936]. Among many examples illuminating this problem (see V. V. FEDORCHUK, V. V. FILIPPOV and B. A. PASYNKOV [1979, §2]) let us mention the following ones constructed by V. V. FILIPPOV in [1970]: for each natural number  $n \geq 1$  there exists a compact space  $F_n$  such that  $\dim F_n = 1$ ,  $\operatorname{ind} F_n = n$ , and  $\operatorname{Ind} F_n = 2n - 1$ .

- ? 401. 2.1. QUESTION Does there exist for each natural number  $n \geq 2$  a compact space  $X_n$  with  $\dim X_n = 1$ ,  $\operatorname{ind} X_n = 2$  and  $\operatorname{Ind} X_n = n$ ?

As was pointed out by FILIPPOV in [1970], a positive answer would provide for an arbitrary triple  $k < l < m$  of positive integers a compact space  $X$  with  $\dim X = k$ ,  $\operatorname{ind} X = l$  and  $\operatorname{Ind} X = m$ . It seems that Question 2.1 is open for  $n = 4$ .

A compact space  $X$  is *chainable*, if for each open cover  $\mathcal{U}$  of  $X$  there exists a continuous map  $f: X \rightarrow I$  such that each fiber  $f^{-1}(t)$  is contained in some element of  $\mathcal{U}$  (this implies  $\dim X \leq 1$ ).

- ? 402. 2.2. QUESTION. Does there exist a chainable compact space  $X$  with  $\operatorname{ind} X < \operatorname{Ind} X$ ?

A space  $X$  is *homogeneous* if for each pair of points  $x, y$  in  $X$  there exists a homeomorphism  $h: X \rightarrow X$  with  $h(x) = y$ .

- ? 403. 2.3. QUESTION. Does there exist a homogeneous compact space  $X$  with  $\operatorname{ind} X < \operatorname{Ind} X$ ?

V. A Chatyrko (see PASYNKOV [1985, p. 234 and p. 241]) constructed for each natural  $n$  a chainable compact space  $X_n$  with  $\text{ind } X_n = n$  and a homogeneous compact space  $Y_n$  with  $\dim Y_n = 1$  and  $\text{ind } Y_n = n$ .

**3.** *Can multiplication by the space of irrationals increase the dimension of a space?* **404. ?**

In [1978] M. WAGE constructed a Lindelöf space  $X$  with  $\dim X = 0$  such that for a certain subset  $B$  of the irrationals,  $\dim(X \times B) > 0$ .

**3.1. QUESTION.** *Does there exist a completely regular space  $X$  such that  $\dim(X \times \mathbb{P}) > \dim X$ , where  $\mathbb{P}$  denotes the space of irrationals?* **405. ?**

An interesting discussion of this problem is given by K. TSUDA [1985, remark 12.1]. Let us notice that if  $\dim(X \times M) > \dim X + \dim M$ , where  $X$  is countably paracompact and  $M$  is metrizable then the product  $X \times M$  is not normal, see NAGATA [1983a, p. 201].

**4.** *Does every  $k$ -dimensional subset of the  $n$ -dimensional cube  $I^n$  have a  $k$ -dimensional compactification embeddable in  $I^n$ ?* **406. ?**

This is a problem of K. Menger [1928]. By a result of SHTANKO [1971] this is equivalent to the question if for each  $k$  the  $k$ -dimensional Menger compactum  $M_k^n$  in  $I^n$  is universal for all  $k$ -dimensional subsets of  $I^n$ .

**5.** *Does there exist a cell-like map of a 2-dimensional compactum onto an infinite-dimensional one?* **407. ?**

A compactum  $K$  is *cell-like* if any continuous map of  $K$  into a polyhedron is null-homotopic. A continuous map  $f: X \rightarrow Y$  of a compactum  $X$  onto a compactum  $Y$  is cell-like if  $f^{-1}(y)$  is cell-like for all  $y \in Y$ . In [1988] DRANISHNIKOV constructed a cell-like map of a 3-dimensional compactum onto an infinite-dimensional one, thus solving a classical problem in dimension theory (no cell-like map of a one-dimensional compactum can raise the dimension see WALSH [1981, Corollary 3.3]).

It follows that there exists a cell-like map of the 7-dimensional cube  $I^7$  onto an infinite-dimensional compactum. G. KOZŁOWSKI and J. J. WALSH proved in [1983] that no cell-like map defined on  $I^3$  can raise the dimension, but for the cubes of dimensions 4, 5 or 6 this remains open.

The following theorem of E. V. Shchepin, see DRANISHNIKOV and SHCHEPIN [1986, Theorem 1 in §5] (closely related to results by R. D. Edwards, see

WALSH [1981]), provides a reformulation of the problem considered in this section.

Let  $K$  be a compactum in the Hilbert space  $\ell_2$ ; the compactum  $K$  is the range of a cell-like map defined on an  $n$ -dimensional compactum if and only if for each map  $f: L \rightarrow K$  where  $L$  is a  $(n+1)$ -dimensional compactum, and every  $\epsilon > 0$  there exists a map  $u: L \rightarrow \ell_2$  such that  $\dim(u[L]) \leq n$  and  $\|u(x) - f(x)\| < \epsilon$  for all  $x \in L$ .

- ? 408. 6. Does a countable union of zero-dimensional sets in the Hilbert cube which has positive dimension contain a one-dimensional subset?

This is a problem of L. A. TUMARKIN [1962].

Let  $p_j: I^\infty \rightarrow I$  be the projection onto the  $j$ th coordinate.

- ? 409. 6.1. QUESTION. Is it true that for any countable-dimensional set  $E \subseteq I^\infty$  there exists a decomposition  $E = E_0 \cup E_1 \cup \dots$  such that  $\dim E_0 \leq 0$ ,  $\dim E_j \leq 1$ , for  $j \geq 1$ , and for each  $j = 1, 2, \dots$  the projection  $p_j[E_j]$  is countable?

If  $E$  provides a negative answer to this question then the Continuum Hypothesis would allow one to define a subset  $M$  of  $E$  giving a negative answer to Tumarkin's problem. Let us sketch the argument.

To begin let us notice that there exists a pair  $A, B$  of disjoint closed sets in the space  $E$  and a countable set  $Q \subseteq I$  such that whenever  $D_j \subseteq E$  are at most one-dimensional sets with countable projection  $p_j[D_j]$  disjoint from  $Q$ , the union  $\bigcup_{j=1}^\infty D_j$  does not separate  $A$  from  $B$  in  $E$  (such a pair can be found in any collection  $\mathcal{A}$  of pairs of disjoint closed sets in  $E$  with the property that given  $F$  closed in  $E$  and  $p \notin F$  there is  $(C, D) \in \mathcal{A}$  with  $p \in C$  and  $F \subseteq D$ ).

By the Continuum Hypothesis, all closed sets separating  $A$  from  $B$  in  $E$  can be arranged in a sequence  $L_1, L_2, \dots, L_\alpha, \dots$ ,  $\alpha < \omega_1$ , and all at most one-dimensional  $G_\delta$ -sets in  $I^\infty$  can be arranged into a sequence  $G_1, G_2, \dots, G_\alpha, \dots$ ,  $\alpha < \omega_1$ . Let us choose by transfinite induction points  $x_\alpha \in L_\alpha$  and distinct points  $q_{j\alpha} \in I \setminus Q$  such that whenever  $p_j[G_\alpha]$  is uncountable,  $q_{j\alpha} \in p_j[G_\alpha]$ , and

$$\{x_\alpha : \alpha \leq \xi\} \cap \bigcup_{j=1}^\infty \bigcup \{p_j^{\leftarrow}(q_{j\alpha}) \cap G_\alpha : \alpha \leq \xi\} = \emptyset$$

for every  $\xi$ . The choice of the pair  $A, B$  guarantees that this procedure does not terminate at a countable stage. Let  $M = \{x_\alpha : \alpha < \omega_1\}$ . Each closed set separating  $A$  and  $B$  in  $E$  hits  $M$ , so  $M$  has positive dimension. Suppose  $N \subseteq M$  is one-dimensional. Then, for some  $j$ , the projection  $p_j[N]$  contains a nontrivial interval  $J$ , see WALSH [1979]. Let  $\alpha$  be an ordinal such that  $p_j^{\leftarrow}[J] \cap N \subseteq G_\alpha \subseteq p_j^{\leftarrow}[J]$  and consider the point  $q_{j\alpha} \in J$ . Then  $p_j^{\leftarrow}(q_{j\alpha}) \cap G_\alpha \cap M \neq \emptyset$  and we arrive at a contradiction with the choice of  $M$ .

**7.** *Does there exist a weakly infinite-dimensional compactum of positive dimension without any subcompactum of dimension one?* **410. ?**

In [1967] D. W. HENDERSON proved that each strongly infinite-dimensional compactum contains an infinite-dimensional compactum all of whose subcompacta of positive dimension are strongly infinite-dimensional (L. R. RUBIN [1980] has shown that “compacta” can be replaced by “sets”; this is related to the result by Walsh quoted after Question 8). The following two questions are related to the question we started with.

**7.1. QUESTION.** *Let  $f: S \rightarrow T$  be a continuous map of a compactum  $S$  onto a compactum  $T$  with  $\dim f^{-1}(t) = 0$  for all  $t \in T$ . If  $T$  is weakly infinite-dimensional, is this also true for  $S$ ?* **411. ?**

**7.2. QUESTION.** *If  $X$  and  $Y$  are weakly infinite-dimensional compacta, is it true that the product  $X \times Y$  is weakly infinite-dimensional?* **412. ?**

If a map  $f: S \rightarrow T$  would provide a negative answer to Question 7.1 then  $T$  would contain a compactum  $K$  giving a negative answer to Question 7.2. To see this, let us consider a countable-dimensional set  $M \subseteq T$  intersecting each one-dimensional subset of  $T$  (POL [1986]). Then, since  $f^{-1}[M]$  is a countable union of zero-dimensional sets in the strongly infinite-dimensional compactum  $S$ , there exists a nontrivial continuum  $C \subseteq S \setminus f^{-1}[M]$  and the compactum  $K = f[C]$  has the required property.

In turn, if a pair  $X, Y$  of compacta provides a negative answer to Question 7.2, then Henderson’s result quoted above yields a strongly infinite-dimensional compactum  $S$  in  $X \times Y$  such that each compactum  $(\{x\} \times Y) \cap S$  is zero-dimensional (being weakly infinite-dimensional) and hence the projection onto the first coordinate, restricted to  $S$  provides a negative answer to Question 7.1

**8.** *Does there exist an infinite-dimensional compactum whose square does not contain one-dimensional subsets?* **413. ?**

J. WALSH [1979] constructed an infinite-dimensional compactum all of whose subsets of positive dimension are strongly infinite-dimensional. The following question was asked by J. VAN MILL [1983, remark 5.7].

**8.1. QUESTION.** *Does there exist an infinite-dimensional compactum  $X$  none of whose finite powers  $X^n$  contain one-dimensional subsets?* **414. ?**

Any infinite-dimensional compactum  $K$  which is the range of a cell-like map defined on a 3-dimensional compactum (after Question 5 we quoted Dranishnikov’s result to that effect) has the property that any finite-dimensional set  $E$  in its  $n$ th power  $K^n$  has dimension at most  $3n$ .

In [1980] D. RANCHIN modified an argument by W. HUREWICZ [1932] to define, assuming the Continuum Hypothesis, a separable metrizable space  $E$  such that all uncountable subsets of every finite power  $E^n$  are infinite-dimensional.

- ? 415. **8.2. QUESTION.** *Does there exist an infinite-dimensional separable metrizable space  $M$  not containing any subset of dimension one whose square  $M \times M$  does contain a one-dimensional subset?*

It is unclear if there exists a finite-dimensional set  $E \subseteq I^\infty \times I^\infty$  such that for each countable-dimensional  $C \subseteq I^\infty$ , the set  $E \setminus (C \times I^\infty \cup I^\infty \times C)$  has positive dimension.

If such a set  $E$  indeed exists, the Continuum Hypothesis would allow one to repeat Hurewicz's argument to define a space  $M$  answering Question 8.2. Let us indicate this argument:

The assumption about  $E$  yields the existence of a point  $p \in E$  and a closed subset  $F$  of  $E$  not containing  $p$ , such that no closed set separating  $p$  from  $F$  in  $E$  can be a subset of a set of the form  $C \times I^\infty \cup I^\infty \times C$  with  $C \subseteq I^\infty$  countable-dimensional.

Let us arrange, using the Continuum Hypothesis, all closed sets separating  $p$  from  $F$  in  $E$  in a sequence  $L_1, L_2, \dots, L_\alpha, \dots, \alpha < \omega_1$  and all zero-dimensional  $G_\delta$ -sets in  $I^\infty$  in a sequence  $G_1, G_2, \dots, G_\alpha, \dots, \alpha < \omega_1$ . For each  $\alpha < \omega_1$  one can choose a point  $\langle x_\alpha, y_\alpha \rangle \in L_\alpha \setminus (C_\alpha \times I^\infty \cup I^\infty \times C_\alpha)$ , where  $C_\alpha = \bigcup_{\beta \leq \alpha} G_\beta$ , and let  $M = \{x_\alpha : \alpha < \omega_1\} \cup \{y_\alpha : \alpha < \omega_1\}$ .

Since  $M$  intersects each  $G_\alpha$  in an at most countable set, the uncountable subsets of  $M$  are infinite-dimensional. On the other hand,  $S = \{\langle x_\alpha, y_\alpha \rangle : \alpha < \omega_1\} \subseteq (M \times M) \cap E$  is a finite-dimensional set of positive dimension, as each closed set separating  $p$  from  $F$  in  $E$  hits  $S$ . Therefore,  $S$  contains a one-dimensional subset, by the inductive character of dimension.

- ? 416. **9.** *If a homogeneous compactum is not finite-dimensional, is it then strongly infinite-dimensional?*

The notion of homogeneity was recalled just before Question 2.3. It is even unclear what the answer is if we assume that the homogeneous compactum contains topologically each finite-dimensional cube  $I^n$ .

- ? 417. **10.** *What is the compactness degree of the  $n$ -dimensional cube  $I^n$  with one open face removed?*

The *compactness degree*  $\text{cmp } X$  of a separable metrizable space  $X$  is defined as follows (ISBELL [1964]):  $\text{cmp } X = -1$  iff  $X$  is compact,  $\text{cmp } X \leq n$  if each point and each closed set not containing the point can be separated in  $X$  by

a closed set  $L$  with  $\text{cmp } L \leq n - 1$  and  $\text{cmp } X$  is the minimal  $n$  for which  $\text{cmp } X \leq n$ , or  $\text{cmp } X = \infty$  if no such  $n$  exists. The following question was asked by J. DE GROOT and T. NISHIURA in [1966].

**10.1. QUESTION.** *Let  $J_n = [0, 1]^{n+1} \setminus \{0\} \times (0, 1)^n$ . What is  $\text{cmp } J_n$ ?*

**418. ?**

It is unknown if  $\text{cmp } J_n \longrightarrow \infty$ . Let  $\text{def } X = \min\{\dim(\tilde{X} \setminus X) : \tilde{X} \text{ is a compact metrizable extension of } X\}$  be the *defect* of a separable metrizable space  $X$ . Then  $\text{cmp } X \leq \text{def } X$  and  $\text{def } J_n = n$ .

In [1988] T. KIMURA defined for each  $n = 1, 2, \dots$  a closed subspace  $K_n$  of the space  $J_{2n+1}$  such that  $\text{cmp } K_n \leq n$  and  $\text{def } K_n \geq 2n$  (exact estimates are not given in KIMURA [1988]). K. P. Hart constructed for each  $n = 1, 2, \dots$  a closed subspace  $H_n$  of  $J_{2n-1}$  with  $\text{cmp } H_n = 1$  and  $\text{def } H_n \geq n$  (unpublished notes, 1985), see also KIMURA [1988].

**11.** *Is the dimension of a compactum determined by the topology of its function space?* **419. ?**

Let  $C_p(X)$  be the space of continuous real-valued functions on a completely regular space  $X$  endowed with the topology of pointwise convergence.

**11.1. QUESTION.** *Let  $X$  and  $Y$  be completely regular spaces such that the function spaces  $C_p(X)$  and  $C_p(Y)$  are homeomorphic. Is it true that  $\dim X = \dim Y$ ?* **420. ?**

The answer is unknown even if  $X$  and  $Y$  are compacta. GUL'KO proved in [19??] that for completely regular spaces  $X$  and  $Y$ , if  $C_p(X)$  and  $C_p(Y)$  are uniformly homeomorphic with respect to their natural uniform structures then  $\dim X = \dim Y$ .

**12.** *What properties of a metric characterize the dimension of the induced topology?* **421. ?**

The following is a conjecture of J. DE GROOT [1957] who proved it for compacta.

**12.1. QUESTION.** *Given a metrizable space  $X$ , is it true that  $\dim X \leq n$  if and only if there exists a metric  $d$  on  $X$  compatible with the topology such that for each point  $x \in X$  and any set  $A \subset X \setminus \{x\}$  of cardinality  $n + 2$  there are distinct points  $a, b \in A$  with  $d(a, b) \leq \min\{d(x, c) : c \in A\}$ ?* **422. ?**

In [1964] J. NAGATA gave a characterization of the dimension in terms of metrics which implies in particular that the condition in this question is necessary.

**13.** *Questions concerning zero-dimensional maps with infinite-dimensional range.*

One such question is Question 7.1; here we state two more.

? **423.** **13.1.** QUESTION. *Let  $f: X \rightarrow Y$  be a continuous map of a compactum  $X$  onto a compactum  $Y$  with  $\dim f^{\leftarrow}(y) = 0$  for all  $y \in Y$ . Does there exist a nontrivial continuous function  $u: X \rightarrow I$  into the unit interval such that  $u[f^{\leftarrow}(y)]$  is zero-dimensional for all  $y \in Y$ ?*

H. Toruńczyk proved (unpublished) that if  $Y$  is a countable-dimensional compactum then almost all maps  $u: X \rightarrow I$ , in the sense of Baire category, have this property.

? **424.** **13.2.** QUESTION. *Let  $f: X \rightarrow Y$  be an open map of a compactum  $X$  onto a compactum  $Y$ , such that all fibers  $f^{\leftarrow}(y)$  are homeomorphic to the Cantor set. Does there exist a continuous map  $u: X \rightarrow I$  such that  $u[f^{\leftarrow}(y)] = I$  for all  $y \in Y$ ?*

This question is taken from BULA [1983], where it is proved that for finite-dimensional  $Y$  the answer is positive.

**14.** *Questions concerning the transfinite extension of the inductive Menger-Urysohn dimension.*

All spaces in this part are metrizable and separable. If we allow  $n$  in the definition of the small inductive dimension  $\text{ind}$  given at the beginning of this note to be an arbitrary ordinal number, we obtain the transfinite extension of the Menger-Urysohn dimension. The transfinite dimension  $\text{ind}$ , if defined—we abbreviate this by  $\text{ind } X \neq \infty$ , has its values in the set of countable ordinals. If  $\text{ind } X \neq \infty$  then  $X$  is countable-dimensional and for complete spaces the converse is also true.

? **425.** **14.1.** QUESTION. *Given a countable ordinal  $\alpha$ , does there exist a compactum  $K_\alpha$  with  $\text{ind } K_\alpha = \alpha$  which contains topologically all compacta  $X$  with  $\text{ind } X \leq \alpha$ ?*

If “compactum” is replaced by “separable metrizable space” then the answer is positive (POL [1986]). W. Hurewicz proved that if  $\text{ind } X \neq \infty$  then  $X$  has a compact metrizable extension  $\tilde{X}$  with  $\text{ind } \tilde{X} \neq \infty$ . The following is a conjecture of L. A. LUXEMBURG [1982, p. 449].

? **426.** **14.2.** QUESTION. *Is it true that every separable metrizable space  $X$  with  $\text{ind } X = \alpha + n$ , where  $\alpha$  is a limit ordinal and  $n$  is a natural number, has a metrizable compact extension  $\tilde{X}$  with*

$\text{ind } \tilde{X} \leq \alpha + (2n + 1)?$

**14.3. QUESTION.** *Let  $f: X \rightarrow Y$  be a continuous map of a compactum  $X$  onto a compactum  $Y$  such that the transfinite dimension  $\text{ind}$  of each fiber  $f^{\leftarrow}(y)$  is defined. Is it true that  $\sup\{\text{ind } f^{\leftarrow}(y) : y \in Y\} < \omega_1$ ?* **427. ?**

Many problems concerning the transfinite dimension can be found in R. ENGELKING'S survey [1980].

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# PART III

## CONTINUA THEORY

### Contents:

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## CHAPTER 19

### **Eleven Annotated Problems About Continua**

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Most of the topics related to continua have rather visible geometric or analytic connotations, and the list of open problems presented in this article is no exception. In a way it also reflects the authors' tastes and experiences. As is always the case when making a selection, these tastes and individual preferences are unavoidable. Here they should only be viewed as implying that the authors consider the problems important and interesting.

By a *continuum* we mean a compact, connected metric space and by a *mapping* we mean a continuous function. The statement that the continuum  $M$  has the *fixed point property* means that, if  $f$  is a mapping of  $M$  into  $M$ , then there exists a point  $x$  of  $M$  such that  $f(x) = x$ . The following problem is a classic one. A great deal of research in the theory of continua during the past fifty years has been motivated by attempts to solve it, and it still remains open.

**Problem 1.** *If  $M$  is a non-separating plane continuum, does  $M$  have the fixed point property?* 428. ?

The roots of this problem certainly lie in the work of BROUWER [1912] who proved that every mapping of the disk, and in general, of the  $n$ -cell, into itself has a fixed point. Some historical comments and additional information with other references, concerning partial solutions of Problem 1, can be found in an article written by Bing in the *Scottish Book* (MAULDIN [1981, pp. 190-192], see also BING [1969]). Related fixed point problems and results are discussed in a survey compiled by LEWIS [1983]. Indeed, work on Problem 1 has led to many discoveries. One milestone in the progress on this problem is an example of BELLAMY [1979] of a tree-like continuum without the fixed point property. It is not known whether the modified example of Bellamy of a tree-like continuum with a fixed point free homeomorphism is planar.

A mapping is said to be an  $\epsilon$ -map provided the preimage of each point has diameter less than  $\epsilon$ . An *arc* is a continuum which has only two non-separating points. A *simple closed curve* is a continuum with the property that each two-point subset separates it. A *tree* is a continuum which is the union of a finite collection of arcs and which contains no simple closed curve. A continuum is *tree-like* if, for each positive number  $\epsilon$ , there is an  $\epsilon$ -map of it into a tree. A continuum is *arc-like* or *chainable* if, for each positive number  $\epsilon$ , there is an  $\epsilon$ -map of it into an arc. If  $f$  is a mapping from  $X$  into  $X$ , a point  $x$  of  $X$  is said to be a *periodic point* for  $f$  provided there exists a positive integer  $n$  such that  $f^n(x) = x$  (here  $f^n$  denotes the  $n$ th composite of  $f$  with itself).

**Problem 2.** *If  $M$  is a tree-like continuum and  $f$  is a mapping of  $M$  into  $M$ , does  $f$  have a periodic point?* 429. ?

The mappings of the examples of Bellamy, although free of fixed points,

nonetheless have periodic points. Problem 2 appears in at least two places: *The Houston Problem Book* [1986, Problem 34] and LEWIS [1983, Problem 35].

If  $M$  is a continuum and  $d$  is the metric on  $M$ , the *span* of  $M$ , denoted  $\sigma M$ , is defined by  $\sigma M = \inf\{\epsilon : \text{there is a subcontinuum } Z \text{ of } M \times M \text{ such that } \pi_1 Z = \pi_2 Z \text{ and } d(x_1, x_2) \geq \epsilon \text{ for each } (x_1, x_2) \text{ in } Z\}$ , where  $\pi_1$  and  $\pi_2$  are the standard projections, that is,  $\pi_1(x_1, x_2) = x_1$  and  $\pi_2(x_1, x_2) = x_2$  for  $(x_1, x_2) \in M \times M$ . A *trioid* is a continuum  $T$  which contains a subcontinuum  $K$  such that  $T - K$  has more than two components. A continuum is said to be *atriodic* if it contains no trioid. A *simple trioid* is a trioid which is the union of three arcs joined at a common end-point. A *simple 4-od* is a tree which is the union of four arcs joined at a common end-point. Simple 4-ods as well as simple triods are special kinds of trees. Suppose  $\mathcal{K}$  is a collection of one or more trees. A continuum is  *$\mathcal{K}$ -like* if, for each positive number  $\epsilon$ , there is an  $\epsilon$ -map of it into a tree from  $\mathcal{K}$ .

? 430. **Problem 3.** *Does there exist a tree-like continuum  $M$  with positive span such that the plane contains uncountably many mutually exclusive homeomorphic copies of  $M$ ?*

The plane does not contain uncountably many mutually exclusive triods (see MOORE [1962, p. 222], originally MOORE [1926]). However, the plane contains an uncountable collection of mutually exclusive, simple-trioid-like continua with positive span (INGRAM [1974]). The members of this collection are far from homeomorphic since the collection has no model (i.e., no continuum can be mapped onto every member of the collection). In [1982] OVERSTEEGEN and TYMCHATYN have results in the direction of answering this problem. In particular, they show that not every atriodic tree-like continuum can be embedded in the plane and that there is a planar, atriodic tree-like continuum  $M$  such that the plane does not contain uncountably many mutually exclusive homeomorphic copies of  $M$ . Problem 3 appears as Problem 3.8 in their abovementioned paper. There they also ask, in particular, if  $X \times C$  is planar, where  $X$  is the continuum of INGRAM [1972] and  $C$  is the Cantor set.

A continuum is *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua. A continuum is *decomposable* if it is the union of two of its proper subcontinua and is *indecomposable* otherwise.

? 431. **Problem 4.** *Does there exist an hereditarily equivalent continuum other than the arc and the pseudo-arc?*

The arc and the pseudo-arc are hereditarily equivalent. In [1948] MOISE constructed the pseudo-arc to obtain an hereditarily equivalent continuum which is not an arc. He called the continuum he constructed a pseudo-arc because of its hereditary equivalence. In [1951b] BING showed that each two hereditarily equivalent indecomposable chainable continua are homeomorphic.

In [1960] HENDERSON showed that the arc is the only decomposable hereditarily equivalent continuum. In [1970] COOK showed that hereditarily equivalent continua are tree-like, and thus there are no infinite-dimensional hereditarily equivalent continua.

**Problem 5.** *Does there exist an atriodic simple-4-od-like continuum which is not simple-triod-like?* **432. ?**

The example of INGRAM [1972] is atriodic, simple-triod-like and not arc-like (i.e., not 2-od-like). Thus, Problem 5 and its obvious modifications are natural in light of this type of example. In [1983] SAM YOUNG raises as Problem 115 essentially this same question. The only difference is that in place of “atriodic” he asks that every proper subcontinuum be an arc. Of course, if every proper subcontinuum of a continuum is an arc, then the continuum is atriodic (INGRAM [1968]).

**Problem 6.** *Do there exist in the plane two simple closed curves  $J_1$  and  $J_2$  such that  $J_1$  lies in the bounded complementary domain of  $J_2$  but the span of  $J_1$  is greater than the span of  $J_2$ ?* **433. ?**

Almost nothing seems to be known about the span of plane continua. Eventually, one wishes to know how span and surjective span of plane continua are related. A good place to start seems to be with this fundamental, unanswered question. It has been published in “The Houston Problem Book” as Problem 173 (dated 1981).

The *symmetric span* of  $M$ , denoted  $sM$ , is defined in a manner similar to that of the span of  $M$  changing only the condition  $\pi_1 Z = \pi_2 Z$  in the definition of span to  $Z = Z^{-1}$ , where  $Z^{-1} = \{(x_2, x_1) : (x_1, x_2) \in Z\}$ . The *surjective span* of  $M$ , denoted  $\sigma^* M$ , is also defined similarly to the span with only one additional requirement, namely that  $\pi_1 Z = \pi_2 Z = M$ .

**Problem 7.** *If  $M$  is a plane continuum, is  $\sigma M = sM$ ?* **434. ?**

The dyadic solenoid has long been known to be an example of a continuum with positive span which has zero symmetric span. In [1984] DAVIS has shown that  $sM \leq \sigma M$  for all continua, and that if  $sM = 0$ , then  $M$  is atriodic and hereditarily unicoherent.

**Problem 8.** *If  $M$  is a continuum and  $\sigma M = 0$ , is  $M$  chainable?* **435. ?**

This is becoming a classic problem in the theory of continua. Problem 8 was first stated in print in LELEK [1971]; the fact that chainable continua have span zero had been known earlier (LELEK [1964]). A positive answer to this problem would complete the classification of homogeneous plane continua (see

OVERSTEEGEN and TYMCHATYN [1982] and also DAVIS [1984]). It is known that continua of span zero, i.e., continua  $M$  such that  $\sigma M = 0$ , are tree-like (OVERSTEEGEN and TYMCHATYN [1984]). A stronger result has been recently established, namely that continua of surjective span zero ( $\sigma^* M = 0$ ) are tree-like, see KATO, KOYAMA and TYMCHATYN [19 $\infty$ ].

A mapping  $f$  from  $X$  onto  $Y$  is said to be *confluent* provided, for each subcontinuum  $K$  of  $Y$ , each component of  $f^{-1}(K)$  is mapped by  $f$  onto  $K$ . If the last condition is satisfied by at least one component rather than by each one, then the mapping is called *weakly confluent*. A *monotone* mapping is a mapping whose preimages of points are connected.

? 436. **Problem 9.** *Is the confluent image of a chainable continuum also chainable?*

This problem also first appeared in LELEK [1971]. In [1972] MCLEAN has shown that the confluent image of a tree-like continuum is tree-like. Problems 8 and 9 then both lie in the general area of determining among tree-like continua which ones are chainable. This question, although not specifically stated in this list of problems, is fundamental to determining the structure of one-dimensional continua and can be seen to have motivated a lot of the work on the problems mentioned here. In [1978] GRACE and VOUGHT provide an example of a chainable continuum and a weakly confluent mapping of it onto a simple triod (see also COOK and LELEK [1978]). In two special cases Problem 9 is known to have an affirmative answer: for monotone mappings (BING [1951a]) and for open mappings (ROSENHOLTZ [1974]).

? 437. **Problem 10.** *Does there exist a tree-like continuum  $M$  such that no monotone image of any subcontinuum of  $M$  is chainable?*

The construction of continua as inverse limits with atomic mappings, as in ANDERSON and CHOQUET [1959], COOK [1967] and INGRAM [1981], will not produce such a continuum. The search for a solution to Problem 4 motivates Problem 10.

Suppose  $M$  is an hereditarily equivalent continuum which is neither an arc nor a pseudo-arc. Then  $M$  is hereditarily indecomposable (HENDERSON [1960]) and  $M$  is tree-like (COOK [1970]), and, in light of Problem 8, we might conjecture that  $M$  has positive span (hereditarily). It can then be shown that there exists in  $M \times M$  a continuum  $Z$  with the property that  $\pi_1 Z = \pi_2 Z = M$  and, for each  $(x_1, x_2)$  in  $Z$ ,  $M$  is irreducible from  $x_1$  to  $x_2$  and, furthermore, that such is the case for every monotone image of  $M$ . Thus,  $M$  would be a continuum as in Problem 10.

? 438. **Problem 11.** *Is it true that  $\sigma X \leq 2\sigma^* X$  for each connected metric space  $X$ ?*

The inequalities which follow directly from the definitions of span, semi-span, surjective span and surjective semi-span (LELEK [1976]) lead to several



questions related to this one (see LELEK [1977, p. 38]). On the other hand, an example shows that  $\sigma^*X$  can be  $\frac{1}{2}\sigma X$  (LELEK [1976]) and the geometry of it, and of a number of related constructions (WEST [1983]), is quite intriguing. Problem 11 was originally published in *The Houston Problem Book* [1986], as part of Problem 83 (dated 1975).

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CHAPTER 20

**Tree-like Curves and Three Classical Problems**

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<sup>1</sup>This work was partially supported by a grant from the National Science Foundation.

This paper deals with three problems that have had a continuous appeal to researchers in the theory of continua since the start of the subject. The three problems are the fixed-point problem for nonseparating plane continua, the classification of hereditarily equivalent continua, and the classification of homogeneous continua. A common theme in these problems is that each requires us to answer certain questions about tree-like curves, sometimes even the same questions; furthermore, these questions seem to capture the essence of continua theory as it has been practiced over the last several years.

I have asked several questions on each of these problems, questions chosen mainly because they appeal to me. I don't think any of them are inadequately worded or already answered, but if so, I hope the reader will adjust for that. I have made little attempt to associate a name with a question, for if I had made an error there under the pressure of a deadline, it's less likely I would be forgiven for that. In particular, many of these questions were first asked by someone else, and if you solve one, I will be glad to help you find out who it was. I do claim that all these questions are interesting to me, and that most continua theorists sense some underlying unity flowing through these questions.

A few problems that appeal to me could not be included under this rubric. I have indulged myself by including them in a final section of miscellaneous questions.

Since this is a selection of problems and not a survey article, I have not included references or many definitions. Surveys of the status of the classification problem for homogeneous continua, however, can be found in ROGERS [1983, 19 $\infty$ ].

A *continuum* is a compact, connected, nonvoid metric space. A *curve* is a one-dimensional continuum. A *map* is a continuous function.

## 1. The Fixed-Point Property

No question has attracted more interest from continua theorists than the following:

**Question 1.** *Does every nonseparating plane continuum have the fixed-point property?* **439. ?**

Apparently the paper of Ayres in 1930 was the first instance in which this problem appeared in print. Ayres called it a "well known problem" and proved that each homeomorphism of a nonseparating Peano plane continuum has a fixed point. In 1932 Borsuk improved this by proving that each map of a nonseparating Peano plane continuum has a fixed point.

This dichotomy between homeomorphisms and maps occurs more than once in the history of this problem, and so the next question is natural.

- ? 440. **Question 2.** *Does each nonseparating plane continuum have the fixed-point property for homeomorphisms?*

In the case the homeomorphism extends to a homeomorphism of the plane, the answer is yes.

R. H. Bing has shown that each one-dimensional, nonseparating plane continuum is tree-like.

- ? 441. **Question 3.** *Does each planar tree-like continuum have the fixed-point property?*

- ? 442. **Question 4.** *Does each planar tree-like continuum have the fixed point property for homeomorphisms?*

D. Bellamy has exhibited a startling example of a tree-like continuum that fails to have the fixed-point property for homeomorphisms.

In 1938, Hamilton proved that each hereditarily decomposable, tree-like continuum has the fixed-point property for homeomorphisms. The appropriate generalization to maps did not occur until 1976, when Manka proved that every hereditarily decomposable, tree-like continuum has the fixed-point property.

In the late 1960's, H. Bell and K. Sieklucki independently showed that any counterexample to Question 1 must contain an indecomposable subcontinuum in its boundary that is invariant under a fixed-point-free map. A natural improvement to the Bell-Sieklucki result would be a positive answer to the following question:

- ? 443. **Question 5.** *Can the Bell-Sieklucki result be improved to state that the indecomposable, invariant subcontinuum in the boundary is tree-like?*

C. L. Hagopian has interesting partial results on these problems, many of which use the Bell-Sieklucki theorem as a tool. A most interesting new result along these lines is due to P. Minc, who showed that each weakly chainable, nonseparating plane continuum has the fixed-point property.

In light of the example of Bellamy and the construction of similar examples by Oversteegen and Rogers, we need to understand more about the fixed-point property for tree-like continua.

- ? 444. **Question 6.** *Does each  $T$ -like continuum (i.e., inverse limit of  $T$ 's) have the fixed-point property?*

- ? 445. **Question 7.** *Does each hereditarily indecomposable, tree-like continuum have the fixed-point property?*

Let  $2^X$  be the space of closed subsets of a continuum  $X$  with the Hausdorff metric, and let  $C(X)$  be the space of all subcontinua of  $X$ .

**Question 8.** *If  $X$  is a tree-like continuum, must  $C(X)$  have the fixed-point property?* **446. ?**

**Question 9.** *When does  $2^X$  have the fixed-point property?* **447. ?**

## 2. Hereditarily Equivalent Continua

A continuum is *hereditarily equivalent* if it is homeomorphic to each of its nondegenerate subcontinua. In 1921, S. Mazurkiewicz asked if each finite-dimensional, hereditarily equivalent continuum is an arc. In 1930, G. T. Whyburn proved that a planar, hereditarily equivalent continuum does not separate the plane. Although the problem was posed as worthy of attention by Klein in 1928 and Wilder in 1937, no further progress occurred until 1948, when E. E. Moise constructed a pseudo-arc. The pseudo-arc is a hereditarily indecomposable, hereditarily equivalent continuum in the plane, and so the answer to Mazurkiewicz's question is no.

The arc and the pseudo-arc are the only known hereditarily equivalent, nondegenerate continua. G. W. Henderson showed that any new example must be hereditarily indecomposable, and H. Cook showed that any new example must be tree-like. J. T. Rogers observed that each continuum of dimension greater than one contains uncountably many topologically distinct subcontinua.

**Question 10.** *Is every hereditarily equivalent, nondegenerate continuum chainable?* **448. ?**

If the answer to this question is yes, then it is known that the arc and the pseudo-arc are the only such examples.

**Question 11.** *Does each hereditarily equivalent continuum have span zero?* **449. ?**

Oversteegen and Tymchatyn have recently shown that planar, hereditarily equivalent continua have symmetric span zero.

**Question 12.** *Does each hereditarily equivalent continuum have the fixed-point property?* **450. ?**

**Question 13.** *Is each indecomposable, hereditarily equivalent continuum homogeneous?* **451. ?**

### 3. Homogeneous Continua

- ? 452. **Question 14.** *Is each homogeneous, nondegenerate nonseparating plane continuum a pseudo-arc?*

If the answer is yes, then it is known that the nondegenerate homogeneous plane continua are the circle, the pseudo-arc, and the circle of pseudo-arcs. Jones and Hagopian have shown that such a continuum must be hereditarily indecomposable. Rogers has shown it must be tree-like. Oversteegen and Tymchatyn have shown that it must have span zero and be weakly chainable. Lewis has shown that it must contain a proper nondegenerate subcontinuum that is not a pseudo-arc.

R. D. Anderson has shown that the circle and the Menger curve are the only homogeneous, locally connected curves. The next step is to classify the important class of so-called Type 2 curves—the aposyndetic homogeneous curves that are not locally connected. All known examples of Type 2 curves can be obtained as inverse limits of universal curves and covering maps. All of them can be obtained as total spaces of Cantor set bundles over the Menger curve.

- ? 453. **Question 15.** *Is each Type 2 curve the total space of a bundle over the universal curve with Cantor sets as the fibers?*
- ? 454. **Question 16.** *Is each Type 2 curve an inverse limit of universal curves? universal curves and fibrations as bonding maps? universal curves and covering maps as bonding maps?*
- ? 455. **Question 17.** *Does each Type 2 curve contain an arc?*
- ? 456. **Question 18.** *Does each Type 2 curve retract onto a solenoid?*
- ? 457. **Question 19.** *Is each pointed-one-movable, aposyndetic homogeneous curve locally connected?*
- ? 458. **Question 20.** *Is each arcwise-connected homogeneous curve locally connected?*
- ? 459. **Question 21.** *Is each hereditarily decomposable homogeneous continuum a simple closed curve?*

An affirmative answer to Question 18 would be especially significant, for it would imply affirmative answers to Questions 19, 20, and 21.

**Question 22.** *Suppose  $X$  is a homogeneous indecomposable curve whose first Čech cohomology group with integral coefficients does not vanish. If  $X$  is not a solenoid, does  $X$  admit a continuous decomposition into tree-like homogeneous curves so that the resulting quotient space is a solenoid?* **460. ?**

**Question 23.** *Is each tree-like, homogeneous curve a pseudo-arc?* **461. ?**

Affirmative answers to Questions 15, 22, and 23 would be especially interesting, for if the answer to each of these three questions is yes, then we can classify homogeneous curves according to the following scheme: Each homogeneous curve would be

- (1) a simple closed curve or a Menger universal curve, or
- (2) the total space of a Cantor set bundle over a type (1) curve, or
- (3) a curve admitting a continuous decomposition into pseudo-arcs such that the quotient space is a curve of type (1) or (2), or
- (4) a pseudo-arc.

**Question 24.** *Is each tree-like, homogeneous curve weakly chainable?* **462. ?**

**Question 25.** *Does each tree-like homogeneous curve have span zero?* **463. ?**

**Question 26.** *Does each tree-like homogeneous curve have the fixed-point property?* **464. ?**

**Question 27.** *Is each decomposable, homogeneous continuum of dimension greater than one aposyndetic?* **465. ?**

**Question 28.** *Must the elements of the Jones aposyndetic decomposition be hereditarily indecomposable?* **466. ?**

**Question 29.** *Can this aposyndetic decomposition raise dimension? lower dimension?* **467. ?**

**Question 30.** *Is each indecomposable, nondegenerate, homogeneous continuum one-dimensional?* **468. ?**



#### 4. Miscellaneous Interesting Questions

? 469. **Question 31.** *Is it true that no indecomposable continuum has a Borel transversal to its composants (i.e., a Borel set that intersects each component in exactly one point)?*

? 470. **Question 32.** *Is every weakly chainable, atriodic, tree-like continuum chainable?*

An affirmative answer would yield an affirmative answer to Question 14 and thus complete the classification of planar homogeneous continua.

? 471. **Question 33.** *Suppose  $G$  is a continuous decomposition of  $E^2$  into nonseparating continua. Must some element of  $G$  be hereditarily indecomposable?*

? 472. **Question 34.** *Is the homeomorphism group of the pseudo-arc totally disconnected?*

? 473. **Question 35.** *Is the homeomorphism group of the pseudo-arc infinite-dimensional?*

? 474. **Question 36.** *If  $\dim X > 1$ , is  $\dim C(X) = \infty$ ? What if  $X$  is indecomposable?*

The answer is known to be yes if any of the following are added to the hypothesis:

- (1)  $X$  is locally connected;
- (2)  $X$  contains the product of two nondegenerate continua;
- (3)  $\dim X > 2$ ;
- (4)  $X$  is hereditarily indecomposable, or
- (5)  $\text{rank } H^1(X) < \infty$ .

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## PART IV

### TOPOLOGY AND ALGEBRAIC STRUCTURES

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## CHAPTER 21

### Problems on Topological Groups and Other Homogeneous Spaces

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## 0. Introduction and Notation

We consider only completely regular, Hausdorff spaces (= Tikhonov spaces). In particular, our topological groups are Tikhonov spaces.

We do not distinguish notationally or grammatically between a topological property  $\underline{\mathbf{T}}$  and the class of spaces with (or, in)  $\underline{\mathbf{T}}$ .

The class of (Tikhonov) spaces is denoted  $\underline{\mathbf{S}}$ ; the class of homogeneous spaces is denoted  $\underline{\mathbf{H}}$ ; the class of topological groups is denoted  $\underline{\mathbf{G}}$ . Sometimes for convenience we use  $\underline{\mathbf{C}}$ ,  $\underline{\mathbf{CC}}$ ,  $\underline{\mathbf{P}}$  and  $\underline{\mathbf{A}}$  to abbreviate the expressions compact, countably compact, pseudocompact and Abelian, respectively. Thus for example the expression  $G \in \underline{\mathbf{CCAG}}$  means that  $G$  is a countably compact Abelian topological group, and if  $\underline{\mathbf{T}}$  is a topological property then the expression  $X \in \underline{\mathbf{TCH}}$  or  $X \in \underline{\mathbf{CTH}}$  means that  $X$  is a compact homogeneous space with property  $\underline{\mathbf{T}}$ .

The least infinite cardinal number is denoted by the symbol  $\omega$ ; the symbols  $\alpha$ ,  $\gamma$ ,  $\kappa$  and  $\lambda$  denote infinite cardinals, and as usual we write  $\alpha = \{\xi : \xi < \alpha\}$ . The symbol  $\alpha$  also denotes the set  $\alpha$  with the discrete topology. The Stone-Ćech remainder of  $X$  is the space  $X^* = \beta X \setminus X$ ; in particular, we write  $\alpha^* = \beta(\alpha) \setminus \alpha$ .

The *Stone extension*  $\bar{f}$  of a continuous function  $f: X \rightarrow Y$  is that continuous function  $\bar{f}: \beta X \rightarrow \beta Y$  such that  $\bar{f}|_X = f$ . For  $p, q \in X^*$  we write  $p \approx q$  if there is a homeomorphism  $h$  of  $X$  onto  $X$  such that  $\bar{h}(p) = q$ .

For a transitive, reflexive relation (i.e., a pre-order)  $\leq_E$  on a set  $S$  and for  $p, q \in S$ , we write  $p =_E q$  if  $p \leq_E q$  and  $q \leq_E p$ . The relation  $\leq_E$  is *directed downward* if for all  $p, q \in S$  there is  $r \in S$  such that  $r \leq_E p$  and  $r \leq_E q$ . The *Rudin-Keisler pre-order*  $\leq_{\mathbf{RK}}$  is defined on  $\omega^*$  by the condition  $p \leq_{\mathbf{RK}} q$  if and only if there is  $f: \omega \rightarrow \omega$  such that  $\bar{f}(q) = p$ . It is known that for  $p, q \in \omega^*$ , the relation  $p =_{\mathbf{RK}} q$  holds if and only if  $p \approx q$  (see COMFORT and NEGREPONTIS [1974, (9.3) and the notes to section 9] for a proof, and for references to the literature).

The cardinality of a set  $X$  is written  $|X|$ . The weight, density character, tightness and cellularity of a space  $X$  are written  $wX$ ,  $dX$ ,  $tX$  and  $cX$ , respectively. A space  $X$  with  $cX \leq \omega$  is a space with the *countable chain condition* or, briefly, a **ccc** space.

The symbols  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{T}$  denote the integers, the reals, and the circle, respectively, in each case with the usual algebraic operations and the usual topology. We write

$$\mathbb{I} = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}.$$

For a group  $G$  and  $A \subseteq G$ , the symbol  $\langle A \rangle$  denotes the subgroup of  $G$  generated by  $A$ ; the torsion subgroup of  $G$  is denoted  $tG$ .

I am grateful to the following mathematicians for suggestions, comments, and references to the literature. Aleksander V. Arkhangel'skiĭ, Alan Dow, Salvador Garcia-Ferreira, Douglass Grant, Anthony W. Hager, Melvin Henriksen, Karl H. Hofmann, R. Daniel Mauldin, Jan van Mill, Sidney A. Morris, James D. Reid, Dieter Remus, Lewis C. Robertson, Kenneth A. Ross, Mary Ellen Rudin, Niel Shell, Rae Michael Shortt, T. Christine Stevens, Michael G. Tkachenko, and F. Javier Trigos.

## 1. Embedding Problems

Of course every space  $X \in \underline{\mathbf{S}}$  embeds into a topological group: The classical result of TIKHONOV [1929] shows how to embed  $X$  into  $\mathbb{R}^{wX}$  or  $\mathbb{T}^{wX}$ . (For results on the embedding of non-Hausdorff spaces into compact homogeneous spaces, see KOROVIN [1987].)

### 1A. $p$ -compact spaces and groups; $p$ -sequential spaces and groups

Following VAN DER SLOT [1966], we say that a topological property  $\underline{\mathbf{U}}$  is a *universal topological property* if

- (a) every compact space has  $\underline{\mathbf{U}}$ ,
- (b) the product of any set of spaces with  $\underline{\mathbf{U}}$  has  $\underline{\mathbf{U}}$ , and
- (c) every closed subspace of a space with  $\underline{\mathbf{U}}$  has  $\underline{\mathbf{U}}$ .

Within the context of this article (Tikhonov spaces), the universal topological properties are exactly the *topological extension properties* defined and studied by WOODS in [1975].

It is well known that for every universal topological property  $\underline{\mathbf{U}}$  and for every space  $X$  there is a space  $\beta_{\underline{\mathbf{U}}}(X)$ , unique up to a homeomorphism fixing  $X$  pointwise, such that  $\beta_{\underline{\mathbf{U}}}(X)$  has  $\underline{\mathbf{U}}$ ,  $X$  is dense in  $\beta_{\underline{\mathbf{U}}}(X)$ , and every continuous function from  $X$  to a space  $Y$  in  $\underline{\mathbf{U}}$  extends to a continuous function from  $\beta_{\underline{\mathbf{U}}}(X)$  to  $Y$ . This result, a special case of the adjoint functor theorem of FREYD [1960, 1964], is accessible through arguments given by KENNISON [1965], VAN DER SLOT [1966, 1968], HERRLICH [1967], HERRLICH and VAN DER SLOT [1967], FRANKLIN [1971], and WOODS [1975]; a systematic exposition is given in COMFORT and NEGREPONTIS [1975].

Universal topological properties are easy to find. For example, given any topological property  $\underline{\mathbf{T}}$  such that  $\mathbb{I} \in \underline{\mathbf{T}}$ , the property  $\underline{\mathbf{U}}$  defined by the condition that  $X$  has  $\underline{\mathbf{U}}$  if and only if  $X$  is homeomorphic to a closed subspace of a product of spaces in  $\underline{\mathbf{T}}$  is a universal topological property. A fertile class of universal topological properties is suggested by the following definition.

**1A.1. DEFINITION.** Let  $p$  be a non-principal ultrafilter on a discrete space  $\alpha$ —that is,  $p \in \alpha^*$ . A space  $X$  is  *$p$ -compact* if for every  $f: \alpha \rightarrow X$  the Stone extension  $\bar{f}: \beta(\alpha) \rightarrow \beta X$  satisfies  $\bar{f}(p) \in X$ .

In our context (Tikhonov spaces) this definition is equivalent to the definition of  $p$ -compactness (for  $p \in \omega^*$ ) given initially by A. BERNSTEIN in [1970] in connection with problems in the theory of non-standard analysis. It should be noted that the  $p$ -limit concept of BERNSTEIN [1970] coincides in important special cases with the “producing” relation introduced by FROLÍK in [1967b, 1967a], with one of the orderings introduced by KATĚTOV in [1961/62, 1968], and with a definition given independently by SAKS in [1972]. For proofs of the fact that (for each  $p \in \omega^*$ )  $p$ -compactness is a universal topological property, see the original paper of BERNSTEIN [1970], GINSBURG and SAKS [1975], SAKS [1978], or VAUGHAN [1984]. In [1981] KANNAN and SOUNDARARAJAN have shown for a vast class of properties  $\underline{U}$  closely related to the universal topological properties—the so-called PCS properties—that a space has  $\underline{U}$  if and only if it is  $p$ -compact for each  $p$  in some set or class of ultrafilters (living perhaps on various discrete spaces). This result has been extended and formulated in a categorical context by HAGER in [1986]. For other results on spaces required to be  $p$ -compact simultaneously for various  $p$ , see WOODS [1975] and SAKS [1978].

It is known (COMFORT and ROSS [1966]) that the product of any set of pseudocompact topological groups is pseudocompact. This makes it natural to ask the following naive question: Is the product of countably compact groups necessarily countably compact? An affirmative answer (for arbitrary products) is equivalent to the condition that there exists  $p \in \omega^*$  such that every countably compact group is  $p$ -compact (see COMFORT [1984, (8.9)]), but it is apparently unknown whether this condition is consistent with the axioms of **ZFC**. In the negative direction VAN DOUWEN [1980] used **MA** to find countably compact subgroups  $G$  and  $H$  of  $\{-1, +1\}^c$  such that  $G \times H$  is not countably compact. (The papers of MALYKHIN [1987] and HART and VAN MILL [19 $\infty$ ] achieve the same conclusion with  $G = H \subseteq \{-1, +1\}^c$ . That such a group  $G$  exists assuming **MA** had been announced, but not proved, by VAN DOUWEN in [1980]. MALYKHIN [1987] assumes **MA**, while HART and VAN MILL [19 $\infty$ ] need only **MA**<sub>countable</sub>.) This positive and this negative result suggest these three questions.

**Question 1A.1.** *Is it consistent with **ZFC** that there exists  $p \in \omega^*$  such that every countably compact group is  $p$ -compact?* **475. ?**

**Question 1A.2.** *Is it a theorem of **ZFC** that there exist two countably compact groups whose product is not countably compact?* **476. ?**

**Question 1A.3.** *Is there, for every (not necessarily infinite) cardinal number  $\alpha \leq 2^c$ , a topological group  $G$  such that  $G^\gamma$  is countably compact for all cardinals  $\gamma < \alpha$ , but  $G^\alpha$  is not countably compact?* **477. ?**

Concerning Question 1A.3, three comments are in order.

(a) The restriction  $\alpha \leq 2^c$  in 1A.3 should not be omitted, since GINSBURG and SAKS have shown in [1975] for each space  $X$  that if  $X^{2^c}$  is countably compact then  $X^\alpha$  is countably compact for all cardinals  $\alpha$ ; indeed, the condition that  $X^{2^c}$  is countably compact is equivalent to the condition that there is  $p \in \omega^*$  such that  $X$  is  $p$ -compact.

(b) Question 1A.3 is the analogue for topological groups of one of the questions I posed some years ago (COMFORT [1976]) in the context of (Tikhonov) spaces. In [1985] YANG has shown in **ZFC** that a space  $X$  exists with  $X^{2^c}$  not countably compact but with  $X^\alpha$  countably compact for all  $\alpha < 2^c$  if and only if for every  $A \subseteq \omega^*$  with  $|A| < 2^c$  there exists  $q \in \omega^*$  which is not  $\leq_{\mathbf{RK}}$ -comparable to any  $p \in A$ . The consistency with **ZFC** of this latter condition has been established by SAKS in [1979]. I do not know whether the existence of such a space as above guarantees the existence of a group as in 1A.3, and I do not know whether the existence of such a space is a theorem of **ZFC**.

(c) I am informed by van Mill that it is easy to augment the argument of his paper with Hart (HART and VAN MILL [19 $\infty$ ]) to answer 1A.3 positively for  $\alpha < \omega$  (assuming **MA**<sub>countable</sub>).

For ultrafilters  $p, q \in \omega^*$  we write  $p \leq_{C, \underline{s}} q$  if every  $q$ -compact space is  $p$ -compact, and  $p \leq_{C, \underline{g}} q$  if every  $q$ -compact group is  $p$ -compact. (The comparison  $C$  on an arbitrary class  $\underline{T}$  of spaces, denoted  $\leq_{C, \underline{T}}$ , would be defined as follows:  $p \leq_{C, \underline{T}} q$  if every  $q$ -compact space in  $\underline{T}$  is  $p$ -compact.) It is clear that  $\leq_{C, \underline{s}} \subseteq \leq_{C, \underline{g}}$  in the sense that if  $p, q \in \omega^*$  and  $p \leq_{C, \underline{s}} q$ , then  $p \leq_{C, \underline{g}} q$ . It is clear also that if for every  $q \in \omega^*$  every  $q$ -compact space  $X$  embeds as a closed subspace of a  $q$ -compact topological group, then  $\leq_{C, \underline{s}} = \leq_{C, \underline{g}}$ .

(PROOF. Let  $p \leq_{C, \underline{g}} q$  and let  $X$  be a  $q$ -compact space. If  $X$  is closed in the  $q$ -compact group  $G$  then  $X$  is  $p$ -compact because  $p \leq_{C, \underline{g}} q$  (so  $G$  is  $p$ -compact) and  $p$ -compactness is closed-hereditary.)

This shows that an affirmative answer to 1A.5 yields an affirmative answer to 1A.4.

? 478. **Question 1A.4.** Are the conditions  $p \leq_{C, \underline{s}} q$  and  $p \leq_{C, \underline{g}} q$  equivalent for all  $p, q \in \omega^*$  (in other words, is the equality  $\leq_{C, \underline{s}} = \leq_{C, \underline{g}}$  valid)?

? 479. **Question 1A.5.** For every  $q \in \omega^*$ , does every  $q$ -compact space embed as a closed subspace into a  $q$ -compact topological group?

Conceivably **ZFC** settles 1A.4 affirmatively, while 1A.5 is independent.

? 480. **Question 1A.6.** If  $M$  is a model of **ZFC** in which  $\leq_{C, \underline{s}} = \leq_{C, \underline{g}}$ , must  $M$  answer 1A.5 affirmatively?

The relation  $\leq_{\mathbf{RK}} \subseteq \leq_{C, \underline{s}}$  is easily established, but the orders  $\leq_{\mathbf{RK}}$  and  $\leq_{C, \underline{s}}$  do not coincide on  $\omega^*/\approx$ : in [1989] GARCIA-FERREIRA has shown in **ZFC** that for every  $p \in \omega^*$  there is  $q \in \omega^*$  such that  $p =_{C, \underline{s}} q$  and  $p \leq_{\mathbf{RK}}$

$q$  and  $p \neq_{\mathbf{RK}} q$ . Nevertheless the relations  $\leq_{\mathbf{RK}}$  and  $\leq_{C, \underline{\mathbf{S}}}$  are intimately related, as the following three theorems (all due to GARCIA-FERREIRA [1989]) make clear.

- (1)  $p \leq_{C, \underline{\mathbf{S}}} q$  if and only if there is  $r \in \omega^*$  such that  $r =_{C, \underline{\mathbf{S}}} q$  and  $p \leq_{\mathbf{RK}} r$ .
- (2) The space  $\omega^*$  is downward directed under  $\leq_{\mathbf{RK}}$  if and only if  $\omega^*$  is downward directed under  $\leq_{C, \underline{\mathbf{S}}}$ . (The first of these conditions has been shown by BLASS and SHELAH in [1987] to be consistent with the axioms of **ZFC**.)
- (3) If  $p$  is a weak P-point, then the three conditions  $p \leq_{\mathbf{RK}} q$ ,  $p \leq_{C, \underline{\mathbf{S}}} q$  and  $p \leq_{C, \underline{\mathbf{G}}} q$  are equivalent.

It follows from (3) that for  $p \in \omega^*$  conditions (a) and (b) below are equivalent, and (c) implies each:

- (a)  $p$  is  $\leq_{\mathbf{RK}}$ -minimal in  $\omega^*$ ;
- (b)  $p$  is  $\leq_{C, \underline{\mathbf{S}}}$ -minimal in  $\omega^*$  and a weak P-point in  $\omega^*$ ;
- (c)  $p$  is  $\leq_{C, \underline{\mathbf{G}}}$ -minimal in  $\omega^*$  and a weak P-point in  $\omega^*$ ;

In the model of BLASS and SHELAH [1987], the three conditions are equivalent.

The following two questions were suggested by Garcia-Ferreira.

**Question 1A.7.** *For  $p \in \omega^*$  are the following conditions equivalent:*

**481. ?**

- (a)  $p$  is  $\leq_{C, \underline{\mathbf{S}}}$ -minimal in  $\omega^*$ ;
- (b)  $p$  is  $\leq_{C, \underline{\mathbf{G}}}$ -minimal in  $\omega^*$ ;
- (c) there is  $q \in \omega^*$  such that  $p =_{C, \underline{\mathbf{S}}} q$  and  $q$  is  $\leq_{\mathbf{RK}}$ -minimal in  $\omega^*$ ?

Garcia-Ferreira has remarked in conversation that in the model of BLASS and SHELAH [1987] the three conditions of 1A.7 are indeed equivalent.

From **MA** it follows—see BOOTH [1969, 1970] and PFISTER [1985]—that there exist (many) pairwise  $\approx$ -inequivalent (that is, pairwise  $=_{\mathbf{RK}}$ -distinct) points in  $\omega^*$  which are  $\leq_{\mathbf{RK}}$ -minimal in  $\omega^*$ . In such a model, of course,  $\omega^*$  is not downward directed under  $\leq_{\mathbf{RK}}$ .

**Question 1A.8.** *Does  $\mathbf{MA} \models [\omega^* \text{ is not downward directed under } \leq_{C, \underline{\mathbf{G}}}]?$*

**482. ?**

Following KOMBAROV [1983, 1985], we say for  $p \in \omega^*$  that a space  $X$  is  $p$ -sequential if for every non-closed subset  $A$  of  $X$  there is  $f: \omega \rightarrow X$  such that  $f[\omega] \subseteq A$  and  $\bar{f}(p) \in X \setminus A$ . It is natural to consider the relation  $\leq_{G-F, \underline{\mathbf{S}}}$  defined on  $\omega^*$  by Garcia-Ferreira as follows:  $p \leq_{G-F, \underline{\mathbf{S}}} q$  if every  $p$ -sequential space is  $q$ -sequential. As it turns out, the relation  $\leq_{G-F, \underline{\mathbf{S}}}$  coincides with  $\leq_{\mathbf{RK}}$  (GARCIA-FERREIRA [1989]); the trick to the proof is to proceed via this “intermediate” equivalent property: the space  $\omega \cup \{p\}$  (in the topology inherited from  $\beta(\omega)$ ) is  $q$ -sequential. The relation  $\leq_{G-F, \underline{\mathbf{G}}}$  is now defined on  $\omega^*$  as expected:  $p \leq_{G-F, \underline{\mathbf{G}}} q$  if every  $p$ -sequential group is  $q$ -sequential. The appropriate analogues to 1A.4 and 1A.5 are these.

**Question 1A.9.** *Is the equality  $\leq_{G-F, \underline{\mathbf{S}}} = \leq_{G-F, \underline{\mathbf{G}}}$  valid?*

**483. ?**



? 484. **Question 1A.10.**

- (a) Does the space  $\omega \cup \{p\}$  embed as a closed subspace into a  $p$ -sequential group?
- (b) Does every  $p$ -sequential space embed as a closed subspace into a  $p$ -sequential group?

(The argument of ORDMAN and SMITH-THOMAS [1980] answers 2(b) affirmatively for  $k_\omega$ -spaces; see GARCIA-FERREIRA [1989].)

An argument similar to the argument cited before 1A.4 shows that an affirmative answer to 2(a) yields an affirmative answer to 1A.9, but there remains the analogue of 1A.6.

? 485. **Question 1A.11.** *If  $M$  is a model of **ZFC** in which  $\leq_{G-F,\underline{s}} = \leq_{G-F,\underline{g}}$ , must  $M$  answer 2 affirmatively?*

It is a theorem of ARKHANGEL'SKIĬ [1980] that the identity  $wG = tG$  is valid for every compact group (indeed, if  $G$  is a compact group and the compact set  $F \subseteq G$  satisfies  $\langle F \rangle = G$ , then the cardinals  $wG$ ,  $tG$ ,  $wF$  and  $tF$  are all equal ARKHANGEL'SKIĬ [1980, 3.8]). Since a  $p$ -sequential space has countable tightness, it follows from Arkhangel'skiĭ's theorem that for a compact group  $G$  the following conditions are equivalent:

- (a)  $tG = \omega$ ;
- (b)  $G$  is  $p$ -sequential for some  $p \in \omega^*$ ;
- (c)  $G$  is  $p$ -sequential for every  $p \in \omega^*$ .

The following three questions, which have evolved in conversations with Garcia-Ferreira, now appear natural:

? 486. **Question 1A.12.** *If  $G$  is a topological group and  $tG = \omega$ , must there exist  $p \in \omega^*$  such that  $G$  is  $p$ -sequential?*

? 487. **Question 1A.13.** (a) *If  $X$  is a homogeneous space such that  $tX = \omega$ , must there exist  $p \in \omega^*$  such that  $X$  is  $p$ -sequential?* (b) *What if  $X$  is compact?*

It has been shown recently by GARCIA-FERREIRA, in [1989], using the fact that the space  $\omega^*$  is  $2^\omega$ -directed in the order  $\leq_{\mathbf{RK}}$  (in the sense that if  $A \subseteq \omega^*$  with  $|A| \leq 2^\omega$  there is  $q \in \omega^*$  such that  $p \leq_{\mathbf{RK}} q$  for all  $p \in A$ ), that every space  $X$  with  $|X| \leq 2^\omega$  and  $tX \leq \omega$  is  $p$ -sequential for some  $p \in \omega^*$ . In [1987] ARKHANGEL'SKIĬ has conjectured that every compact homogeneous space  $X$  such that  $tX \leq \omega$  must satisfy  $|X| \leq 2^\omega$ . It is clear that if Arkhangel'skiĭ's conjecture is correct, then from this theorem of Garcia-Ferreira will follow an affirmative answer to 1A.13(b).

The question whether every compact space of countable tightness is sequential, raised by R. C. MOORE and MROWKA in [1964], is answered in the negative assuming diamond by OSTASZEWSKI [1976] and independently, under the same assumption, by FEDORCHUK [1976]. Since the compact spaces of Ostaszewski and Fedorchuk are of cardinality less than or equal to  $2^\omega$ , they are  $p$ -sequential (for some  $p \in \omega^*$ ) by the theorem of GARCIA-FERREIRA [1989] cited above. Recently, in [1988], BALOGH, DOW, FREMLIN and NYIKOS have used the proper forcing axiom to answer the Moore-Mrowka question in the affirmative; see BALOGH [1989] for a detailed proof.

Since there exist spaces of countable tightness which are  $p$ -sequential for no  $p \in \omega^*$ , a positive answer to the following question would yield a negative answer to 1A.12.

**Question 1A.14.** *Can every space  $X$  be embedded as a closed subspace into a topological group  $G$  such that  $tG = tX$ ?* **488. ?**

The answer is affirmative when  $X$  is a  $k_\omega$ -space (GARCIA-FERREIRA [1989]).

It is immediate from the theorem of ARKHANGEL'SKIĬ from [1980] cited just after Question 1A.11 that for a non-degenerate compact group  $G$  and  $\alpha > \omega$  the power  $G^\alpha$  is  $p$ -sequential for no  $p \in \omega^*$ .

To answer in the negative (in **ZFC**) the question whether the product of finitely many groups of countable tightness again has countable tightness, it is enough to cite the argument of MALYKHIN [1987]. First, recall two nice theorems from the literature:

(a) PRZYMUSIŃSKI [1980]. There exist spaces  $X$  and  $Y$  such that  $X^n$  and  $Y^n$  are Lindelöf spaces for all  $n < \omega$ , but  $X \times Y$  is not Lindelöf; and

(b) Arkhangel'skiĭ and Pytkeev (see ARKHANGEL'SKIĬ [1978, 4.1.2]). For each space  $Z$ , the group  $C(Z)$  (in the topology inherited from  $\mathbb{R}^Z$ ) has countable tightness if and only if  $Z^n$  is Lindelöf for all  $n < \omega$ .

Now with  $X$  and  $Y$  chosen as in (a), the topological groups  $C(X)$  and  $C(Y)$  satisfy  $tC(X) \leq \omega$  and  $tC(Y) \leq \omega$ ; but  $C(X) \times C(Y)$ , which is topologically isomorphic to  $C(Z)$  with  $Z$  the topological sum  $Z = X \oplus Y$ , satisfies  $t(C(X) \times C(Y)) > \omega$  since  $X \times Y$  is not Lindelöf.

The following question, considered by MALYKHIN in [1987], remains unsolved.

**Question 1A.15.** *Is there a topological group  $G$  such that  $\omega = tG < t(G \times G)$ ?* **489. ?**

In [1986] and [1987] MALYKHIN has shown that in the model of **ZFC** obtained by adding a single Cohen real, there is in  $\{-1, 1\}^{\omega^+}$  an S-group  $G$  (that is, a hereditarily separable, non-Lindelöf group  $G$ ) such that ( $tG = \omega$  and)  $t(G \times G) > \omega$ . If also **MA** holds and **CH** fails in the ground model, then  $G$

may be chosen Frechet-Urysohn; in this case,  $G$  is  $p$ -sequential for all  $p \in \omega^*$ , and  $G \times G$  is  $p$ -sequential for no  $p \in \omega^*$ .

? 490. **Question 1A.16.** *Is it consistent with the axioms of ZFC that for  $p \in \omega^*$  the class of  $p$ -sequential groups is closed under finite products? Countable products?*

It must be mentioned that the questions listed above owe much to theorems, conjectures and questions contributed by ARKHANGEL'SKIĬ in [1978, 1980, 1981b, 1987]. For example, in [1980, 1.17] ARKHANGEL'SKIĬ has conjectured that every compact homogeneous space of countable tightness is first countable; ARKHANGEL'SKIĬ [1978, section 4] and [1987] contains fundamental "positive" results on the stability of tightness under the formation of products, and ARKHANGEL'SKIĬ [1978] and [1980, page 21] raise questions about embedding spaces into groups of countable tightness.

Most of the theorems enunciated and the problems posed in this section can be phrased for other classes. For example, the class G of topological groups may be profitably replaced throughout by the class AG of Abelian groups.

### 1B. Closed Embeddings

Suppose that T (= TS) is a topological property not closed under products. In an effort to investigate the behavior of TG under products, it is natural to raise two questions: (a) Is the class T closed-hereditary? (b) Are there  $X$  and  $Y \in \underline{T}$  such that  $X \times Y \notin \underline{T}$  and there exist embeddings  $X \subseteq H \in \underline{TG}$  and  $Y \subseteq G \in \underline{TG}$  with  $X$  closed in  $H$  and  $Y$  closed in  $G$ ? When (a) and (b) can be answered affirmatively, the proof is complete that the class TG is not closed under products. A similar strategy, of course, may be attempted with respect to the class TH.

This was the technique employed by van Douwen who, using the construction of PRZYMUSIŃSKI [1980] referred to above, found two Lindelöf groups  $G$  and  $H$  such that  $G \times H$  is not Lindelöf.

(In COMFORT [1984, 8.4] I indicated on the basis of a letter received from van Douwen that his proof would appear in his paper VAN DOUWEN [19∞b]. More recently I have learned that the version of this paper now scheduled for posthumous publication does not include this argument; it appears therefore that COMFORT [1984, 8.4] is the most accessible published source for van Douwen's construction of two Lindelöf groups whose product whose product is not Lindelöf.) Similarly, in [1983] OKROMESHKO has shown that for each of the following properties T, every  $X \in \underline{T}$  embeds as a closed subspace (indeed, as a retract) into a space  $H(X) \in \underline{TH}$ : T = Lindelöf, T = paracompact, T = hereditarily paracompact, T = of tightness less than  $\alpha$ . Since each of these classes T is closed-hereditary and there is  $X \in \underline{T}$  such that  $X \times X \notin \underline{T}$ ,

it follows not only that  $\mathbf{TH}$  is not closed under products but even that (for each such  $\mathbf{T}$ ) there is  $Y = H(X) \in \mathbf{TH}$  such that  $Y \times Y \notin \mathbf{TH}$ .

**Question 1B.1.** (ARKHANGEL'SKIĬ [1980, 1981a, 1988]) (a) *Is there a Lindelöf group  $G$  such that  $G \times G$  is not Lindelöf?* (b) *Can every Lindelöf space be embedded as a closed subspace of a Lindelöf group? What about the Sorgenfrey line?* 491. ?

To find a Lindelöf group  $G$  such that  $G \times G$  is not Lindelöf it would suffice to find a Lindelöf group  $G$  with a closed subspace which maps continuously onto some Lindelöf space  $X$  such that  $X \times X$  is not Lindelöf. (For if  $A$  is such a subspace of  $G$  then, since  $A \times A$  maps continuously onto  $X \times X$ , and  $A \times A$  is closed in  $G \times G$ , the group  $G \times G$  cannot be Lindelöf.) This suggests the following variant of Question 1B.1.

**Question 1B.2.** (a) *Are there a Lindelöf group  $G$  and a space  $X$  such that  $X \times X$  is not Lindelöf and some closed subspace of  $G$  maps continuously onto  $X$ ?* (b) (ARKHANGEL'SKIĬ [1988]) *Is the Sorgenfrey line the continuous image of a Lindelöf group?* 492. ?

Questions 1B.1(a) and 1B.2(a) should be considered in  $\mathbf{ZFC}$ , since MALYKHIN, in [1986], using a  $\mathbf{ZFC}$ -consistent axiom he calls  $N(\aleph_1)$ , has constructed a dense, hereditarily Lindelöf subgroup of  $\{-1, +1\}^{\omega^+}$  such that  $G \times G$  is not a Lindelöf space.

A nice construction of USPENSKIĬ [1983] shows (in just a few lines) how to embed each space  $X$  as a retract into a suitable homogeneous space  $U(X)$ —indeed, with  $U(X)$  homeomorphic to  $X \times U(X)$ . Using his construction repeatedly, COMFORT and VAN MILL [1985] found a number of results similar in spirit to those of OKROMESHKO [1983]. For example, there are pseudocompact homogeneous spaces  $X_0$  and  $X_1$  such that  $X_0 \times X_1$  is not pseudocompact; if  $\mathbf{MA}$  is assumed (in order to find two  $\leq_{\mathbf{RK}}$ -incomparable,  $\leq_{\mathbf{RK}}$ -minimal points in  $\omega^*$ ), then  $X_0$  and  $X_1$  may be chosen countably compact.

**Question 1B.3.** *Is there a pseudocompact, homogeneous space  $X$  such that  $X \times X$  is not pseudocompact?* 493. ?

**Question 1B.4.** *Is there a countably compact, homogeneous space  $X$  such that  $X \times X$  is (a) not countably compact? (b) not pseudocompact?* 494. ?

Question 1B.4(a) is answered affirmatively using  $\mathbf{MA}_{\text{countable}}$  by the result of HART and VAN MILL [19 $\infty$ ] cited above. Since the product of pseudocompact groups is pseudocompact (COMFORT and ROSS [1966]), questions 1B.3 and 1B.4(b) must not be carried from the class  $\mathbf{H}$  over to the class  $\mathbf{G}$ .

In March, 1985, D. B. Motorov reported to a topological seminar at Moscow State University (USSR) that the closure  $K$  in Euclidean 2-space of the graph of the function  $f(x) = \sin(1/x)$  (with  $0 < x \leq 1$ ) does not embed as a retract into any compact homogeneous space. As a consequence, there is no compact space  $X$  such that  $K \times X$  is homogeneous. (Our reference below to the paper of MOTOROV [1985], like the reference given by ARKHANGEL'SKIĬ [1987, (3.2)], is a trifle misleading, since it reduces to a single line with no substantial mathematical content.) It is known, however, that every compact space embeds as a retract into a homogeneous space which may be chosen  $\sigma$ -compact (OKROMESHKO [1983], COMFORT and VAN MILL [1985]) or (for  $p \in \omega^*$  chosen in advance)  $p$ -compact (COMFORT and VAN MILL [1985]). This shows that for every cardinal  $\alpha$  there is a homogeneous  $\sigma$ -compact space  $X$ , and there is a homogeneous countably compact space  $Y$ , such that  $cX > \alpha$  and  $cY > \alpha$ . (For a proof, apply the theorems just cited to some compactification of the discrete space  $\alpha^+$ .) The following question remains open.

? 495. **Question 1B.5.** (van Douwen) *For what cardinal numbers  $\alpha$  is there a compact, homogeneous space with cellularity greater than  $\alpha$ ? What about the case  $\alpha = \mathfrak{c}$ ?*

MAURICE [1964] and VAN MILL [1982] have given examples in **ZFC** of compact homogeneous spaces containing  $\mathfrak{c}$ -many pairwise disjoint non-empty open subsets.

The statement that every compact space  $X$  embeds as a retract into (a) a homogeneous  $\sigma$ -compact space, and as a retract into (b) a homogeneous  $p$ -compact space, raises the question whether these enveloping spaces may be chosen even to be topological groups. In general the answer in each case is No.

- (a) In [1983] TKACHENKO has shown that every  $\sigma$ -compact group is a **ccc** space. In a slightly different direction, USPENSKIĬ [1982] has shown that if  $\alpha \geq \omega$  and the group  $G$  has the property that for every non-empty open subset  $U$  of  $G$  there is  $A \subseteq G$  such that  $|A| \leq \alpha$  and  $G = AU$ , then  $cG \leq 2^\alpha$ ; further, the upper bound  $cG = 2^\alpha$  is realized for suitable  $G$ . The principal result of TKACHENKO [1983] is generalized in USPENSKIĬ [1985].
- (b) Every  $p$ -compact group, since it is countably compact and hence pseudocompact and hence totally bounded (COMFORT and ROSS [1966]), is a **ccc** space (being a subgroup of its Weil completion).

? 496. **Question 1B.6.** *Does every countably compact space embed as a retract into a countably compact homogeneous space?*

The existence of Haar measure shows  $cG \leq \omega$  for every compact group  $G$ , and TKACHENKO's theorem from [1983] gives the same inequality for  $\sigma$ -compact groups  $G$ . The inequality  $cG \leq \omega$  cannot be shown for Lindelöf

groups, since there exist Lindelöf groups  $G$  such that  $cG = \omega^+$ . To see this it is enough to embed the 1-point Lindelöfication (call it  $X$ ) of the discrete space  $\omega^+$  into a topological group  $K$  which is a  $P$ -space, for then the subgroup  $G = \langle X \rangle$  of  $K$  is a Lindelöf group such that  $cG = \omega^+$ . (A theorem of NOBLE [1971] is helpful here: The product of countably many Lindelöf  $P$ -spaces is Lindelöf. In particular  $X^\omega$ , hence  $X^n$  for each  $n < \omega$ , is a Lindelöf space; thus the map  $x = \langle x_1, \dots, x_n \rangle \rightarrow x_1 \cdot \dots \cdot x_n$  from  $X^n$  to  $\langle X \rangle$  has Lindelöf image, so  $\langle X \rangle$  itself, the union of countably many Lindelöf spaces, is Lindelöf.) The required embedding  $X \subseteq K$  may be achieved as in TKACHENKO [1983] by taking for  $K$  the free topological group over  $X$ . (The fact that the free topological group over a  $P$ -space is itself a  $P$ -space is easily established; TKACHENKO [1983] cites ARKHANGEL'SKIĬ [1980, (6.9)] in this connection.) Alternatively, as suggested in conversation by Jan van Mill, one may note that the  $P$ -space modification  $K = P(\{0, 1\}^{\omega^+})$  of the group  $\{0, 1\}^{\omega^+}$  (defined as in 2B below) contains a natural copy of  $X$ .

The following natural question is apparently unsolved.

**Question 1B.7.** *Is the relation  $cG \leq \omega^+$  valid for every Lindelöf topological group?* **497. ?**

Of course, the theorem cited above from USPENSKIĬ [1982] answers 1B.7 affirmatively in case **CH** is assumed.

Following COMFORT and VAN MILL [1988], for subclasses **U** and **V** of **G** and a space  $X$ , we say that a topological group  $G$  is a *free* (**U**, **V**)-group over  $X$  if

- (a)  $X$  is a subspace of  $G$ ,
- (b)  $G \in \mathbf{U}$ , and
- (c) every continuous  $f: X \rightarrow H$  with  $H \in \mathbf{V}$  extends uniquely to a continuous homomorphism  $\bar{h}: G \rightarrow H$ .

It is shown in COMFORT and VAN MILL [1988] that (a) there is a free (**PAG**, **PAG**)-group over  $X$  if and only if  $X = \emptyset$ , and (b) for every space  $X$  there is a free (**PAG**, **CAG**)-group over  $X$  in which  $X$  is closed. These results suggest the following question.

**Question 1B.8.** *For what non-empty spaces  $X$  does there exist a free (**CCAG**, **CCAG**)-group over  $X$ ? For some  $X$ ? For all  $X$ ?* **498. ?**

**Question 1B.9.** *Does every countably compact space  $X$  admit a free (**CCAG**, **CCAG**)-group over  $X$  in which  $X$  is closed?* **499. ?**

Evidently, 1B.9 is an “ambitious” question. An affirmative answer would answer 1A.2 and 1B.4(a) affirmatively, and 1A.1 negatively.

For constructions related to the one just cited, and for helpful references to the literature, the reader might consult MORRIS [1982, 1984].

## 2. Proper Dense Subgroups

Some topological groups do, and some do not, have a proper dense subgroup. In this section we discuss some of the relevant literature and cite some unsolved problems. For a proof of some of the theorems we quote or use, and for additional references, see HEWITT and ROSS [1963] and COMFORT [1984].

### 2A. The General Case

The relation  $|G| = 2^{wG}$  holds for each infinite topological group which is either (a) compact or (b)  $\sigma$ -compact and locally compact and non-discrete, so each such group admits a proper dense subgroup  $H$  (indeed, even with  $|H| < |G|$ ). It is tempting to conjecture that every non-discrete, locally compact group admits a proper dense subgroup, but in [1976] RAJAGOPALAN and SUBRAHMANIAN have described in detail a number of (divisible, Abelian) counterexamples.

A topological group  $G$  is said to be *totally bounded* (by some authors: *pre-compact*) if, for every non-empty open subset  $U$  of  $G$ ,  $G$  is covered by a finite number of translates of  $U$ . It is a theorem of WEIL [1937] that a topological group  $G$  is totally bounded if and only if  $G$  is a dense subgroup of a compact group. When these conditions hold, the enveloping compact group is unique in the obvious sense. It is denoted  $\overline{G}$  and is called the *Weil completion* of  $G$ . Since a locally compact, totally bounded group is compact, the groups of RAJAGOPALAN and SUBRAHMANIAN [1976] cannot be totally bounded. Accordingly one may ask whether every infinite totally bounded group has a proper dense subgroup, but this question also is excessively naive: When an Abelian group  $G$  is given its largest totally bounded topological group topology (as defined in 3F below), the resulting topological group  $G^\#$  has the property that every subgroup of  $G$  is closed in  $G^\#$ . Since  $w(G^\#) = 2^{(|G|)} > |G|$ , it then becomes proper to ask: Does every infinite totally bounded Abelian group  $G$  such that  $wG \leq |G|$  have a proper dense subgroup? For  $wG = \omega$ , anything can happen: For  $1 \leq i \leq 4$  there are totally bounded Abelian groups  $G_i$  such that  $wG_i = |G_i| = \omega$ ,  $G_1$  and  $G_2$  are torsion groups,  $G_3$  and  $G_4$  are torsion-free,  $G_1$  and  $G_3$  have proper, dense subgroups, and  $G_2$  and  $G_4$  have none (COMFORT and VAN MILL [19 $\infty$ ]). If  $G$  is a totally bounded Abelian group such that  $|G/tG| \geq wG > \omega$  (in particular, if  $G$  is torsion-free with  $|G| \geq wG > \omega$ ) then  $G$  has a proper dense subgroup, but for every strong limit cardinal  $\alpha$  of countable cofinality there is a totally bounded Abelian torsion group  $G$  such that  $wG = |G| = \alpha$  and  $G$  has no proper dense subgroup. These restrictions on  $\alpha$  are not known to be essential, and the following questions are left unsolved in COMFORT and VAN MILL [19 $\infty$ ].

? 500. **Question 2A.1.** *Let  $\alpha$  be an infinite cardinal number. Are there totally*

bounded Abelian torsion groups  $G_0$  and  $G_1$  such that  $G_i$  has no proper dense subgroup and

- (a)  $w(G_0) = |G_0| = \alpha$ ?
- (b)  $w(G_1) = |G_1| = 2^{<\alpha}$ ?

What if  $\alpha$  is assumed to be a (strong) limit cardinal?

## 2B. The Pseudocompact Case

For certain topological properties  $\underline{\mathbf{T}}$ , there exist topological groups  $G$  with  $\underline{\mathbf{T}}$  (that is,  $G \in \underline{\mathbf{TG}}$ ) such that  $wG > \omega$  and no proper dense subgroup of  $G$  has  $\underline{\mathbf{T}}$ . (Among known examples of such classes  $\underline{\mathbf{T}}$  are the class of  $\omega$ -bounded groups and the class  $\underline{\mathbf{CCG}}$  of countably compact groups.)

What happens with respect to pseudocompactness? Every pseudocompact metric space is compact, so the case  $G \in \underline{\mathbf{PG}}$  with  $wG = \omega$  has no interest. When  $G$  is Abelian (that is,  $G \in \underline{\mathbf{PAG}}$ ) with  $wG = \alpha > \omega$ , it is known that  $G$  has a proper dense pseudocompact subgroup provided either that  $G$  is zero-dimensional (COMFORT and ROBERTSON [1988]) or that  $G$  is connected and one of the following five conditions holds:

- (i)  $wG \leq \mathfrak{c}$ ;
- (ii)  $|G| \geq \alpha^\omega$ ;
- (iii)  $\alpha$  is a strong limit cardinal and  $cf(\alpha) > \omega$ ;
- (iv)  $|tG| > \mathfrak{c}$ ;
- (v)  $G$  is not divisible (COMFORT and VAN MILL [1989]).

Several questions arise, the following being typical.

**Question 2B.1.** Does every pseudocompact group  $G$  of uncountable weight **501.** ? have a proper dense pseudocompact subgroup? What if  $G$  is Abelian? Connected and Abelian?

If a counterexample is sought, perhaps the most accessible candidate not excluded by known results lies in the torus of dimension  $\mathfrak{c}^+$ :

**Question 2B.2.** Let  $G$  be a dense, pseudocompact subgroup of  $\mathbb{T}^{\mathfrak{c}^+}$ . Must **502.** ?  $G$  have a proper dense subgroup?

Concerning an infinite compact group  $K$  with  $wK = \alpha$ , two powerful and remarkable statements are available:

- (1) There is a continuous surjection  $f: \{0, 1\}^\alpha \rightarrow K$ ;
- (2) there is a continuous surjection  $g: K \rightarrow \mathbb{I}^\alpha$ .

Statement (1) is due to IVANOVSKIĬ [1958] (the Abelian case) and KUZ' MINOV [1959] (the general case); see also HEWITT and ROSS [1963, (9.15, 25.35)] and USPENSKIĬ [1985, 1988]. Statement (2) is due to SHAPIROVSKIĬ [1975, 1980]; see also BALCAR and FRANĚK [1982], GERLITS [1976, 1980, 1978/81] and JUHÁSZ [1980]. We note that (1) and (2) furnish very brief



proofs of the useful identities  $|K| = 2^\alpha$  and  $dK = \log \alpha$ ; the original proofs of these results were achieved over a period of years, before (1) and (2) were available, by direct arguments.

Now for a space  $X = (X, \mathfrak{S})$  let  $P(X) = (P(X), \mathfrak{S}')$  denote the set  $X$  with the smallest topology  $\mathfrak{S}'$  such that  $\mathfrak{S}' \supset \mathfrak{S}$  and every  $\mathfrak{S}$ - $G_\delta$ -set is  $\mathfrak{S}'$ -open. That is,  $P(X)$  carries the  $P$ -space topology generated by the topology of  $X$ . It is clear that for  $K \in \underline{\mathbf{CG}}$  as above the continuous functions  $f: \{0, 1\}^\alpha \rightarrow K$  and  $g: K \rightarrow \mathbb{I}^\alpha$  remain continuous when the spaces  $\{0, 1\}^\alpha$ ,  $K$  and  $\mathbb{I}^\alpha$  are replaced by  $P(\{0, 1\}^\alpha)$ ,  $P(K)$  and  $P(\mathbb{I}^\alpha)$ , respectively. Since the first and third of these spaces are homeomorphic, it follows that  $dP(K) = dP(\{0, 1\}^\alpha)$ . (In fact all three of those  $P$ -spaces are homeomorphic; see CHOBAN [1976] for a proof.) Beginning with a dense subgroup  $D$  of  $K$  such that  $|D| = \log \alpha$ , a routine induction over the countable ordinals yields a countably compact subgroup  $G$  of  $K$  such that  $D \subseteq G \subseteq K$  and  $|G| \leq (\log \alpha)^\omega$ . Since for any compact space  $X$  a dense, countably compact subspace of  $X$  is  $G_\delta$ -dense in  $X$ —i.e., is dense in  $P(X)$ —we have

$$\log \alpha = dK = d(\{0, 1\}^\alpha) \leq dP(\{0, 1\}^\alpha) = dP(K) \leq (\log \alpha)^\omega \quad (*)$$

for every  $K \in \underline{\mathbf{CG}}$  with  $wK = \alpha \geq \omega$ .

? 503. **Question 2B.3.** Is  $dP(\{0, 1\}^\alpha) = dP(K) = (\log \alpha)^\omega$  a theorem of **ZFC**?

Question 2B.3, which together with (\*) is taken from COMFORT and ROBERTSON [1985], is a special case of a question raised in a very general context by CATER, ERDŐS and GALVIN in [1978]. As observed in this paper, the singular cardinals hypothesis ( $\kappa^\lambda \leq 2^\lambda \cdot \kappa^+$  for all infinite cardinals) is enough to settle 2B.3 affirmatively. Indeed for our limited purposes one needs only  $(\log \alpha)^\omega \leq \mathfrak{c} \cdot (\log \alpha)^+$ .

For  $K$  as above, the cardinal  $dP(K)$  is the least cardinality of a dense pseudocompact subgroup of  $K$  (COMFORT and ROBERTSON [1985]). Present attempts to find a compact group  $K$  with  $wK = \alpha$  and a dense pseudocompact subgroup  $G$  for which  $|G| = dP(K) < (\log \alpha)^\omega$  seem to fail because present methods produce a group  $G$  which is perhaps “too large”— $G$  is even countably compact. We are led to this question.

? 504. **Question 2B.4.** Let  $G$  be a dense pseudocompact subgroup of a compact group  $K$ . Must  $K$  contain a dense countably compact subgroup  $C$  such that  $|C| \leq |G|$ ? Can one choose  $C \supset G$ ?

### 3. Miscellaneous Problems

#### 3A. The Structure of LCA Groups

As is well-known (see for example HEWITT and ROSS [1980]), the group  $\hat{G}$  of continuous homomorphisms from a locally compact Abelian group  $G$  to

the circle group  $\mathbb{T}$  is itself a locally compact Abelian group in the compact-open topology. A locally compact Abelian group  $G$  is said to be *self-dual* if the groups  $G$  and  $\hat{G}$  are topologically isomorphic. The self-dual torsion-free locally compact Abelian groups which satisfy certain additional conditions (e.g., metrizable or  $\sigma$ -compact) have been identified and classified by RAJAGOPALAN and SOUNDARARAJAN in [1969]. (As is remarked by ARMACOST [1981] and ROSS [1968] the stronger result announced earlier (RAJAGOPALAN and SOUNDARARAJAN [1967]) was overly optimistic.) Shortly thereafter, as ARMACOST writes in [1981, (4.37)]: “CORWIN [1970] initiated a new and interesting approach to the problem of classifying the self-dual LCA groups.” In its full generality, the following question is apparently still open.

**Question 3A.1.** *Classify the self-dual locally compact Abelian groups.* **505. ?**

For a related investigation of “duality” in a (not necessarily Abelian) context, see MUKHIN [1985].

### 3B. Infinite Compact Groups

I first heard the following question from Kenneth A. Ross about 20 years ago. I do not know if it remains open today.

**Question 3B.1.** *Does every infinite compact group contain an infinite Abelian subgroup?* **506. ?**

In dealing with 3B.1 it is enough to consider groups in which every element has finite order. It is natural then to consider the following question, which also dates back at least 20 years (HEWITT and ROSS [1970, (28.23(b))]) and is apparently still open.

**Question 3B.2.** *Must a compact group in which each element has finite order have the property that the orders of its elements are bounded?* **507. ?**

MCMULLEN [1974] contributes to both 3B.1 and 3B.2, while HERFORT shows in [1979] that a compact group hypothesized as in 3B.2 is of bounded order if and only if each of its Sylow subgroups is of bounded order.

It is perhaps worthwhile to remark that an elementary argument based on the Baire category theorem answers 3B.2 affirmatively in the case of Abelian groups. Indeed, the compact Abelian torsion groups have been classified in concrete form; see for example HEWITT and ROSS [1963, (25.9)]. The Baire category argument applies readily to Abelian torsion groups which are assumed only to be pseudocompact; see COMFORT and ROBERTSON [1988, §7] for a proof and an application.

### 3C. The Free Abelian Group

Let  $G$  denote the free Abelian group on  $\mathfrak{c}$ -many generators. According to recent correspondence from Michael G. Tkachenko, the group  $G$  admits no compact (Hausdorff) topological group topology but, assuming **CH**,  $G$  does admit a countably compact topological group topology; this latter may be chosen hereditarily separable, hereditarily normal, connected and locally connected. It is unknown whether this or a similar construction is available in **ZFC**.

- ? 508. **Question 3C.1.** (M. G. Tkachenko (**ZFC**)) *Can the free Abelian group on  $\mathfrak{c}$ -many generators be given a countably compact topological group topology?*

### 3D. A Universal Topological Group

Responding to a question posed by A. V. Arkhangel'skiĭ and leaning on an idea of V. G. Pestov, USPENSKIĬ [1986] has shown that there is a separable metrizable topological group  $G$  which contains (up to topological isomorphism) every separable metrizable topological group. Indeed the group  $\text{Homeo}(\mathbb{I}^\omega)$  of homeomorphisms of the Hilbert cube into itself, in the topology of uniform convergence, is a realization of such a group  $G$ ; as a space, this  $G$  is homeomorphic to sequential Hilbert space  $\ell^2$ .

It is unclear whether the method of USPENSKIĬ [1986] can be adapted to higher cardinal numbers, and the following question remains unsettled.

- ? 509. **Question 3D.1.** *For what cardinal numbers  $\alpha$  is there a topological group  $G(\alpha)$  of weight  $\alpha$  with this property: Every topological group of weight  $\alpha$  is topologically isomorphic to a subgroup of  $G(\alpha)$ ?*

Even in the case  $\alpha = \omega$ , the Abelian version of the question answered by Uspenskii's theorem remains open.

- ? 510. **Question 3D.2.** (ARKHANGEL'SKIĬ [1987, (Problem VI.14)]) *For what cardinals  $\alpha$  is there an Abelian topological group  $G(\alpha)$  of weight  $\alpha$  with this property: Every Abelian topological group of weight  $\alpha$  is topologically isomorphic to a subgroup of  $G(\alpha)$ ? Is  $\alpha = \omega$  such a cardinal?*

There are other contexts in which the Abelian version of a natural question appears to be less tractable than the general case. As motivation for question 3D.3 below, Sidney A. Morris points to the well known fact that for  $1 < n < \omega$  the free group  $F(n)$  on  $n$  generators contains  $F(m)$  for all  $m < \omega$ —indeed, even for  $m = \omega$ —while the free Abelian group  $FA(n)$  on  $n$  generators contains  $FA(m)$  if and only if  $m \leq n$ . Now for a space  $X$ , let  $F(X)$  and  $FA(X)$  denote respectively the free topological group, and the free

Abelian topological group, over  $X$ . (In the terminology and notation preceding Question 1B.8 above,  $F(X)$  is a free  $(\underline{\mathbf{G}}, \underline{\mathbf{G}})$ -group over  $X$ , and  $FA(X)$  is a free  $(\underline{\mathbf{AG}}, \underline{\mathbf{AG}})$ -group over  $X$ .) Early constructions of the groups  $F(X)$  and  $FA(X)$  or of closely related groups are given by MARKOV [1941, 1962], GRAEV [1962, 1950], NAKAYAMA [1943], KAKUTANI [1981] and SAMUEL [1948]; see also THOMAS [1974]. The papers of MORRIS [1982, 1984] offer new results in several directions concerning free groups and related topics, and they contain extensive bibliographies.

It is known (NICKOLAS [1976]) that the free topological group  $F(\mathbb{I})$  contains (up to topological isomorphism) the group  $F(X)$  for every finite-dimensional compact metric space. Proceeding by analogy with the results cited above concerning  $F(n)$  and  $FA(n)$ , Morris has conjectured that the group  $FA(\mathbb{I})$  contains  $FA(X)$  (for  $X$  compact metric) if and only if the dimension of  $X$  is 0 or 1. As an arresting special case, he proposes the following test question.

**Question 3D.3.** (MORRIS [1984]) *Is  $FA(\mathbb{I} \times \mathbb{I})$  topologically isomorphic with a subgroup of  $FA(\mathbb{I})$ ?* **511. ?**

The negative answer to 3D.3 anticipated by Morris will strengthen the “only if” part of his conjecture. For the proof of the existence of an embedding of  $FA(\mathbb{T})$  into  $FA(\mathbb{I})$ , and for other evidence supporting the “if” direction of the conjecture, see KATZ, MORRIS and NICKOLAS [1984].

### 3E. Epimorphisms

A continuous homomorphism  $h: H \rightarrow G$ , with  $H$  and  $G$  topological groups, is said to be an *epimorphism* if for every two continuous homomorphisms  $f$  and  $g$  from  $G$  to a topological group, the equality  $f \circ h = g \circ h$  guarantees  $f = g$ . Because of our standing restriction here to Hausdorff topological groups, it is obvious that every continuous homomorphism  $h: H \rightarrow G$  with  $h[H]$  dense in  $G$  is an epimorphism. It is an intriguing question, raised years ago by Karl H. Hofmann and considered subsequently by many workers, whether the “dense image” homomorphisms are the only epimorphisms. The question may be phrased as follows.

**Question 3E.1.** (Hofmann) *Given a proper closed subgroup  $H$  of a (Hausdorff) group  $G$ , must there exist a topological group  $K$  and continuous homomorphisms  $f, g: G \rightarrow K$  such that  $f \neq g$  and  $f|_H = g|_H$ ?* **512. ?**

It is obvious that the answer to Question 3E.1 is “Yes” when  $G$  has a proper closed normal subgroup  $N$  containing  $H$  (in particular, when  $G$  is Abelian); in this case one may take  $K = G/N$ ,  $f$  the canonical homomorphism and  $g$  the trivial homomorphism.

The answer to Question 3E.1 also is “Yes” when  $G$  and  $K$  are required to belong to the class of compact groups (POGUNTKE [1970]) or to the class of  $k_\omega$ -groups (LAMARTIN [1977]). For these and additional results, and for comprehensive bibliographical surveys of the literature, see THOMAS [1973, 1977], LAMARTIN [1976, 1977] and NUMMELA [1978].

### 3F. The Finest Totally Bounded Topological Group Topology

When  $G$  is Abelian the group  $\text{Hom}(G, \mathbb{T})$  of homomorphisms from  $G$  to  $\mathbb{T}$  separates points and accordingly the evaluation function  $i: G \rightarrow \mathbb{T}^{\text{Hom}(G, \mathbb{T})}$  induces a (Tikhonov) topology on  $G$ ; the group  $G$  with this topology is denoted  $G^\#$ . The Abelian group  $G^\#$  is totally bounded, and every homomorphism from  $G$  to a totally bounded topological group  $H$  is continuous as a function from  $G^\#$  to  $H$ . In particular, the topology of  $G^\#$  is the finest topology for  $G$  relative to which  $G$  is a totally bounded topological group.

The following three questions are taken from VAN DOUWEN [19 $\infty$ a].

? 513. **Question 3F.1.** For  $|G| > \omega$ , is  $G^\#$  a normal topological space? Always? Sometimes? Never?

If  $X$  is a space such that  $|X| < 2^\omega$ , and if  $p, q \in \beta X$  with  $p \neq q$ , then  $p$  and  $q$  are separated in  $\beta X$  by a complementary pair of open-and-closed sets. (For, having chosen a continuous  $f: \beta X \rightarrow \mathbb{I}$  such that  $f(p) = 0$  and  $f(q) = 1$ , find  $r \in \mathbb{I} \setminus f[X]$ . The set  $f^{-1}([0, r])$  is open-and-closed in  $X$ , so its closure in  $\beta X$  is open-and-closed in  $\beta X$ .) It follows that every  $X$  with  $|X| < 2^\omega$  is strongly zero-dimensional in the sense that  $\beta X$  is zero-dimensional. It is known, further, see VAN DOUWEN [19 $\infty$ a] and COMFORT and TRIGOS [1988], that the groups  $G^\#$  are zero-dimensional.

? 514. **Question 3F.2.** For  $|G| \geq 2^\omega$ , is  $G^\#$  strongly zero-dimensional?

Perhaps the boldest of van Douwen's questions is this.

? 515. **Question 3F.3.** Does  $|G|$  determine  $G^\#$  up to homeomorphism?

Now for a locally compact Abelian group  $G = \langle G, \mathfrak{S} \rangle$ , let  $G^+$  denote the set  $G$  with the topology induced by the set of  $\mathfrak{S}$ -continuous homomorphisms from  $G$  to  $\mathbb{T}$ . ( $G^+$  may be viewed as  $G$  with the topology inherited from its *Bohr compactification*. This is the finest totally bounded topological group topology for  $G$  coarser than the locally compact topology  $\mathfrak{S}$ .) F. J. Trigos has noted that if  $G$  and  $H$  are both locally compact Abelian groups such that  $G^+$  and  $H^+$  are homeomorphic as spaces, then  $G$  and  $H$  are also homeomorphic. (This fact follows easily from work of GLICKSBERG [1962]; see TRIGOS [19 $\infty$ ] for a direct

treatment.) This suggests the following extended version of van Douwen's question 3F.3.

**Question 3F.4.** (TRIGOS [19 $\infty$ ]) *If  $G$  and  $H$  are locally compact Abelian groups which are homeomorphic as spaces, must  $G^+$  and  $H^+$  be homeomorphic?* **516. ?**

### 3G. Markov's Fifth Problem

One of the questions posed in MARKOV's celebrated paper [1962] on free topological groups is this: If  $G$  is a group in which each unconditionally closed subgroup  $H$  satisfies  $|G/H| \geq 2^\omega$ , must  $G$  admit a connected topological group topology? (A subset  $X$  of  $G$  is said to be *unconditionally closed* if  $X$  is closed in each (Hausdorff) group topology for  $G$ .) The question intrigued Markov as a potential characterization of groups which admit a connected topological group topology: It is obvious that if  $H$  is a closed subgroup of such a group  $G$  then, since the coset space  $G/H$  is a continuous image of  $G$ , one has  $|G/H| \geq 2^\omega$ .

Remaining open for over 40 years, Markov's problem has been recently solved in the negative by PESTOV [1988] and, independently, by REMUS [19 $\infty$ ]. Pestov's construction proceeds through the theory of pre-norms, locally convex topological linear spaces, semidirect products, and equicontinuous group actions, while Remus' construction is more simple and direct. (Remus uses two facts: (a) For  $\alpha \geq \omega$  every proper subgroup  $H$  of the group  $S(\alpha)$  of permutations of  $\alpha$  satisfies  $|S(\alpha)/H| \geq \alpha$ , and (b) every topological group topology on groups of the form  $S(\alpha)$  is totally disconnected.)

The examples of Pestov and Remus are non-Abelian, and according to Remus the following variant of Markov's problem remains open and is worthy of investigation.

**Question 3G.1.** (REMUS [19 $\infty$ ], following MARKOV [1962]) *Is there an Abelian group  $G$ , with no connected topological group topology, such that every unconditionally closed subgroup  $H$  of  $G$  satisfies  $|G/H| \geq 2^\omega$ ?* **517. ?**

### 3H. Compact Images

The following somewhat specialized question has been suggested by Michael G. Tkachenko (letter of April, 1989).

**Question 3H.1.** *Suppose that  $\alpha$  is an infinite cardinal number and  $X$  is a compact space with  $wX \leq 2^\alpha$  such that  $X$  is the continuous image of a  $\sigma$ -compact topological group  $G$ . Does it follow that  $dX \leq \alpha$ ? What if  $wX \leq \alpha^+$ ?* **518. ?**

The question is appealing since the answer is “Yes” in case  $G$  is assumed compact. Indeed in this case, since  $G$  is dyadic by the result of Vilenkin and Kuz'minov, the space  $X$  is itself the continuous image of  $\{0, 1\}^{2^\alpha}$  (which, by the Hewitt-Marczewski-Pondiczery theorem, has density character less than or equal to  $\alpha$ ).

### 3I. Minimal Topological Groups

It is an old question of MARKOV [1962] whether every infinite group admits a non-discrete topological group topology. Assuming **CH**, SHELAH [1980] found a group of cardinality  $\omega^+$  with no such topological group topology. The definitive solution of Markov's problem—that is, the proof in **ZFC** of the existence of a countable group with no non-discrete topological group topology—is given by A. Ju. Ol'shanskiĭ; an account of his construction is given by ADIAN [1980, section 13.4]. For the groups of Shelah and Ol'shanskiĭ, of course, the discrete topology is a minimal topological group topology (and is not totally bounded).

It was for many years an unsolved problem, aggressively pursued by the Bulgarian school of topology and finally settled affirmatively by PRODANOV and STOJANOV in [1984], whether every minimal topological group topology on an Abelian group is totally bounded. Earlier, in [1979], DIEROLF and SCHWANENGEL had given an example of a non-Abelian group with a minimal topological group topology which is not totally bounded, and in [1971/72] PRODANOV had shown that the group  $\mathbb{Q}$  of rational numbers admits no topological group topology which is both totally bounded and minimal (among all topological group topologies); it had been known for some years before the appearance of PRODANOV and STOJANOV [1984] that every minimal topological group topology on a divisible Abelian group is totally bounded, and in [1984] DIKRANJAN had characterized those divisible Abelian groups which admit a minimal topological group topology.

For an infinite cardinal  $\alpha$ , let  $F(\alpha)$  and  $FA(\alpha)$  denote respectively the free group, and the free Abelian group, on  $\alpha$ -many generators. It is known that each of the groups  $F(\alpha)$  admits a minimal topological group topology (SHAKMATOV [1985]), but there do exist in **ZFC** groups of the form  $FA(\alpha)$  with no minimal topological group topology. Indeed STOJANOV [1981] has shown that  $FA(\alpha)$  has a minimal topological group topology if and only if  $\alpha$  is *admissible* in the sense that there is a sequence  $\alpha_n$  of cardinals such that  $\sum_{n < \omega} 2^{\alpha_n} \leq \alpha \leq \prod_{n < \omega} 2^{\alpha_n}$ . (It is not difficult to see that for every cardinal number  $\beta$  there is an admissible cardinal  $\alpha > \beta$  and there is a non-admissible cardinal  $\alpha > \beta$ .)

The following two questions are perhaps the remaining outstanding questions in the theory of minimal topological group topologies. The second of these was brought to my attention by D. B. Shakhmatov.

**Question 3I.1.** (ARKHANGEL'SKIĬ [1987]) *Is every Hausdorff group a quotient of a minimal group?* **519. ?**

**Question 3I.2.** *For each group  $G$ , let  $\mathfrak{R}(G)$  denote the set of (Hausdorff) topological group topologies, partially ordered by inclusion. Is it true for each  $\alpha$  that the partially ordered sets  $\mathfrak{R}(F(\alpha))$  and  $\mathfrak{R}(FA(\alpha))$  are non-isomorphic?* **520. ?**

### 3J. Almost Periodicity

A topological group is said to be (a) *maximally almost periodic* (b) *minimally almost periodic* if the continuous homomorphisms on  $G$  to compact groups (a) separate points (b) are all constant. The existence of a minimally almost periodic group is given by VON NEUMANN in [1934, section 18]. For a proof that the special linear group  $SL(2, \mathbb{C})$  is such a group, even in its discrete topology, see VON NEUMANN and WIGNER [1940], and COMFORT [1984, section 9.8] for an expository treatment. AJTAI, HAVAS and KOMLÓS [1983] have shown that every infinite Abelian group admits a topological group topology which is not maximally almost periodic, and HEWITT and ROSS [1963, section 23.32] describe in detail a number of topological vector spaces which are minimally almost periodic Abelian groups. In [1988] REMUS has shown that every free Abelian group, and every divisible Abelian group, admits a minimally almost periodic topological group topology. In a preliminary version of the present manuscript the question was posed, following PROTASOV [1984] and REMUS [1988], whether every infinite Abelian group admits a minimally almost periodic topological group topology. I am indebted to D. Remus for suggesting (letter of September, 1989) the following elementary construction, showing that the answer to this very general question is “No.” For an arbitrary infinite cardinal  $\alpha$  and for distinct prime numbers  $p$  and  $q$  set  $G = (\bigoplus_{\alpha} \mathbb{Z}(p)) \times \mathbb{Z}(q)$ ; here  $\mathbb{Z}(p) = \{t \in \mathbb{T} : t^p = 1\}$ . Now define  $h: G \rightarrow \mathbb{T}$  by  $h(u, v) = v$  with  $u \in \bigoplus_{\alpha} \mathbb{Z}(p)$ ,  $v \in \mathbb{Z}(q)$ . Then  $h$  is a non-constant homomorphism from  $G$  to the compact group  $\mathbb{Z}(q)$ , and  $h$  is continuous with respect to any topological group topology  $\mathfrak{I}$  for  $G$  since the kernel of  $h$ , which is  $\bigoplus_{\alpha} \mathbb{Z}(p) \times \{1\}$ , is the kernel of the (necessarily  $\mathfrak{I}$ -continuous function)  $G \rightarrow G$  given by  $x \rightarrow x^p$ . (We use here the fact that a homomorphism from a topological group to a finite group is continuous if and only if its kernel is closed.)

The argument just given shows that for every infinite cardinal  $\alpha$  there is an Abelian group  $G$  of bounded order such that  $|G| = \alpha$  and  $G$  admits no minimally almost periodic topological group topology. With the general question cited from PRODANOV [1971/72] and REMUS [1988] thus dispatched, there remains this residue.

**Question 3J.1.** *Does every Abelian group which is not of bounded order* **521. ?**



*admit a minimally almost periodic topological group topology? What about the countable case?*

### *3K. Unique Polish Topological Group Topology*

It is a theorem of KALLMAN [1986] that for many (locally) compact metric spaces  $X$ , including the Cantor set and the Hilbert cube, the group  $\text{Homeo}(X)$  of homeomorphisms from  $X$  onto  $X$  admits a unique complete separable metrizable topological group topology. The paper KALLMAN [1986] contains a number of questions, both general and specific, related to this result; the following is one of the former.

- ? 522. **Question 3K.1.** *For what spaces  $X$  does the conclusion of Kallman's theorem hold?*

### *3L. Algebraic Structures Weaker Than Groups*

At the annual meeting of the American Mathematical Society in Baltimore, Maryland in December, 1953, A. D. WALLACE [1955] noted that several authors had advanced arguments sufficient to prove that a compact topological semigroup with two-sided cancellation is a topological Group. (By definition, a topological semigroup is a semigroup  $S$  with a topology relative to which multiplication from  $S \times S$  to  $S$  is continuous.) As to whether “compact” may be legitimately weakened to “countably compact”, WALLACE [1955] remarked that despite “several published assertions ... [the issue] remains in doubt”. The question, known commonly as “Wallace’s question”, remains unsettled today, 35 years later.

- ? 523. **Question 3L.1.** *Is every countably compact topological semigroup with two-sided cancellation a topological group?*

According to MUKHERJEA and TSERPES [1972], the answer is affirmative for semigroups which in addition are assumed to be first countable. The same conclusion is given by GRANT in [19 $\infty$ ] for cancellative semigroups which are weakly first countable in the sense of NYIKOS [1981] (these are the cancellative semigroups which satisfy Arkhangel’skiĭ’s gf-axiom of countability ARKHANGEL’SKIĬ [1966]).

Several authors have considered conditions under which a group with a topology relative to which multiplication from  $G \times G$  to  $G$  is continuous must be a topological group. (The Sorgenfrey line shows that the Lindelöf property, and the property of Baire, are inadequate to make inversion continuous.) The best-known theorem in this circle of ideas is due to ELLIS [1957b, 1957a]: It is enough that the group be locally compact (and that multiplication be continuous in each variable ELLIS [1957a]). RAGHAVAN and REILLY [1978] have collected and contributed several results of this type; see also PFISTER [1985].

### 3M. Algebraic Structures Stronger Than Groups

For  $F$  a field and  $\mathfrak{S}$  a topology for  $F$ , the pair  $\langle F, \mathfrak{S} \rangle$  is a *topological ring* if subtraction and multiplication are  $\mathfrak{S}$ -continuous; and  $\langle F, \mathfrak{S} \rangle$  is a *topological field* if multiplicative inversion is also continuous. Since the closure of  $\{0\}$  is an ideal in  $F$ , every nontrivial ring topology for  $F$  is a Hausdorff topology. Since every nontrivial ring topology contains a nontrivial field topology (GELBAUM, KALISH and OLMSTED [1951]), each minimal ring topology for  $F$  is a field topology.

A subset  $A$  of  $F$  is *bounded* if for every neighborhood  $U$  of 0 there is a neighborhood  $V$  of 0 such that  $VA \subseteq U$ ; and  $F$  is *locally bounded* if 0 has a bounded neighborhood. It is a theorem achieved by TURYN [1951], by FLEISCHER [1953b, 1953a], and by KOWALSKY and DÜRBAUM [1953] that the topology  $\mathfrak{S}$  of a topological field  $\langle F, \mathfrak{S} \rangle$  is locally bounded and minimal if and only if  $\mathfrak{S}$  is induced by an absolute value or by a non-Archimedean valuation. (A *non-Archimedean valuation*  $v$  on  $F$  is a function  $v$  from  $F$  to an ordered group  $G$  with largest element  $\infty$  adjoined such that  $v(0) = \infty$ ,  $v(ab) = v(a) + v(b)$ , and  $v(a+b) \geq \min(v(a), v(b))$  for all  $a, b \in F$ ; the topology given by  $v$  has as a base at 0 all sets of the form  $N_p(0) = \{a \in F : v(a) > p\}$  for  $p > 0$ ,  $p \in G$ . Versions of the theorem just cited are given in WIECŁAW [1988, (Theorem 5.3.8)] and in SHELL [19 $\infty$ , (§16.5)].) In his extensive list of open problems concerning topological fields, WIECŁAW [1988, Chapter 15] begins with an “old problem”:

**Question 3M.1.** *Is there a minimal topological field  $\langle F, \mathfrak{S} \rangle$  such that  $\mathfrak{S}$  is not locally bounded? What about the case  $F = \mathbb{Q}$ ?* **524. ?**

In view of the characterization of locally bounded minimal field topologies cited above, this question may be phrased as follows.

**Question 3M.2.** (KOWALSKY [1954]) *Is every minimal Hausdorff field topology on a (commutative) field induced by an absolute value or by a non-Archimedean valuation?* **525. ?**

The analogous question for noncommutative fields has been answered in the negative by HARTMANN [1988].

For background on topological fields and the theory of valuations, the reader may consult JACOBSON [1980], SHELL [19 $\infty$ ], or WIECŁAW [1988].

It is well known that every topological field is either connected or totally disconnected. Among the latter, all known examples are in fact zero-dimensional. This suggests the following natural question.

**Question 3M.3.** (Niel Shell) *Is every totally disconnected topological field  $F$  zero-dimensional?* **526. ?**

zero-dimensional? What if  $F = \langle F, \mathcal{I} \rangle$  is assumed simply to be a topological ring?

For background and relevant recent results, see SHELL [1987].

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## CHAPTER 22

### Problems in Domain Theory and Topology

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Domain theory is an area which has evolved from two separate impetuses. The first and most prominent has been denotational semantics of high-level programming languages. It was the pioneering work of Dana Scott which led to the discovery that algebraic lattices, and their generalization, continuous lattices could be used to assign meanings to programs written in high-level programming languages. When Gordon Plotkin pointed out the need for more general objects to use as mathematical models, the notion of a domain was formulated, and the structure theory of domains has been a focal point for research in denotational semantics ever since.

On the purely mathematical side, research into the structure theory of compact semilattices led Lawson and others to consider the category of those compact semilattices which admit enough continuous semilattice morphisms into the unit interval to separate the points. In an effort to give a purely algebraic description of these objects, Hofmann and Stralka were lead to the definition of certain complete lattices, and it was soon noted that these objects were precisely the continuous lattices of Scott. This occurred in the mid-1970's, and a flurry of research activity arose which culminated in the comprehensive treatise GIERZ, HOFMANN, KEIMEL, LAWSON, MISLOVE and SCOTT [1980], hereafter called *the Compendium*. A more complete discussion of this facet is provided in the Forward and Historical Notes of the Compendium.

Our goal here is to outline some of those areas where domain theory and topology interact. This has been one of the central features of the theory, since the most important topology on a domain—the Scott topology—has a completely algebraic characterization. In fact, all of the topologies which are useful for domains are determined by their algebraic structure.

Let  $(P, \leq)$  be a partially ordered set. A subset  $D$  of  $P$  is *directed* if given  $x, y \in P$ , there exists  $z \in P$  such that  $x, y \leq z$ . The order on  $P$  is a *directed complete partial order* if every directed subset of  $P$  has a least upper bound. In this case we refer to  $P$  as a *directed complete partially ordered set* or DCPO for short.

A significant contribution of the theory of continuous partially ordered sets has been the explicit definition and use of a new order relation, one that sharpens the traditional notion of order.

Let  $P$  be a DCPO and  $x, y \in P$ . We say  $x$  is *way below*  $y$ , written  $x \ll y$ , if given a directed set  $D \subseteq P$  such that  $y \leq \sup D$ , then  $x \leq d$  for some  $d \in D$ . A partially ordered set  $P$  is a *continuous* DCPO if it is a DCPO and satisfies

$$y \in P \quad \Rightarrow \quad y = \sup\{x : x \ll y\} = \sup \downarrow y,$$

and the set on the right is directed. If  $P$  is simultaneously a complete lattice and a continuous DCPO, then it is called a *continuous lattice*.

The most important structures in the theory of continuous DCPO's from the viewpoint of computer science have been what are usually referred to as Scott domains.

An element  $k \in P$  is *compact* if  $k \ll k$ , i.e., if  $\sup D \geq k$  for  $D$  directed, then  $k \leq d$  for some  $d \in D$ . A DCPO  $P$  is *algebraic* if every element is a directed sup of compact elements. Alternately, algebraic DCPO's are referred to as *domains*. If a domain (i.e., an algebraic DCPO) is a complete lattice, then it is called an *algebraic lattice*.

Note that algebraic DCPO's are a special subclass of the class of continuous DCPO's. In an algebraic DCPO the relation  $\ll$  is characterized by  $x \ll y$  iff there exists a compact element  $k$  such that  $x \leq k \leq y$ .

The following are basic properties of the relation  $\ll$  in a continuous DCPO.

- (1)  $a \ll b \Rightarrow a \leq b$
- (2)  $a \ll d, b \ll d \Rightarrow \exists c$  such that  $a, b \leq c$  and  $c \ll d$
- (3)  $a \leq b \ll c \leq d \Rightarrow a \ll d$
- (4)  $a \ll c \Rightarrow \exists b$  such that  $a \ll b \ll c$
- (5)  $\perp \ll a$ , where  $\perp$  is the least element.

The fourth property plays a crucial role in the theory and is referred to as the "interpolation" property. A continuous DCPO is said to be *countably based* if there exists a countable subset  $B$  of  $P$  such that  $p \ll q$  in  $P$  implies there exists  $b \in B$  such that  $p \ll b \ll q$ .

When one is working in the context of algebraic DCPO's, properties of continuous DCPO's can generally be given alternate characterizations in terms of the partially ordered set of compact elements. For example, an algebraic DCPO is countably based iff the set of compact elements is countable.

## 1. Locally compact spaces and spectral theory

We consider an illustrative topological example of naturally occurring continuous orders. The next results are mainly drawn from HOFMANN and LAWSON [1978] or Chapter V of the Compendium.

Let  $X$  be a topological space, let  $O(X)$  denote the lattice of open sets ordered by inclusion, and let  $U, V \in O(X)$ . Then  $U \ll V$  iff for every open cover of  $V$ , there is a finite subcollection that covers  $U$ . In this context it seems appropriate to say that  $U$  is *compact* in  $V$ .

We say that  $X$  is *core compact* if given  $x \in V \in O(X)$ , there exists  $U$  open,  $x \in U \subseteq V$ , such that  $U$  is compact in  $V$ .

**1.1. THEOREM.**  $X$  is core compact if and only if  $O(X)$  is a continuous lattice.

□

For Hausdorff spaces, the core compact spaces are precisely the locally compact spaces. Core compactness appears to be the appropriate generalization of local compactness to the non-Hausdorff setting, in the sense that basic mapping properties of locally compact spaces are retained in this setting. For example,  $X$  is core compact iff  $1_X \times f: X \times Y \rightarrow X \times Z$  is a quotient mapping whenever  $f: Y \rightarrow Z$  is a quotient mapping (DAY and KELLY [1970]). Also

appropriate modifications of the compact-open topology for function spaces exist so that one gets an equivalence between  $[X \times Y \rightarrow Z]$  and  $[X \rightarrow [Y \rightarrow Z]]$  if  $Y$  is core compact (see Chapter II of the Compendium, and for later developments, SCHWARZ and WECK [1985] or LAMBRINOS and PAPADOPOULOS [1985]). Of course this equivalence is closely related to the categorical notion of Cartesian closedness, a topic to which we return at a later point.

The spectral theory of lattices seeks to represent a lattice as the lattice of open sets of a topological space. However, the constructions are more intuitive if one works with the lattice of closed sets. We take this approach initially, and set everything on its head at a later stage.

Suppose that  $X$  is a  $T_1$ -space, and let  $L$  be the lattice of closed subsets of  $X$  (ordered by inclusion). We let  $\hat{X}$  denote the set of atoms in  $L$  (which correspond to the singleton subsets of  $X$ ) and topologize  $\hat{X}$  by defining a closed set to be all the atoms below a fixed member of the lattice  $L$ , i.e.,  $\{\{x\} : \{x\} \subseteq A\}$  where  $A$  is a closed subset of  $X$ . Then the mapping from  $X$  to  $\hat{X}$  which sends an element to the corresponding singleton set is a homeomorphism. Thus  $X$  may be recovered (up to homeomorphism) from the lattice of closed sets.

The situation becomes more complex (and more interesting) for a  $T_0$ -space  $X$ . In this case we let an element of  $X$  correspond to the closure of the corresponding singleton set in the lattice  $L$  of closed sets. The fact that  $X$  is  $T_0$  is precisely the condition needed for this correspondence to be one-to-one. But how does one distinguish in a lattice-theoretic way the closed sets that arise in this fashion? One easily verifies that sets that are closures of points are *irreducible*, i.e., not the union of two strictly smaller closed sets. We are thus led to define the cospectrum,  $\text{Cospec}(L)$ , to be the set of coprime elements ( $p$  is *coprime* if  $p \leq \sup\{x, y\}$  implies  $p \leq x$  or  $p \leq y$ ) equipped with the *hull-kernel* topology with closed sets of the form  $hk(a) = \{p \in L : p \text{ is coprime, } p \leq a\}$ , for  $a \in L$ .

A space is *sober* if every irreducible closed set is the closure of a unique point. In precisely this case the embedding of  $X$  into the cospectrum of the closed sets is a homeomorphism. For any topological space  $X$ , there is a largest  $T_0$ -space  $\hat{X}$  having the same lattice of closed (open) sets as  $X$ , called the *sobrification* of  $X$ . The sobrification of  $X$  can be obtained by taking  $\hat{X}$  to be the cospectrum of the closed sets;  $X$  maps to the sobrification by sending a point to its closure. It can be shown that a space is core compact iff its sobrification is locally compact. (A space is *compact* if every open cover has a finite subcover, and *locally compact* if every (not necessarily open) neighborhood of a point contains a compact neighborhood of that point.)

We now dualize the preceding notions to the lattice of open sets. An element  $p \in L$ ,  $p \neq 1$  is *prime* (resp. *irreducible*) if  $x \wedge y \leq p \Rightarrow x \leq p$  or  $y \leq p$  (resp.  $x \wedge y = p \Rightarrow x = p$  or  $y = p$ ). It can be shown that the irreducible elements of a continuous lattice order generate (i.e., every element is an infimum of such



elements) and that the prime elements of a distributive continuous lattice order generate.

If  $\text{PRIME } L$  denotes the set of prime elements of  $L$ , then the collection of sets of the form  $\text{PRIME } L \cap \uparrow x$  (where  $\uparrow x = \{y : x \leq y\}$ ) for  $x \in L$  forms the closed sets for a topology on  $\text{PRIME } L$ , called the *hull-kernel* topology.  $\text{PRIME } L$  equipped with the hull-kernel topology is called the *spectrum* of  $L$ , and is denoted  $\text{Spec } L$ . The following theorem results by showing that the spectrum is sober (which is always the case) and locally compact when  $L$  is continuous.

**1.2. THEOREM.** *Given any continuous distributive lattice  $L$ , there exists a (unique) locally compact sober space  $X$  namely the spectrum  $\text{Spec } L$  such that  $L$  is order-isomorphic to  $O(X)$ .  $\square$*

As a consequence of the preceding considerations there results a duality between distributive continuous lattices and locally compact sober spaces.

## 2. The Scott Topology

A distinctive feature of the theory of continuous orders is that many of the considerations are closely interlinked with topological and categorical ideas. The result is that topological considerations and techniques are basic to significant portions of the theory.

The Scott topology is the topology arising from the convergence structure given by  $D \rightarrow x$  if  $D$  is a directed set with  $x \leq \sup D$ . Thus a set  $A$  is *Scott closed* if  $A = \downarrow A = \{z : z \leq x \text{ for some } x \in A\}$  and if  $D \subseteq A$  is directed, then  $\sup D \in A$ . Similarly  $U$  is *Scott open* if  $U = \uparrow U = \{y : x \leq y \text{ for some } x \in U\}$  and  $\sup D \in U$  for a directed set  $D$  implies  $d \in U$  for some  $d \in D$ .

By means of the Scott topology one can pass back and forth between an order-theoretic viewpoint and a topological viewpoint in the study of DCPO's. Generally order-theoretic properties have corresponding topological properties and vice-versa. For example, *continuous morphisms* between DCPO's may be defined either as those order preserving functions which also preserve sups of directed sets or as those functions which are continuous with respect to the Scott topologies.

**2.1. EXAMPLE.** The Scott-open sets in  $\mathbb{R}^* = [-\infty, \infty]$  consist of open right rays. For a topological space  $X$ , the set of Scott-continuous functions  $[X, \mathbb{R}^*]$  consists of the lower semicontinuous functions.

Suppose that a DCPO  $P$  is equipped with the Scott topology, so that it is now a topological space. Then the original order may be recovered from the topological space as the *order of specialization*, which is defined by  $x \leq y$  iff  $x \in \overline{\{y\}}$ . Note that any topological space has an order of specialization, and that this order is a partial order precisely when the space is  $T_0$ .

There are useful alternate descriptions of the Scott topology for special classes of DCPO's. For a continuous DCPO  $P$ , let  $\uparrow z = \{x : z \ll x\}$ . It follows from the interpolation property that  $\uparrow z$  is a Scott open set. That  $\{\uparrow z : z \in P\}$  forms a basis for the Scott topology follows from the fact that each  $x \in P$  is the directed supremum of  $\downarrow x$ . It follows that a continuous DCPO is countably based iff the Scott topology has a countable base. Alternately the Scott open filters also form a basis for the Scott topology in a continuous DCPO.

For domains, a basis for the Scott open sets is given by all sets of the form  $\uparrow z$ , where  $z$  is a compact element. The argument is analogous to the continuously ordered case.

**Problem.** *Characterize those DCPO's*

**527. ?**

- (i) *for which the Scott topology has a basis of open filters, and*
- (ii) *for which the topology generated by the Scott open filters is  $T_0$ .*

*Analogously, characterize those  $T_0$  topological spaces  $X$  for which the Scott topology on the lattice  $O(X)$  of open sets satisfies (i) or (ii). (Both are true in the first case if the DCPO is continuous and in the second case if  $X$  is core compact.)*

Given a partially ordered set  $P$ , there are a host of topologies on  $P$  for which the order of specialization agrees with the given order. The finest of these is the *Alexandroff discrete* topology, in which every upper set is an open set, and the coarsest of these is the *weak* topology, in which  $\{\downarrow x : x \in P\}$  forms a subbasis for the closed sets. The Scott topology is the finest topology giving back the original order with the additional property that directed sets converge to their suprema. It is this wealth of topologies that makes the study of DCPO's from a topological viewpoint (as opposed to an order-theoretic viewpoint) both richer and more complex.

What spaces arise by equipping continuous DCPO's with the Scott topology? In general, a continuous DCPO equipped with the Scott topology gives rise to a locally compact, sober  $(T_0)$ -space. (A base of compact neighborhoods of  $x$  in this case is given by  $\uparrow z$  for all  $z \ll x$ .) Indeed, the lattice of Scott-open sets in this case is a completely distributive lattice (a lattice is completely distributive if arbitrary joins distribute over arbitrary meets and vice-versa; these are a special class of distributive continuous lattices). Conversely the spectrum of a completely distributive lattice turns out to be a continuous DCPO (with respect to the order of specialization) equipped with the Scott topology. Hence another characterization of continuous DCPO's equipped with their Scott topologies is that they are the spectra of completely distributive lattices (see LAWSON [1979] or HOFMANN [1981a]). These results were generalized to a class of DCPO's called quasicontinuous posets in GIERZ, HOFMANN and STRALKA [1983].

? 528. **Problem.** Characterize those DCPO's for which the lattice of open sets for the Scott topology (alternately the Scott open filter topology) is a continuous lattice, i.e., characterize those DCPO's that are core compact with respect to the Scott topology. In the opposite direction, characterize those distributive continuous lattices for which the spectrum is a DCPO equipped with the Scott topology.

A result of SCOTT [1972] asserts that continuous lattices equipped with the Scott topology are precisely the injective  $T_0$ -spaces (any continuous function from a subspace  $A$  of a  $T_0$ -space  $X$  into  $L$  extends to a continuous function on all of  $X$ ). Thus these are a generalization to the non-Hausdorff setting of the absolute retracts of topology.

? 529. **Problem.** Characterize those spaces which would be generalizations of absolute neighborhood retracts, i.e., those  $T_0$ -spaces  $Y$  such that any continuous function from a subspace  $A$  of a  $T_0$ -space  $X$  into  $Y$  extends to a continuous function on some neighborhood of  $A$  in  $X$ .

Retracts play an important role in the theory of continuous DCPO's. We consider some of their most basic properties.

Let  $P$  be a DCPO. An (internal) *retraction* is a continuous morphism  $r: P \rightarrow P$  such that  $r \circ r = r$ . It was Scott's observation that a continuous retract of a continuous lattice is again a continuous lattice (SCOTT [1972]), and the proof carries over to continuous DCPO's.

**2.2. PROPOSITION.** Let  $P$  be a continuous DCPO and let  $r: P \rightarrow P$  be a retraction. Then  $r(P)$  is a continuous DCPO, and the inclusion  $j: r(P) \rightarrow P$  is continuous.  $\square$

A DCPO  $A$  is a *retract* of a DCPO  $P$  if there exist continuous morphisms  $r: P \rightarrow A$  and  $j: A \rightarrow P$  such that  $r \circ j = 1_A$ . In this case the function  $r$  is called an (external) *retraction*. Note that  $j \circ r$  is an internal retraction on  $P$  and that  $j: A \rightarrow j(A)$  is an order isomorphism. Thus the previous proposition yields

**2.3. COROLLARY.** A retract of a continuous DCPO is a continuous DCPO.

A special type of (external) retraction is the *projection*, where in addition to the preceding conditions we require that  $j \circ r \leq 1_P$ . In this case we write  $P \stackrel{r}{\rightrightarrows} Q$ . If  $r$  is a projection, then  $j$  is unique, is automatically continuous, and is given by  $j(y) = \inf\{x : r(x) \geq y\}$ .

Continuous DCPO's have an alternate characterization in terms of their ideal completions, namely a DCPO  $P$  is continuously ordered iff the mapping  $\text{SUP}: Id(P) \rightarrow P$  is a projection. The continuous embedding  $j: P \rightarrow Id(P)$  is given by  $j(x) = \Downarrow x$ , which is the smallest ideal with supremum greater than or equal to  $x$ .

It follows that every continuously ordered set is the retract of a domain and that the class of continuously ordered sets is the smallest class of DCPO's that contains the domains and is closed with respect to taking retracts.

There is an inclusion functor from the category of sober spaces into the category of  $T_0$ -spaces, and there is a functor from the category of  $T_0$ -spaces into the category of partially ordered sets which sends a space to the order of specialization. Both of these functors have left adjoints. The adjoint functor for the order of specialization functor equips a partially ordered set with the Alexandroff discrete topology. The functor sending a space to its sobrification is the adjoint of the inclusion of sober spaces into  $T_0$ -spaces. The composition of these two functors sends a partially ordered set to the sobrification of the Alexandroff discrete topology, which turns out to be the ideal completion equipped with the Scott topology. Thus the sobrification of the Alexandroff discrete topology gives a topological analog of the ideal completion. The topological retracts of these sobrified Alexandroff discrete spaces are the retracts in our earlier sense and, as we have seen previously, are the continuous DCPO's. (These results appear in HOFMANN [1981b].)

**Problem.** Investigate those “varieties” of topological spaces that are generated by a certain class of spaces by taking the smallest class closed under retracts and products (e.g., the continuous lattices endowed with the Scott topology make up the variety generated by the two element lattice endowed with the Scott topology, sometimes called the Sierpiński space). When do all members of the variety arise as a retract of a product of generating spaces? What classes arise when one starts with a set of finite  $T_0$  spaces? In the latter case is the variety generated Cartesian closed (see later sections)? Is it finitely generated? 530. ?

### 3. Fixed Points

If  $f: D \rightarrow D$  is a self-map defined on the domain  $D$ , then a *fixed point* for  $f$  is an element  $x \in D$  satisfying  $f(x) = x$ . Because they provide a method to assign meanings to recursive constructs, the existence of fixed points for a continuous self-map  $f: D \rightarrow D$  defined on a DCPO  $D$  is crucial for the application of domain theory to the semantics of programming languages. They can also be used to solve domain equations by considering domains of domains (see WINSKEL and LARSEN [1984]).

It was a basic result of Tarski's that any monotone self-map  $f: L \rightarrow L$  defined on a complete lattice  $L$  has a least fixed point, and, in fact the set  $Fix(f)$  of fixed points of  $f$  is a complete lattice. For a DCPO, the least fixed point of a continuous map  $f: D \rightarrow D$  exists, and is given by  $x = \bigvee_{n \geq 0} f^n(\perp)$ , where  $\perp$  is the least element of  $D$ . Surprisingly, little attention has been paid to the structure of the set  $Fix(f)$  of fixed points of such a function  $f$ . Recently

in [1988] HUTH has characterized the conditions under which the set  $Fix(f)$  is a consistently complete domain (that is, a complete algebraic semilattice) if  $D$  is, and in this case, the map  $x \mapsto \bigwedge(\uparrow x \cap Fix(f)): D \rightarrow Fix(f)$  is a continuous retraction. The question is whether this result can be generalized.

**? 531. Problem.** *For which classes of continuous DCPO's  $D$  and continuous self-maps  $f: D \rightarrow D$  is the set  $Fix(f)$  a continuous DCPO? When this is the case, is  $Fix(f)$  a retract of  $D$ ? In particular, one might investigate certain classes to be introduced latter such as strongly algebraic and finitely separated DCPO's.*

#### 4. Function Spaces

A crucial and characteristic property of countably based continuous DCPO's is that they are closed under a wide variety of set-theoretic operations. This allows one to carry along a recursive theory. Such constructions break down in the category of sets because one obtains sets of larger cardinality. Also one can employ these stability features of continuous DCPO's to obtain examples which reproduce isomorphic copies of themselves under a variety of set-theoretic operations. (This is essentially the idea of solving domain equations.) It is these features that provide strong motivation for moving from the category of sets to some suitable category of domains or continuous DCPO's.

One of the most basic constructs is that of a function space. If  $X$  and  $Y$  are DCPO's, then  $[X \rightarrow Y]$  denotes the set of continuous morphisms (the order preserving functions which preserve suprema of directed sets) from  $X$  to  $Y$ . For a directed family of continuous morphisms, the pointwise supremum is again continuous. So the set  $[X \rightarrow Y]$  with the pointwise order is again a DCPO.

For topological spaces  $X$  and  $Y$  let  $[X \rightarrow Y]$  denote the set of continuous functions from  $X$  to  $Y$ . If  $X$  or  $Y$  is a DCPO, then we identify it with the topological space arising from the Scott topology. If  $Y$  is a DCPO, then  $[X \rightarrow Y]$  is also a DCPO with respect to the pointwise order on functions. One verifies that the supremum of a directed family of continuous functions is again continuous, so directed suprema are computed pointwise in  $[X \rightarrow Y]$ . If  $X$  and  $Y$  are both DCPO's equipped with the Scott topology, then the function space  $[X \rightarrow Y]$  is just the set of continuous morphisms of the previous paragraph.

Suppose additionally that  $X$  is a continuous DCPO. Let  $f: X \rightarrow Y$  be a (not necessarily continuous) order preserving function. Then there exists a largest continuous morphism  $\underline{f}: X \rightarrow Y$  which satisfies  $\underline{f} \leq f$ ;  $\underline{f}$  is given by  $\underline{f}(x) = \sup\{f(z) : z \ll x\}$ . Thus if  $Y^X$  denotes the set of all order-preserving functions from  $X$  to  $Y$ , the mapping  $f \mapsto \underline{f}$  from  $Y^X$  to  $[X \rightarrow Y]$  is a projection. If  $X$  is an algebraic DCPO, then  $\underline{f}$  is the unique continuous extension of the restriction of  $f$  to the set of compact elements  $K(X)$ .

Under what conditions will  $[X \rightarrow Y]$  be a continuous DCPO? Let us first consider the case that  $Y = 2$ , where  $2 = \{0, 1\}$  denotes the two-element chain with  $0 < 1$  equipped with the Scott topology. Then  $f: X \rightarrow 2$  is continuous iff  $f$  is the characteristic function of an open set of  $X$ . Hence there is a natural order isomorphism between  $O(X)$ , the lattice of open sets, and  $[X \rightarrow 2]$ . Since  $O(X)$  is a continuous lattice iff  $X$  is core compact, we conclude that the same is true for  $[X \rightarrow 2]$ .

More generally, let us suppose that  $X$  is core compact and that  $Y$  is a continuous DCPO with least element  $\perp$ . Let  $f \in [X \rightarrow Y]$ ,  $a \in X$ , and  $f(a) = b$ . Let  $z \ll b$ . Pick  $U$  open in  $X$  containing  $a$  such that  $f(U) \subseteq \uparrow z$  (which we can do since  $f$  is Scott continuous). Pick  $V$  open with  $a \in V$  such that  $V \ll U$ . Define  $g \in [X \rightarrow Y]$  by  $g(x) = z$  if  $x \in V$  and  $g(x) = \perp$  otherwise. It is straightforward to verify that  $g \ll f$  in  $[X \rightarrow Y]$  (see Exercise II.4.20 in the Compendium) and that  $f$  is the supremum of such functions. However, one needs additional hypotheses on  $X$  and/or  $Y$  to be able to get a directed set of such functions. If  $L$  is a continuous lattice, then one can take finite suprema of such functions  $g$  and obtain the principal implication of

**4.1. THEOREM.** *Let  $L$  be a non-trivial DCPO equipped with the Scott topology. Then  $[X \rightarrow L]$  is a continuous lattice iff  $X$  is core compact and  $L$  is a continuous lattice.* □

**Problem.** Suppose  $P$  is a DCPO endowed with the Scott topology. Characterize those  $P$  for which  $[X \rightarrow P]$  is a continuous DCPO for all core compact spaces  $X$ . A likely candidate is the class of continuous  $L$ -domains, that is, all continuous DCPO's  $P$  in which the principal ideals  $\downarrow x$  are all continuous lattices. Does one get the same answer if one restricts to the core compact spaces which are also compact? **532. ?**

It is frequently desirable to model the notion of self-application (we may think of programs that act on other programs, including themselves, or programming languages that incorporate the  $\lambda$ -calculus, where objects are also functions and vice-versa). This involves building spaces  $X$  homeomorphic to  $[X \rightarrow X]$ . These can be constructed in suitable subcategories of continuously ordered sets by using projective limit constructions, where the bonding maps are projections. This was the original approach of SCOTT in [1972], where the lattice  $2^\omega$  was shown to satisfy this equation. It has also been shown that any domain  $D \simeq [D \rightarrow D]$  must contain a copy of  $2^\omega$  (cf., MISLOVE [1986]).

**Problem.** How extensive is the class of countably based spaces for which  $X$  is homeomorphic to  $[X \rightarrow X]$ ? One such model is  $2^\omega$ , with the Scott topology. Are other such spaces which are algebraic DCPO's locally homeomorphic to  $2^\omega$ ? If not, are there natural restrictions that one can impose so that this is the case. One might be led here to a theory of manifolds modelled on  $2^\omega$ . **533. ?**

## 5. Cartesian Closedness

Let  $X$ ,  $Y$  and  $Z$  be sets and let  $\alpha: X \times Y \rightarrow Z$ . Define  $\hat{\alpha}: X \rightarrow [Y \rightarrow Z]$  by  $\hat{\alpha}(x)(y) = \alpha(x, y)$ . Then the exponential (or currying) function  $E_{XYZ} = E: [X \times Y \rightarrow Z] \rightarrow [X \rightarrow [Y \rightarrow Z]]$  sending  $\alpha: X \times Y \rightarrow Z$  to the associated function  $\hat{\alpha}: X \rightarrow [Y \rightarrow Z]$  is a bijection (a type of exponential law). In general, we call a category *Cartesian closed* if products and function spaces are again in the category and the exponential function is always a bijection. This is a convenient property for constructions such as those in the preceding section and for other purposes.

Note that  $E$  restricted to the category of DCPO's and continuous morphisms is still a bijection, for if  $X$ ,  $Y$  and  $Z$  are all DCPO's, then one verifies directly that  $\alpha$  preserves directed sups if and only if  $\hat{\alpha}$  does (where  $[Y \rightarrow Z]$  is given the pointwise order). Hence the category of DCPO's and continuous morphisms is also Cartesian closed.

Again things rapidly become more complicated when one moves to a topological viewpoint. First of all, one has to have a means of topologizing the function spaces  $[Y \rightarrow Z]$ . In this regard we recall certain basic notions from topology (see e.g., DUGUNDJI [1964, Chapter XII]).

A topology  $\tau$  on  $[Y \rightarrow Z]$  is *splitting* if for every space  $X$ , the continuity of  $\alpha: X \times Y \rightarrow Z$  implies that of the associated function  $\hat{\alpha}: X \rightarrow [Y \rightarrow Z]_\tau$  (where  $\hat{\alpha}(x)(y) = \alpha(x, y)$ ). A topology  $\tau$  on  $[Y \rightarrow Z]$  is called *admissible* (or *conjoining*) if for every space  $X$ , the continuity of  $\hat{\alpha}: X \rightarrow [Y \rightarrow Z]_\tau$  implies that of  $\alpha: X \times Y \rightarrow Z$ . Thus for fixed  $Y$ ,  $Z$  we have that  $E_{XYZ}$  is a bijection for all  $X$  if and only if the topology  $\tau$  on  $[Y \rightarrow Z]$  is both splitting and admissible.

We list some basic facts about splitting and admissible topologies. A topology  $\tau$  is admissible iff the evaluation mapping  $\epsilon: [X \rightarrow Y]_\tau \times X \rightarrow Y$  defined by  $\epsilon(f, x) = f(x)$  is continuous. A topology larger than an admissible topology is again admissible, and a topology smaller than a splitting topology is again splitting. Any admissible topology is larger than any splitting topology, and there is always a unique largest splitting topology. Thus a function space can have at most one topology that is both admissible and splitting, and such a topology is the largest splitting topology and the smallest admissible topology.

A standard function space topology is the compact-open topology. We need a slight modification of this that is suitable for core compact spaces. Let  $X$  and  $Y$  be spaces, let  $H$  be a Scott open set in the lattice  $O(X)$  of open sets on  $X$ , and let  $V$  be an open subset of  $Y$ . We define the *Isbell topology* on  $[X \rightarrow Y]$  by taking as a subbase for the open sets all sets of the form

$$N(H, V) = \{f \in [X \rightarrow Y] : f^{-1}(V) \in H\}.$$

If  $X$  is locally compact, then the Isbell topology is just the compact-open

topology. The next theorem asserts that the core compact spaces are the exponentiable spaces (see ISBELL [1975], SCHWARZ and WECK [1985] or LAMBRINOS and PAPADOPOULOS [1985]).

**5.1. THEOREM.** *Let  $Y$  be a core compact space. Then for any space  $Z$  the space  $[Y \rightarrow Z]$  admits an (unique) admissible, splitting topology, the Isbell topology, and with respect to this topology the exponential function  $E_{XYZ}$  is a bijection for all  $X$ .*

What happens if  $Y$  is not core compact? Then results of DAY and KELLY [1970] show that the Scott topology on  $[Y \rightarrow 2]$  is not admissible, but it is the infimum of admissible topologies. Thus there is no smallest admissible topology on  $[Y \rightarrow 2]$ , hence no topology that is both admissible and splitting. In this case there is no topology on  $[Y \rightarrow Z]$  such that  $E_{XYZ}$  is a bijection for all  $X$ . Thus any category of topological spaces which contains 2, is closed with respect to taking function spaces with respect to some appropriate topology, and is Cartesian closed must be some subcategory of core compact spaces. These considerations reduce the search for large Cartesian closed categories in Top to the following central problem (to which we return at a later point):

**Problem.** *Find the maximal subcategories of the category of core compact spaces which contain 2 and which are closed with respect to taking finite products and function spaces equipped with the Isbell topology (since this is the one that yields that the exponential function is a bijection).* **534. ?**

Suppose now that  $Z$  is a DCPO equipped with the Scott topology. Then  $[Y \rightarrow Z]$  is again a DCPO, and one can investigate how the Scott and Isbell topologies compare on  $[Y \rightarrow Z]$ . A direct argument from the definition of the Isbell topology yields that a directed set of functions converges to its pointwise supremum in the Isbell topology, and hence the Isbell topology is coarser than the Scott topology. Since we have seen that the Isbell topology is an admissible topology if  $Y$  is core compact, it follows that the Scott topology is also admissible. In [1982] GIERZ and KEIMEL have shown that if  $Y$  is locally compact and  $Z$  is a continuous lattice, then the compact-open and Scott topologies agree on  $[Y \rightarrow Z]$ . Analogously SCHWARZ and WECK [1985] have shown that if  $Y$  is core compact and  $Z$  is a continuous lattice, then the Isbell topology agrees with the Scott topology on  $[Y \rightarrow Z]$ . More general recent results may be found in LAWSON [1988].

**Problem.** *Let  $X$  be a core compact space and let  $P$  be a DCPO equipped with the Scott topology. Under what conditions on  $P$  do the Isbell and Scott topologies on  $[X \rightarrow P]$  agree?* **535. ?**

If  $Y$  is core compact and second countable (i.e., the topology has a countable base) and if  $Z$  is also second countable, then  $[Y \rightarrow Z]$  equipped with the Isbell



topology is second countable (see LAMBRINOS and PAPADOPOULOS [1985, Proposition 2.17]). Hence if  $Y$  is core compact and second countable (e.g.,  $Y$  is a countably based continuous DCPO),  $Z$  is a countably based continuous DCPO, and  $[Y \rightarrow Z]$  is a continuous DCPO on which the Scott and Isbell topologies agree, then  $[Y \rightarrow Z]$  is a countably based continuous DCPO (since being countably based is equivalent to the second countability of the Scott topology).

The category of locally compact Hausdorff spaces is not Cartesian closed since function spaces are not back in the category. This deficiency can be overcome by considering the category of Hausdorff  $k$ -spaces with  $k$ -products and  $k$ -function spaces taken in that category. Here, a topological space  $X$  is a  $k$ -space (sometimes also called compactly generated) if a subset  $U$  of  $X$  is open if and only if  $U \cap K$  is open in  $K$  for every compact subset  $K$ . In the  $T_0$  setting, compactness is too weak of a notion to use to define  $k$ -spaces precisely as in the Hausdorff setting. There have been attempts to find an appropriate alternate notion of a  $k$ -space in the  $T_0$ -setting, for example HOFMANN and LAWSON [1984], but it is not clear that the definitive word has yet been spoken.

? 536. **Problem.** *Is there a Cartesian closed category of sober spaces (with appropriately modified products and function spaces) which provides the appropriate generalization of the category of  $k$ -spaces? Does this theory encompass all DCPO's so that they are endowed with some topology making them  $k$ -spaces and the continuous function spaces between them are precisely the Scott continuous functions?*

## 6. Strongly algebraic and finitely continuous DCPO's

The category of finite partially ordered sets and order preserving functions is Cartesian closed. The full subcategories with objects lattices or (meet) semilattices are also Cartesian closed. One can extend these categories by taking projective limits where the bonding mappings are projections. For the finite lattices (resp. semilattices), one gets the algebraic lattices (resp. the algebraic semilattices). For all finite partially ordered sets one obtains objects which are called *strongly algebraic* DCPO's. They form a larger Cartesian closed category than the algebraic semilattices and were introduced by PLOTKIN in [1976] to have a Cartesian closed category available where one could carry out certain power domain constructions and remain in the category. The morphisms in these categories (as earlier) are the Scott continuous morphisms, and the function spaces are the DCPO's arising from the pointwise order of functions. In the section on supersober and compact ordered spaces we will relate these function spaces to the topological considerations of the previous section.

One can consider all retracts of strongly algebraic DCPO's and obtain an even larger Cartesian closed category. These objects have been called *finitely*

*continuous* DCPO's by Kamimura and Tang and studied in several of their papers (see in particular KAMIMURA and TANG [1986]) (an alternate terminology for such DCPO's is "bifinite"). A DCPO  $P$  is a finitely continuous DCPO iff there exists a directed family  $D$  of continuous functions from  $P$  into  $P$  with supremum the identity function on  $P$  such that the  $f(P)$  is finite for each  $f \in D$ . The strongly algebraic DCPO's are characterized by requiring in addition that each member of  $D$  be a projection. We take these characterizations for our working definition of these concepts. Frequently one's attention is restricted to the countably based case. Here the directed family of functions, respectively, projections with finite range may be replaced by an increasing *sequence* of functions.

A potentially larger class of DCPO's that share many of the properties of the finitely continuous ones has recently been introduced by Achim Jung.

**6.1. DEFINITION.** Let  $D$  be a DCPO. A continuous function  $f: D \rightarrow D$  is *finitely separated* if there exists a finite set  $M \subseteq D$  such that for all  $x \in D$ , there exists  $m \in M$  such that  $f(x) \leq m \leq x$ . A DCPO  $D$  with least element is called a *finitely separated* domain if there exists a directed collection of finitely separated functions with supremum the identity map on  $D$ .

**Problem.** Give an internal description of a finitely continuous (finitely separated) DCPO that one can apply directly to determine whether a given continuous DCPO is finitely continuous. Find a topological description of the spaces obtained by endowing a finitely continuous (finitely separated) DCPO with the Scott topology. A finitely continuous DCPO is finitely separated. Under what conditions does the reverse containment hold? (Currently one lacks any counterexample to the reverse containment.) 537. ?

We list some basic properties of finitely continuous DCPO's (derived by Kamimura and Tang) and finitely separated DCPO's (derived by Jung).

**6.2. PROPOSITION.**

- (i) Continuous lattices and complete continuous semilattices are finitely continuous DCPO's.
- (ii) A finitely continuous DCPO is finitely separated, and these in turn are continuous.
- (iii) A retract of a finitely continuous (finitely separated) DCPO is again a finitely continuous (finitely separated) DCPO.
- (iv) Let  $P$  and  $Q$  be finitely continuous (finitely separated) DCPO's. Then  $[P \rightarrow Q]$  is a finitely continuous (finitely separated) DCPO.

It follows directly from the last proposition that the finitely continuous and finitely separated DCPO's form Cartesian closed subcategories of the DCPO category. Alternately if they are viewed as topological spaces endowed

with the Scott topology, then they form a Cartesian closed subcategory of topological spaces.

In [1976] PLOTKIN gave an alternate characterization of strongly algebraic DCPO's in terms of the partially ordered set of compact elements, which we do not pursue here. In [1983a] SMYTH used these to derive the following result:

**6.3. THEOREM.** *Let  $P$  be a countably based algebraic DCPO with  $\perp$ . If  $[P \rightarrow P]$  is also an algebraic DCPO, then  $P$  is a strongly algebraic DCPO.*

This theorem shows that the largest Cartesian closed full subcategory of countably based algebraic DCPO's consists of the strongly algebraic DCPO's. This result has been significantly generalized by JUNG [1988]. He has shown that there are two maximal Cartesian closed categories of domains with  $\perp$ , those which are strongly algebraic and those for which each principal ideal  $\downarrow x$  is a complete lattice (the  $L$ -domains). When one moves to domains in general, then the strongly continuous and  $L$ -domains each split into two maximal Cartesian closed categories so that one obtains four altogether. One expects analogous results to carry over to finitely continuous DCPO's, but only partial results exist at this time.

- ? **538. Problem.** *Do the finitely separated DCPO's which are countably based form the largest Cartesian closed full subcategory contained in the category of countably based continuous DCPO's with least element?*
- ? **539. Problem.** *Characterize the maximal Cartesian closed full subcategories of the category of continuous DCPO's. When these are viewed as spaces (equipped with the Scott topology), are they maximal Cartesian closed full subcategories of the category of topological spaces and continuous maps?*

We remark that A. Jung has recently shown that the category of finitely separated DCPO's with largest and smallest elements forms the largest Cartesian closed full subcategory in the category of continuous DCPO's with largest and smallest elements.

## 7. Dual and patch topologies

An alternate topological approach (from domain theory) to the construction of various semantic models has been via the theory of metric spaces (see LAWVERE [1973] for one of the pioneering efforts in this direction). One may consult, for example, the articles of Kent, Smyth, America and Rutten, and Reed and Roscoe in MAIN, MELTON, MISLOVE and SCHMIDT [1988] for recent examples of this approach and for attempts to find comprehensive theories that encompass both approaches. In this problem survey we have

made no attempt to list open problems arising from this approach. However, we include a brief description of one way of interrelating the two approaches.

Suppose  $d: X \times X \rightarrow \mathbb{R}^+$  satisfies the triangle inequality. We use  $d$  to generate a topology on  $X$  by declaring a set  $U$  open if for each  $x \in U$ , there exists a *positive* number  $r$  such that  $N(x; r) \subseteq U$ , where  $N(x; r) = \{y : d(x, y) \leq r\}$ . (This is slightly at variance with the usual approach, but allows us momentarily a useful generalization.) Then  $d^*(x, y) = d(y, x)$  gives rise to a *dual* topology.

**7.1. EXAMPLE.** Define  $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  by  $d(x, y) = \max\{0, x - y\}$ . Then  $d$  generates the Scott topology on  $\mathbb{R}$ ,  $d^*$  gives the reverse of the Scott topology (the Scott topology on the order dual), and the join of the two topologies is the usual topology.

The situation can be considerably generalized by considering functions satisfying the triangle inequality into much more general semigroups than the positive reals  $\mathbb{R}^+$  (see e.g., KOPPERMAN [19∞]). In this case we need to specify an ordered semigroup  $S$  and a subset of positive elements  $S^+$  for the codomain of the “distance” function. Suppose that  $P$  is a continuous DCPO. We set  $S$  equal to the power set of  $P$  with addition being the operation of union. We let  $S^+$ , the set of positive elements, be the cofinite subsets. We define the metric  $d$  by  $d(x, y) = \downarrow x \setminus \downarrow y$ , and then define the open sets precisely as in the earlier paragraph for real metrics. This metric is called the *canonical generalized metric* for a continuous DCPO.

**7.2. PROPOSITION.** *The topology generated by  $d$  is the Scott topology.*

Given a  $T_0$ -topology, each open set is an upper set and each closed set is a lower set with respect to the order of specialization  $x \leq y \Leftrightarrow x \in \overline{\{y\}}$ . There are topological methods (as opposed to the previous metric approach) for creating “complementary” topologies from the given topology in which open sets in the new complementary topology are now lower sets (with respect to the original order of specialization). “Patch” topologies then arise as the join of a topology and a complement.

One specific topological approach has been the following (see HOFMANN and LAWSON [1978] or SMYTH [1983b]). Let  $X$  be a  $T_0$ -topological space. A set is said to be *saturated* if it is the intersection of open sets (this is equivalent to being an upper set in the order of specialization). One defines the *dual* topology by taking as a subbasis for the closed sets all saturated compact sets. The join of these two topologies is called the *patch* topology.

**Problem.** *Characterize those topologies that arise as dual topologies. If one continues the process of taking duals, does the process terminate after finitely many steps with topologies that are duals of each other?* **540. ?**

For a partially ordered set  $P$ , the *weak* topology is defined by taking as a subbase for the closed sets all principal lower sets  $\downarrow x$  for  $x \in P$ . The *weak<sup>d</sup>*

topology is defined to be the weak topology on the dual of  $P$ , the set  $P$  with the order reversed. All sets of the form  $\uparrow x$  form a subbasis for the closed sets for the weak<sup>d</sup> topology.

**7.3. PROPOSITION.** *Let  $P$  be a continuous DCPO. Then the dual topology for the canonical generalized metric and the dual topology for the Scott topology both agree and both yield the weak<sup>d</sup> topology.*

The  $\lambda$ -topology (or *Lawson topology*) on a DCPO is obtained by taking the join of the Scott topology and the weak<sup>d</sup> topology. It follows from the last proposition that if  $P$  is a continuous DCPO, then the  $\lambda$ -topology is the patch topology defined from the canonical generalized metric and it is also the patch topology arising from the Scott topology. We refer the reader to LAWSON [1988] for this result and a majority of the following results on the  $\lambda$ -topology.

The  $\lambda$ -topology on a continuous DCPO  $P$  is Hausdorff, for if  $x \not\leq y$ , then there exists  $z \ll x$  such that  $z \not\leq y$ , and  $\uparrow z$  and  $P \setminus \uparrow z$  are disjoint neighborhoods of  $x$  and  $y$ , respectively. Indeed the set  $\uparrow z \times P \setminus \uparrow z$  misses the graph of the order relation  $\leq$ , so that the order relation is closed in  $P \times P$ . Such spaces (in which the order is closed) are called *partially ordered spaces*.

If  $P$  is an algebraic DCPO, then the  $\lambda$ -topology is generated by taking all sets  $\uparrow x$  for compact elements  $x$  to be *both* open and closed. It follows that  $P$  with the Lawson topology is a 0-dimensional space. Hence it is the continuous (as opposed to the algebraic) DCPO's that can give rise to continuum-like properties with respect to the  $\lambda$ -topology.

If  $S$  is a complete semilattice, then one can take all complete subsemilattices which are upper sets or lower sets as a subbase for the closed sets and again obtain the  $\lambda$ -topology. If  $S$  is a continuous complete semilattice, then the  $\lambda$ -topology is compact and Hausdorff, the operation  $(x, y) \mapsto x \wedge y$  is continuous, and each point of  $S$  has a basis of neighborhoods which are subsemilattices. Conversely, if a semilattice admits a topology with these properties, then the semilattice is a continuous complete semilattice and the topology is the  $\lambda$ -topology (see the Compendium, VI.3).

**7.4. EXAMPLE.** Let  $X$  be a compact Hausdorff space and let  $L$  be the semilattice of closed non-empty subsets ordered by reverse inclusion and with the binary operation of union. Then  $X$  is a continuous complete semilattice, the traditional Vietoris topology on  $L$  agrees with the  $\lambda$ -topology, and this is the unique compact Hausdorff topology on  $L$  for which the binary operation of union is continuous.

Let  $L$  be a distributive continuous lattice. Then its spectrum is a locally compact space, and it is known that the patch topology on the spectrum agrees with the relative topology that the spectrum inherits from the  $\lambda$ -topology on  $L$ . It is also known that the spectrum equipped with the patch topology is

a Baire space and hence a polish space in the case that  $L$  is countably based (see the Compendium).

**Problem.** *Investigate those topological spaces which arise as the spectra of distributive continuous lattices equipped with the patch topology. Is the class of all complete separable metric spaces included?* **541. ?**

If it could be ascertained that a large class of spaces arise from the preceding construction, then the fact they arise from locally compact  $T_0$  spaces might be quite useful in studying their structure. For example, they have a natural compactification, sometimes called the Fell compactification, that arises by taking the closure in the  $\lambda$ -topology in the lattice  $L$  in which they arose (see the article of R.-E. HOFFMANN [1982]).

**Problem.** *Given a compact metric space  $X$  and a dense open subset  $U$ , is there a topology on  $U$  making it a core compact sober space such that the metric topology on  $U$  is the patch topology and  $X$  is the Fell compactification?* **542. ?**

## 8. Supersober and Compact Ordered Spaces

A *compact supersober* topological space  $X$  is one in which the set of limit points of an ultrafilter is the closure of a unique point. These spaces are in particular sober and also turn out to be locally compact (and hence the lattice of open sets is a continuous lattice). The patch topology on such a space is compact and Hausdorff, and the order of specialization is closed in  $(X, \text{patch}) \times (X, \text{patch})$ . Hence in a natural way a compact ordered space results.

Conversely, if  $X$  is a compact ordered space, consider the space  $(X, \mathcal{U})$ , where  $\mathcal{U}$  consists of all open *upper* sets. Then  $(X, \mathcal{U})$  is a compact supersober space (with the set of limit points of an ultrafilter being the lower set of the point to which the ultrafilter converged in the original topology). The dual topology consists of all open lower sets, the patch topology is the original topology, and the order of specialization is the original order (see VII.1 Exercises in the Compendium for the preceding results). Specializing to DCPO's and the Scott topology, we obtain

**8.1. THEOREM.** *A DCPO  $P$  is compact supersober with respect to the Scott topology iff the  $\lambda$ -topology is compact. In this case  $P$  is a compact ordered space with respect to the  $\lambda$ -topology.*

We note that the order dual of a compact partially ordered space is another such. Hence the topology consisting of the open lower sets is also a compact supersober space with dual topology the open upper sets.

The preceding theorem quickly yields

**8.2. PROPOSITION.** *If the  $\lambda$ -topology is compact for a DCPO  $P$ , then the same is true for any retract.*

It was shown in the Compendium that a continuous lattice or continuous complete semilattice is compact in the  $\lambda$ -topology. This result extends to finitely separated DCPO's.

**8.3. PROPOSITION (Jung).** *A finitely separated DCPO is compact in the  $\lambda$ -topology.*

? **543. Problem.** *Characterize those continuous DCPO's for which the  $\lambda$ -topology is locally compact. Characterize those distributive continuous lattices  $L$  for which  $\text{Spec } L$  with the patch topology is locally compact.*

? **544. Problem.** *Characterize those pairs  $(X, P)$  such that  $X$  is a core compact space,  $P$  is a continuous DCPO, and  $[X \rightarrow P]$  is a continuous DCPO for which the  $\lambda$ -topology is compact.*

In this connection A. Jung has recently shown that if  $D$  and  $E$  are DCPO's with least element and if  $[D \rightarrow D]$ ,  $[E \rightarrow E]$ , and  $[D \rightarrow E]$  are continuous, then either  $D$  is  $\lambda$ -compact or  $E$  is an  $L$ -domain.

## 9. Adjunctions

Let  $f^+: P \rightarrow Q$  and  $f^-: Q \rightarrow P$  be order-preserving functions between the partially ordered sets  $P$  and  $Q$ . The pair  $(f^+, f^-)$  is called an *adjunction* if  $\forall x \in P, \forall y \in Q, y \leq f^+(x) \Leftrightarrow f^-(y) \leq x$ . (Such pairs are also sometimes referred to as Galois connections, but many authors prefer to define Galois connections in terms of antitone functions.) Adjunctions can be alternately characterized by the property that  $1_Q \leq f^+ \circ f^-$  and  $1_P \geq f^- \circ f^+$ . Hence  $f^+$  is called the *upper adjoint* and  $f^-$  the *lower adjoint*. The mapping  $f^-$  is sometimes referred to as a *residuated* mapping.

The upper adjoint  $f^+$  has the property that the inverse of a principal filter  $\uparrow q$  in  $Q$  is again a principal filter in  $P$  (indeed this property characterizes mappings that arise as upper adjoints). Hence if  $P$  and  $Q$  are DCPO's, then  $f^+$  is Scott continuous iff it is  $\lambda$ -continuous. If  $Q$  is a continuous DCPO, then  $f^+$  is Scott continuous iff  $f^-$  preserves the relation  $\ll$  (see Exercise IV.1.29 in the Compendium). Note that projections are upper adjoints (with the lower adjoint being the inclusion mapping), and hence are continuous in the  $\lambda$ -topology.

The preceding remarks show that the Scott continuous upper adjoints form a good class of morphisms to consider if one is working with the  $\lambda$ -topology. If

$P$  and  $Q$  are both continuous lattices, then these mappings are precisely the  $\lambda$ -continuous  $\wedge$ -homomorphisms, which in turn are the mappings that preserve infima of non-empty sets and suprema of directed sets. As we have seen in the previous paragraph, there results a dual category consisting of the same objects with morphisms the lower adjoints which preserve the relation  $\ll$ . If one restricts to algebraic lattices, then the lower adjoint must preserve the compact elements. Its restriction to the compact elements is a  $\vee$ -preserving and  $\perp$ -preserving mapping. In this way one obtains the Hofmann-Mislove-Stralka duality (HOFMANN, MISLOVE and STRALKA [1974]) between the category of algebraic lattices with morphisms the Scott continuous upper adjoints and the category of sup-semilattices with  $\perp$  and morphisms preserving  $\perp$  and the  $\vee$ -operation.

## 10. Powerdomains

A *powerdomain* is a DCPO together with extra algebraic structure for handling nondeterministic values. Their consideration is motivated by the desire to find semantic models for nondeterministic phenomena. Examples are frequently obtained by taking some appropriate subset of the power set of a given DCPO  $P$  (hence the terminology “powerdomain”). We think of the subsets as keeping track of the possible outcomes of a nondeterministic computation. Again one is motivated to find categories where powerdomain constructions remain in the category.

We quickly overview some of the standard powerdomain constructions. If  $P$  is a DCPO with  $\perp$ , then one can construct the Hoare powerdomain as all non-empty Scott closed subsets. If  $P$  is a continuous DCPO, then this set is anti-isomorphic to the lattice of open sets, and hence forms a continuous (indeed completely distributive) lattice. The Smyth powerdomain is obtained by taking all the upper sets which are compact in the Scott topology. (We refer to SMYTH [1983b] for a nice topological development of these ideas in a general setting.) In the case of a continuous DCPO for which the  $\lambda$ -topology is compact, these are just the closed sets in the weak<sup>d</sup> topology, which is again anti-isomorphic to the lattice of weak<sup>d</sup> open sets. We have seen previously that in the case that the  $\lambda$ -topology is compact, this topology is compact supersober, hence locally compact, and hence the lattice of open sets is continuous.

One of the most interesting of the powerdomain constructions is the so-called Plotkin powerdomain. This again lends itself to nice description in the case that  $D$  is a continuous DCPO for which the  $\lambda$ -topology is compact (which we assume henceforth). It will also be convenient to assume certain basic facts about compact partially ordered spaces (see the Compendium, VI.1]). Let  $P(D)$  denote the set of all non-empty  $\lambda$ -closed order-convex subsets. If  $A \in P(D)$ , then  $A$  is compact, and hence  $\downarrow A$  and  $\uparrow A$  are closed. Since  $A$  is



order convex,  $A = \downarrow A \cap \uparrow A$ . Hence  $A \in P(D)$  iff it is the intersection of a closed upper set and closed lower set. We order  $P(D)$  with what is commonly referred to as the Egli-Milner ordering:  $A \leq B \Leftrightarrow A \subseteq \downarrow B$  and  $B \subseteq \uparrow A$ .

**10.1. THEOREM.**  *$(P(D), \leq)$  is a continuous DCPO for which the  $\lambda$ -topology is compact, provided the same is true of  $D$ .*

We remark that Plotkin introduced the strongly algebraic (countably based) DCPO's because the Plotkin powerdomain is another such (PLOTKIN [1976]). The same is true for finitely continuous DCPO's, as has been shown by KAMIMURA and TANG in [1987]. To get the directed family of functions which approximate the identity and have finite range on  $P(D)$  from those on  $D$ , simply consider  $A \mapsto h(f(A))$  for each  $f$  in the approximating family on  $D$ . The same technique works to obtain projections if  $D$  is strongly algebraic, and in the countably based case one obtains a sequence of functions.

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# PART V

## TOPOLOGY AND COMPUTER SCIENCE

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## CHAPTER 23

### Problems in the Topology of Binary Digital Images

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## 1. Background

In computer graphics and image processing a scene is often represented as an array of 0's and 1's. The set of 1's represents the object or objects in the scene and the set of 0's represents the background. The array is usually two-dimensional, but three-dimensional image arrays are produced by reconstruction from projections in applications such as computer tomography and electron microscopy (see ROSENFELD and KAK [1982, Chapter 11]). The array elements are called *pixels* in the 2D case and *voxels* in the 3D case.

We identify each array element with the lattice point in the plane or 3-space whose coordinates are the array indices of the element. The lattice points that correspond to array elements with value 1 are called *black points* and the other lattice points are called *white points*.

Let  $S$  be the set of black points. In pattern recognition one sometimes wants to reduce the black point set to a "skeleton"  $S' \subseteq S$  with the property that the inclusion of  $S'$  in  $S$  is "topology-preserving". This is called *thinning*. Figure 1 shows what effect a thinning algorithm might have on a digitized '6'. In Figure 1 the large black dots represent points in  $S$ , and the boxed black dots represent points in the skeleton  $S' \subseteq S$ .

In this paper we are mainly concerned with the requirement that a thinning algorithm must preserve topology. However, it has to be pointed out that a thinning algorithm must satisfy certain non-topological conditions as well. (For example, the skeleton produced by thinning the digitized '6' in Figure 1 must look like a '6', which means that the 'arm' of the 6 must not be shortened too much.) The non-topological requirements of thinning are hard to specify precisely<sup>1</sup> and are beyond the scope of this paper.

For  $n = 2$  or  $3$  write  $E^n$  for  $n$ -dimensional Euclidean space and write  $\mathbb{Z}^n$  for the set of lattice points in  $E^n$ .

## 2. Two-Dimensional Thinning

The topological requirements of two-dimensional thinning are well understood. Let  $S \supseteq S'$  be finite subsets of  $\mathbb{Z}^2$ . In this section we define what it means for the inclusion of  $S'$  in  $S$  to preserve topology. In fact we shall give three different but equivalent definitions.

Given any set  $T \subseteq \mathbb{Z}^2$  we can construct a plane polyhedron  $C(T) \subseteq E^2$  from  $T$  as follows. For each unit lattice square  $K$  let  $C(T, K)$  denote the convex hull of the corners of  $K$  that are in  $T$ . Let  $C(T)$  be the union of the sets  $C(T, K)$  for all unit lattice squares  $K$ . (See Figure 2.) Then one satisfactory definition of topology preservation is:

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<sup>1</sup>See DAVIES and PLUMMER [1981] for an approach to thinning which incorporates a definition of the non-topological requirements. However, a possible drawback of that approach is pointed out in HILDITCH [1983, page 121].

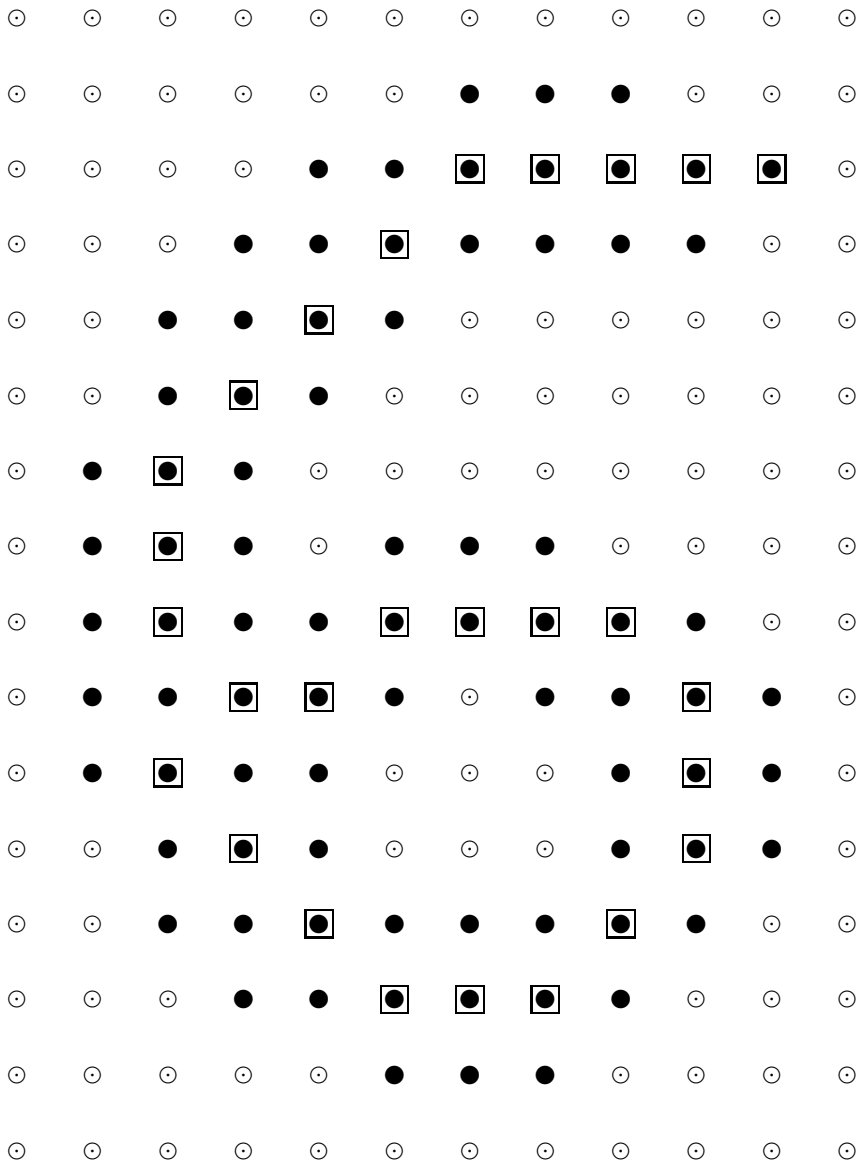


Figure 1: Possible effect of a thinning algorithm on a digitized '6'.

**2.1. DEFINITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^2$ . Then the inclusion of  $S'$  in  $S$  *preserves topology* if  $C(S')$  is a deformation retract of  $C(S)$ .

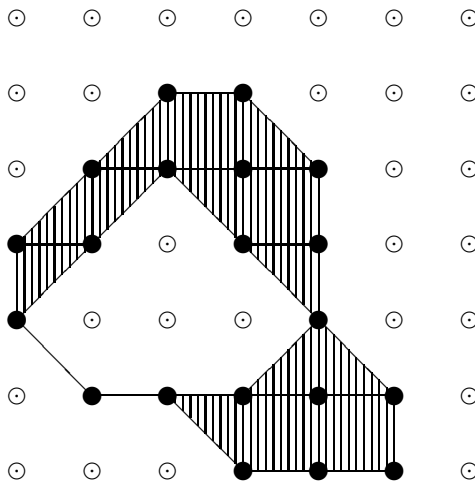


Figure 2: An example of the polyhedron  $C(T)$ . The points in  $T$  are represented by the large black dots.

Although  $S$  and  $S'$  are sets of lattice points, this definition involves continuous maps between polyhedra. We now give an alternative definition of topology preservation that is equivalent to Definition 2.1 but is entirely discrete. This second definition is more accessible to non-mathematicians than Definition 2.1. It is also easier to use in proofs that proposed thinning algorithms preserve topology<sup>2</sup>.

Two points in  $\mathbb{Z}^2$  are said to be *8-adjacent* if they are distinct and each coordinate of one differs from the corresponding coordinate of the other by at most 1; two points in  $\mathbb{Z}^2$  are *4-adjacent* if they are 8-adjacent and differ in exactly one of their coordinates. For  $p$  in  $\mathbb{Z}^2$  we write  $N(p)$  for the 3 by 3 neighborhood of  $p$  consisting of  $p$  and the points that are 8-adjacent to  $p$ .

We say a set of lattice points  $T$  is *n-connected* if  $T$  cannot be partitioned into two (disjoint) subsets  $A$  and  $B$  such that no point in  $A$  is  $n$ -adjacent to a point in  $B$ . Thus every 4-connected set is 8-connected, but an 8-connected set need not be 4-connected. An *n-component* of a non-empty set of lattice points  $T$  is a maximal  $n$ -connected subset of  $T$  — in other words, a non-empty  $n$ -connected subset  $X$  of  $T$  such that no point in  $X$  is  $n$ -adjacent to a point in  $T - X$ .

<sup>2</sup>For an example of such a proof see STEFANELLI and ROSENFELD [1971].



Note that  $T$  is 8-connected if and only if  $C(T)$  is connected. In fact, the set of lattice points in each component of  $C(T)$  is an 8-component of  $T$ , and the set of lattice points in each component of  $E^2 - C(T)$  is a 4-component of  $\mathbb{Z}^2 - T$ .

The discrete definition of topology preservation is:

**2.2. ALTERNATIVE DEFINITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^2$ . Then the inclusion of  $S'$  in  $S$  preserves topology if each 8-component of  $S$  contains just one 8-component of  $S'$ , and each 4-component of  $\mathbb{Z}^2 - S'$  contains just one 4-component of  $\mathbb{Z}^2 - S$ .

There is another natural definition of topology preservation, which is analogous to Definition 2.1 but is based on collapsing<sup>3</sup> rather than deformation retraction:

**2.3. ALTERNATIVE DEFINITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^2$ . Then the inclusion of  $S'$  in  $S$  preserves topology if there is a geometric simplicial complex  $K_S$  with a subcomplex  $K_{S'}$  such that  $|K_S| = C(S)$ ,  $|K_{S'}| = C(S')$ , and  $K_S$  collapses to  $K_{S'}$ .

Note that in this definition  $|K_S| = C(S)$  and  $|K_{S'}| = C(S')$  indicate equality and not just homeomorphism.

It is not hard to show that the Definitions 2.1, 2.2 and 2.3 are equivalent. Perhaps the easiest way is to show that  $2.1 \Rightarrow 2.2 \Rightarrow 2.3 \Rightarrow 2.1$ .

A point  $p$  in a set of lattice points  $T \subseteq \mathbb{Z}^2$  is called a *simple point* of  $T$  if the inclusion of  $T - \{p\}$  in  $T$  preserves topology. This is an important concept in the theory of image thinning<sup>4</sup>. One can determine whether or not a point  $p$  in  $T$  is a simple point just by looking at  $N(p)$ . In fact, if  $p \in T$  then  $p$  is a simple point of  $T$  if and only if  $p$  is 4-adjacent to at least one point in  $N(p) - T$  and  $(N(p) - \{p\}) \cap T$  is non-empty and 8-connected.

The following proposition gives a fourth characterization of a topology preserving inclusion. It is a special case of a result proved by RONSE in [1986].

**2.4. PROPOSITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^2$ . Then the inclusion of  $S'$  in  $S$  preserves topology if and only if there exist sets  $S_1, S_2 \dots S_n$  with  $S_1 = S$ ,  $S_n = S'$  and, for  $0 < i < n$ ,  $S_{i+1} = S_i - \{p_i\}$  where  $p_i$  is a simple point of  $S_i$ .

The “if” part of this proposition is clear. The more interesting “only if” part is proved by showing that if the inclusion of  $S'$  in  $S$  preserves topology then  $S - S'$  contains a simple point of  $S$ .

The concept of a simple point can be used to give a useful sufficient condition for topology preservation by a parallel thinning algorithm (ROSENFELD [1975]). A point in  $T \subseteq \mathbb{Z}^2$  with coordinates  $(x, y)$  is called a *north border point* of  $T$  if the point with coordinates  $(x, y + 1)$  is not in  $T$ .

<sup>3</sup>as defined in MAUNDER [1980, page 77].

<sup>4</sup>Simple points have been called *deletable points* by some authors.

**2.5. PROPOSITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^2$  and each point in  $S - S'$  is a simple north border point of  $S$  that is 8-adjacent to at least two other points in  $S$ . Then the inclusion of  $S'$  in  $S$  preserves topology.

Many 2D thinning algorithms consist of a number of passes, where each pass deletes some black points, but from one side of the picture only. (Thus the first pass may delete only north border points, while the second pass deletes only “south border” points etc.) If all the points deleted in each pass of such an algorithm are simple points that are 8-adjacent to at least two other black points, then Proposition 2.5 shows that the algorithm is topologically sound.

### 3. Three-Dimensional Thinning

We now consider three-dimensional generalizations of Definitions 2.1, 2.2 and 2.3.

It is easy to generalize the definition of  $C(T)$  to three dimensions. If  $T \subseteq \mathbb{Z}^3$  then for each unit *lattice cube*  $K$  let  $C(T, K)$  denote the convex hull of the corners of  $K$  that are in  $T$ . Let  $C(T)$  be the union of the sets  $C(T, K)$  for all unit lattice cubes  $K$ .

Say that  $S$  is *deformable* to  $S'$  if the inclusion of  $S'$  in  $S$  preserves topology in the sense of Definition 2.1:

**3.1. DEFINITION.** Suppose  $S' \subseteq S \subseteq \mathbb{Z}^3$ . Then  $S$  is *deformable* to  $S'$  if  $C(S')$  is a deformation retract of  $C(S)$ .

For plane polyhedra  $P$  and  $Q$ ,  $Q$  is a deformation retract of  $P$  if and only if  $E^2 - P$  is a deformation retract of  $E^2 - Q$ . However, this is not true of polyhedra in 3-space<sup>5</sup>. So the following is another valid generalization of Definition 2.1:

**3.2. DEFINITION.** Suppose  $S' \subseteq S \subseteq \mathbb{Z}^3$ . Then  $S'$  is *complement deformable* to  $S$  if  $E^3 - C(S)$  is a deformation retract of  $E^3 - C(S')$ .

We have seen (in Definition 2.2) that for sets of lattice points in the plane one can give a discrete formulation of the concepts of deformability and complement deformability (which are equivalent in the 2D case). It turns out that this is also possible for sets of lattice points in 3-space.

Two points in  $\mathbb{Z}^3$  are said to be *26-adjacent* if they are distinct and each coordinate of one differs from the corresponding coordinate of the other by at most 1; two points in  $\mathbb{Z}^3$  are *6-adjacent* if they are 26-adjacent and differ in exactly one of their coordinates. For  $p$  in  $\mathbb{Z}^3$  we write  $N(p)$  for the 3 by 3 by 3 neighborhood of  $p$  consisting of  $p$  and the points that are 26-adjacent to  $p$ .

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<sup>5</sup>For a counterexample, let  $P$  be a solid torus and let  $Q$  be a knotted simple closed curve in  $P$  that winds around the hole of  $P$  just once. By Proposition 3.4 the polyhedron  $Q$  is a deformation retract of  $P$ , but  $E^3 - P$  is not a deformation retract of  $E^3 - Q$ .

With the same definitions of  $n$ -connectedness and  $n$ -components as before, the set of lattice points in each component of  $C(T)$  is a 26-component of  $T$ , and the set of lattice points in each component of  $E^3 - C(T)$  is a 6-component of  $\mathbb{Z}^3 - T$ .

One can define a discrete analog of the fundamental group for a set of lattice points  $T \subseteq \mathbb{Z}^3$  with one point  $p$  in  $T$  chosen as the base point. We call this group the *digital fundamental group* of  $T$  with base point  $p$ , and denote it by  $\pi(T, p)$ . (See KONG [1989] for the definition of the group. In KONG, ROSCOE and ROSENFELD [19 $\infty$ ] its basic mathematical properties are established. In these references, the group  $\pi(T, p)$  is denoted by  $\pi((\mathbb{Z}^3, 26, 6, T), p)$ .)

The digital fundamental group has the property that for each base point  $p$  in  $T \subseteq \mathbb{Z}^3$  the inclusion of  $T$  in  $C(T)$  induces an isomorphism of  $\pi(T, p)$  to  $\pi_1(C(T), p)$ , and for each base point  $q$  in  $\mathbb{Z}^3 - T$  the inclusion of  $\mathbb{Z}^3 - T$  in  $E^3 - C(T)$  induces an isomorphism of  $\pi(\mathbb{Z}^3 - T, q)$  to  $\pi_1(E^3 - C(T), q)$ . Here  $\pi_1$  denotes the ordinary fundamental group.

Discrete characterizations of deformability and complement deformability can be given in terms of the digital fundamental group:

**3.3. PROPOSITION.** *Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^3$ . Then  $S$  is deformable to  $S'$  if and only if the following conditions all hold:*

- (1) *each 26-component of  $S$  contains just one 26-component of  $S'$*
- (2) *each 6-component of  $\mathbb{Z}^3 - S'$  contains just one 6-component of  $\mathbb{Z}^3 - S$*
- (3) *for all  $p$  in  $S'$  the inclusion of  $S'$  in  $S$  induces an isomorphism of  $\pi(S', p)$  to  $\pi(S, p)$*

There is an analogous discrete characterization of complement deformability. The validity of these characterizations is a consequence of the following recently discovered result in geometric topology:

**3.4. PROPOSITION (C. Gordon, private communication, May 1989).** *Suppose  $Q \subseteq P \subseteq E^3$ , where both  $P$  and  $Q$  are finite polyhedra or both  $E^3 - P$  and  $E^3 - Q$  are finite polyhedra. Then  $Q$  is a deformation retract of  $P$  if and only if the following conditions all hold:*

- (1) *each component of  $P$  contains just one component of  $Q$*
- (2) *each component of  $E^3 - Q$  contains just one component of  $E^3 - P$*
- (3) *for each point  $q$  in  $Q$  the inclusion of  $Q$  in  $P$  induces an isomorphism of  $\pi_1(Q, q)$  to  $\pi_1(P, q)$*

One could call a 3D thinning algorithm topologically sound if the input black point set is always deformable to the skeleton, and the skeleton is always complement deformable to the input black point set. But from a theoretical viewpoint such a definition would arguably be too weak<sup>6</sup>.

---

<sup>6</sup>For example, if  $S' \subseteq S \subseteq \mathbb{Z}^3$  are any sets such that  $C(S)$  is a solid torus and  $C(S')$  is an unknotted simple closed curve that winds around the hole of the torus just once then  $S$  is deformable to  $S'$  and  $S'$  is complement deformable to  $S$  — regardless of how  $C(S')$  may be linked with the hole of the torus.

A definition of topology preservation based on Definition 2.3 may be more appropriate. Say that  $S$  is *collapsible* to  $S'$  if the inclusion of  $S'$  in  $S$  preserves topology in the sense of Definition 2.3:

**3.5. DEFINITION.** Suppose  $S' \subseteq S \subseteq \mathbb{Z}^3$ . Then  $S$  is *collapsible* to  $S'$  if there is a geometric simplicial complex  $K_S$  with a subcomplex  $K_{S'}$  such that  $|K_S| = C(S)$ ,  $|K_{S'}| = C(S')$ , and  $K_S$  collapses to  $K_{S'}$ .

If  $S$  is collapsible to  $S'$  then  $S$  is deformable to  $S'$  and  $S'$  is complement deformable to  $S$ . The converse is true in the plane (as we have seen) but not in 3-space<sup>7</sup>.

However, it turns out that if  $p \in T \subseteq \mathbb{Z}^3$  then  $T$  is collapsible to  $T - \{p\}$  if and only if  $T$  is deformable to  $T - \{p\}$ . (For a proof, see KONG [1985, Chapter 4].) This result suggests a natural generalization of the concept of a simple point to three dimensions:

**3.6. DEFINITION.** A point  $p$  in  $T \subseteq \mathbb{Z}^3$  with the property that  $T$  is deformable (and hence collapsible) to  $T - \{p\}$  is called a *simple point* of  $T$ .

As in the 2D case, one can determine whether or not a point  $p$  in  $T$  is a simple point just by looking at the points in its neighborhood  $N(p)$ . (See KONG and ROSENFELD [1989, section 9].)

## 4. Open Problems

Is there a discrete characterization of collapsibility? The following conjecture, if true, would provide just such a characterization:

**Conjecture 1.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^3$  and  $S$  is collapsible to  $S'$ . Then there are sets  $S_1, S_2 \dots S_n$  with  $S_1 = S$ ,  $S_n = S'$  and, for  $0 < i < n$ ,  $S_{i+1} = S_i - \{p_i\}$  where  $p_i$  is a simple point of  $S_i$ . 545. ?

Note that this conjecture is certainly true in the 2D case, by Proposition 2.4.

The problem of defining topology preservation in 3D thinning was first considered by MORGENTHALER [1981]. He used a discrete approach, which leads to a definition of topology preservation that is quite similar to the discrete characterization of deformability given in Proposition 3.3. But the definition may also be stated in continuous terms as follows:

**4.1. DEFINITION.** Suppose  $S' \subseteq S$  are finite subsets of  $\mathbb{Z}^3$ . Then the inclusion of  $S'$  in  $S$  is *topology preserving in the sense of Morgenthaler* if the following conditions hold:

- (1) each component of  $E^3 - C(S')$  contains just one component of  $E^3 - C(S)$

---

<sup>7</sup>For a counterexample, let  $S$  be such that  $C(S)$  is an embedding of the dunce hat (see MAUNDER [1980, page 352]) in  $E^3$ , and  $S'$  consists of a single point in  $S$ . Then  $S$  is deformable to  $S'$  and  $S'$  is complement deformable to  $S$ , but  $S$  is not collapsible to  $S'$ .

- (2) the inclusion of  $S'$  in  $S$  induces a bijection of the *free* homotopy classes of loops in  $C(S')$  to the free homotopy classes of loops in  $C(S)$

If  $S$  is deformable to  $S'$  then the inclusion of  $S'$  in  $S$  preserves topology in the sense of Morgenthaler. Is the converse true? To establish the converse, it would suffice to prove the following conjecture, which may be regarded as a strengthened version of Proposition 3.4:

- ? 546. **Conjecture 2.** *Suppose  $P \supseteq Q$  are finite polyhedra in 3-space such that each component of  $E^3 - Q$  contains just one component of  $E^3 - P$ , and such that the inclusion of  $Q$  in  $P$  induces a bijection of the free homotopy classes of loops in  $Q$  to the free homotopy classes of loops in  $P$ . Then  $Q$  is a deformation retract of  $P$ .*

Finally, here is an open-ended problem whose solution could provide a useful tool for verifying the topological soundness of a large class of 3D parallel thinning algorithms:

- ? 547. **Problem 3.** *Find a 3D version of Proposition 2.5.*

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We are grateful to Prof. Amer Bešliagić for communicating Proposition 3.4 as a conjecture to a number of prominent geometric topologists; as a result of this the problem reached Prof. Cameron Gordon who found a proof. We wish to thank Prof. Gordon for solving the problem and telling us about his proof. We also thank Prof. Connor Lazarov and Dr. S. Wylie for useful discussions and comments on this topic.

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## CHAPTER 24

### **On Relating Denotational and Operational Semantics for Programming Languages with Recursion and Concurrency**

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## 1. Introduction

In this paper we present a uniform method to show the relationship between two well-known methods of assigning meaning to programming languages in which both recursion and concurrency—for simplicity in the sense of interleaving—are expressible. These two methods are called denotational and operational semantics.

Denotational semantics associates with each expression in the language an element of a semantical domain, a denotation, in a compositional or homomorphic way, i.e., the meaning of an expression is a combination of the meanings of its proper subexpressions. Moreover, fixed point techniques are used to handle recursion. Operational semantics, on the other hand, is assigning meaning to an expression of the language in an entirely different way. A so-called transition system models the computation steps of an abstract machine that is to execute the programs of the language. The operational semantics of an expression then, is the collection of observable behaviours of this abstract machine running the expression. From this brief description of denotational and operational semantics, it may be clear that these two semantical methods are totally different in nature.

When for a particular language both an operational and a denotational semantics are defined, it is, of course, a natural question how these are related. Since the operational semantics refers to a notion of computability via an abstract machine, it is assumed to embody the actual behaviour of the statements in the language, and therefore it is often considered as the more basic semantics in the sense that it can (or even must) be used as a yard-stick to ‘measure’ the adequacy of the denotational semantics that is proposed. So, in this view, establishing a relationship between the operational and denotational model obtains the nature of a test of justifiability or correctness of the denotational semantics. Ideally the operational and denotational meanings should coincide, i.e.,  $\mathcal{O} = \mathcal{D}$ . However, the requirement of denotational semantics to be compositional often enforces the denotations to be more informative (and hence more complicated) than the result of an operational semantics. If so, we want to have  $\mathcal{O} = \alpha \circ \mathcal{D}$  for some suitable abstraction operator  $\alpha$ . This is especially the case in the context of languages with concurrency operators involving synchronisation.

Often the relationship between  $\mathcal{O}$  and  $\mathcal{D}$  must be obtained via hard technical work in an *ad hoc* fashion, not unlike the way in which completeness proofs of logical systems are obtained. In this paper we propose a general technique which, as we believe, can be used for many different languages with recursion. The idea is to introduce an intermediate semantics  $\mathcal{I}$ , which is very similar to the operational semantics, in that it is also based on transition systems. However, now every configuration—representing a state of the abstract machine—is provided with information w.r.t. recursion. Indices which



accompany statements indicate their calling level. The intermediate semantics can be represented as a least upper bound of a chain of approximations. The  $n$ -th approximation is induced by the same transition systems but now restricted to those configurations with indices less than or equal to  $n$ . The approximations will improve by allowing a higher nesting of calls. As can be seen by continuity arguments, the denotational semantics  $\mathcal{D}$  is also a least upper bound of chains of approximations. The equality of  $\mathcal{I}$  and  $\mathcal{D}$  will follow from the equality—for each  $n$ —of the  $n$ -th approximation of  $\mathcal{I}$  and  $\mathcal{D}$ , respectively.

We shall illustrate the method for a very simple basic language with—besides recursion—operators for sequential, nondeterministic and parallel composition, where the last one is to be interpreted as an interleaving or shuffle operator, i.e., the execution of its arguments progresses alternatively and not simultaneously.

Finally, we remark that in both our denotational model and our relating technique we introduce elementary order theoretical notions inspired by metric topology, thus showing a fruitful application of this area of mathematics in a perhaps somewhat unexpected field.

This paper is organised as follows: §2 provides some mathematical preliminaries concerning domains, operators and transition systems. In §3 we introduce the sample programming language and provide an operational semantics for it. §4 contains a denotational definition. In §5 we present our equivalence result by introducing an intermediate model.

## 2. Mathematical Preliminaries

In this section we collect the mathematical prerequisites for the construction and equivalence of the several semantical definitions in this paper.

To start with we appoint the mathematical structure compelling enough coherence to serve as a domain of denotations. This is the notion of complete partial order, which is generally used in the area of semantics of programming languages. See e.g., PLOTKIN [1976], STOY [1977], BROOKES, HOARE and ROSCOE [1984]. We will present the slightly simpler notion of an  $\omega$ -complete partial order,  $\omega$ -cpo for short, which already serves our purposes, cf. DE BAKKER [1980]. The ordering will be interpreted as an approximation relation between partial and total meanings. The  $\omega$ -completeness captures a notion of computability. Morphisms between  $\omega$ -cpo's are the so-called continuous mappings. The main tool for making fixed point constructions in this context is known as (a simplified version of) the Knaster-Tarski theorem. See TARSKI [1955].

### 2.1. DEFINITION.

- (i) An  $\omega$ -cpo is a partial order  $(D, \leq)$  which has a least element  $\perp_D$  and in which every  $\omega$ -chain  $\langle x_i \rangle_i$  has a least upper bound  $\text{lub}_i x_i$ .

- (ii) Let  $D, E$  be  $\omega$ -cpo. A mapping  $f: D \rightarrow E$  is called continuous iff  $f$  is monotonic and moreover, for each chain  $\langle x_i \rangle_i$  in  $D$  it holds that  $f(\text{lub}_i x_i) = \text{lub}_i f(x_i)$ .

**2.2. THEOREM (Knaster-Tarski).** *Let  $D$  be an  $\omega$ -cpo. Suppose  $f: D \rightarrow D$  is continuous. Then  $f$  has a least fixed point  $\mu f$ . Moreover,  $\mu f = \text{lub}_i f^i(\perp)$ .*

PROOF. See DE BAKKER [1980]. □

To specify the objects in the semantical domain slightly more already we introduce streams in the sense of BACK [1983], BROY [1986] and MEYER [1985]. Streams are finite or infinite sequences of (abstract) actions, possibly ending in a distinguished marker  $\perp$  (indicating that this stream is not completed yet). A stream represents just one computation sequence. In order to deal with the nondeterminism that will be incorporated in our programming language, we have to resort to certain sets of streams, to a so-called powerdomain. Cf. PLOTKIN [1976], SMYTH [1978]. (On powerdomain constructions or more generally the solution of reflexive domain equations a vast amount of research on the border line of mathematics and computer science has emerged. See e.g., DE BAKKER and ZUCKER [1982], MAIN [1987], LAWSON [1988], AMERICA and RUTTEN [1988].) Here we will follow the more concrete approach of MEYER and DE VINK [1988] that is more suitable to the type of denotational model employed in this paper because of the availability of the extension and lifting lemma for the construction of semantical operators.

### 2.3. DEFINITION.

- (i) Let  $A$  be a set. Distinguish  $\perp \notin A$ . The set of streams  $A^{st}$  over  $A$  is given by  $A^{st} = A^* \cup A^* \cdot \perp \cup A^\omega$ . For  $x \in A^{st}$  and  $n \in \mathbb{N}$  we define  $x[n] \in A^{st}$  as follows:  $x[0] = \perp$ ,  $\perp[n+1] = \perp$ ,  $\epsilon[n+1] = \epsilon$  and  $(a.x')[n+1] = a.(x'[n])$ . We stipulate  $x[\infty] = x$ . The stream ordering  $\leq_{st}$  on  $A^{st}$  is defined as follows:  $x \leq_{st} y \Leftrightarrow \exists \alpha \in \mathbb{N}_\infty: x = y[\alpha]$ , with  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ .
- (ii) Let  $X \subseteq A^{st}$ .  $X$  is *flat* if  $\neg(x <_{st} x')$  whenever  $x, x' \in X$ .  $X[n] = \{x[n] \mid x \in X\}$ .  $X$  is *closed* if for every  $x \in A^{st}$ ,  $x \in X$  whenever  $x[n] \in X[n]$  for all  $n \in \mathbb{N}$ .  $X$  is *bounded* if  $X[n]$  is finite for all  $n \in \mathbb{N}$ .  $X$  is *compact* if  $X$  is flat, closed and bounded.
- (iii) Let  $\mathcal{P}^*(A^{st})$  denote the collection of all compact subsets of  $A^{st}$ . The Smyth-ordering  $\leq_S$  on  $\mathcal{P}^*(A^{st})$  is defined by  $X \leq_S Y \Leftrightarrow \forall y \in Y \exists x \in X: x \leq_{st} y$ .

### 2.4. THEOREM.

- (i)  $(\mathcal{P}^*(A^{st}), \leq_S)$  is an  $\omega$ -cpo.
- (ii) (*Extension Lemma*) Put  $A^f = A^* \cup A^* \cdot \perp$ . Let  $D$  be an  $\omega$ -cpo. Suppose  $f: (A^f)^k \rightarrow D$  is monotonic. Then  $\bar{f}: (A^{st})^k \rightarrow D$  defined by  $\bar{f}(\vec{x}) = \text{lub}_i f(\vec{x}[i])$  is well-defined and continuous.

(iii) (*Lifting Lemma*) If  $f: (A^{st})^k \rightarrow \mathcal{P}^*(A^{st})$  is continuous and

$$F: (\mathcal{P}^*(A^{st}))^k \rightarrow \mathcal{P}^*(A^{st})$$

is given by  $F(\vec{x}) = \min(\bigcup \{f(\vec{x}) | \vec{x} \in \vec{X}\})$ , then  $F$  is well-defined and continuous.

PROOF. See MEYER and DE VINK [1988]. □

Next we will give the definition of a transition system that will be used to define the operational semantics presented in the next section. Intuitively, transition systems describe the behaviours of abstract machines. Here a transition system is just a relation of type  $C \times L \times C$  where  $C$  and  $L$  are arbitrary sets (of configurations and labels, respectively). So a transition is a triple, say  $(c, l, c')$ , representing a computation step from a source configuration  $c$  to a target configuration  $c'$  with observable activity  $l$ . For a transition system we will require that from an arbitrary configuration finitely many configurations can be reached in one step involving finitely many labels. (Hence we consider ourselves in a situation of bounded nondeterminism, i.e., at any moment there are only finitely many alternatives.)

Sequences of transitions will be called computations. What can be observed from all maximal computations starting from an initial configuration  $c$  w.r.t. a transition system  $t$  constitutes the so-called yield of  $t$  for  $c$ . We shall find it convenient to have available a characterisation of this yield function in terms of a fixed point construction.

In §5, where the equivalence of the operational and denotational semantics of the particular programming language is addressed we will consider computations that only involve source configurations of a certain type. Such a restriction to a subclass of configurations induces derived notions of computation and yield. We observe that this restriction is a continuous operation from the cpo of subsets of configurations (ordered by set-inclusion) into the cpo of compact stream sets over labels.

## 2.5. DEFINITION.

- (i) Fix two sets  $C$  and  $L$ , the elements of which are called configurations and labels, respectively. A subset  $t \subseteq C \times L \times C$  is called a transition system over  $C$  and  $L$  if  $\#\{(\tilde{c}, l, \tilde{c}') \in t | \tilde{c} = c\} < \infty$  whenever  $c \in C$ .  $TS(C, L)$  denotes the collection of all transition systems over  $C$  and  $L$ .
- (ii) For  $t \in TS(C, L)$  we define the set  $IC_t$  of intermediate configurations by  $IC_t = \{c \in C | (\exists l \in L)(\exists c' \in C) : (c, l, c') \in t\}$  and the set  $FC_t$  of final configurations by  $FC_t = C \setminus IC_t$ . We often write  $c \xrightarrow{\ell}_t c'$  if  $(c, l, c') \in t$ .
- (iii) Let  $t \in TS(C, L)$ . A  $t$ -computation  $\gamma$  for  $c$  is a sequence of pairs  $\langle \ell_i, c_i \rangle_{i \in I}$  such that  $I$  is an initial segment of  $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$  and  $\forall i \in I : c_{i-1} \xrightarrow{\ell_i}_t c_i$ , where  $c_0 = c$ .  $lgt(\gamma)$  denotes the length of the computation

- $\gamma$ ; we put  $lgt(\gamma) = \#I \in \mathbb{N}_\infty$ . We say that  $\gamma$  is finished, if  $lgt(\gamma)$  is finite and  $c_{lgt(\gamma)}$  is a final configuration. We then define the yield of  $\gamma$  by  $yld(\gamma) = \ell_1 \cdots \ell_{lgt(\gamma)} \in L^*$ . If  $lgt(\gamma)$  is finite but  $c_{lgt(\gamma)}$  is non-final, we call  $\gamma$  unfinished and put  $yld(\gamma) = \ell_1 \cdots \ell_{lgt(\gamma)}. \perp \in L^*$ .  $\perp$ . We say that  $\gamma$  is infinite if  $lgt(\gamma) = \infty$ . In this case we put  $yld(\gamma) = \ell_1 \ell_2 \cdots \in L^\omega$ .
- (iv) For  $c \in C$  we define  $comp_t(c) = \{\gamma \mid \gamma \text{ is a finished or is an infinite } t\text{-computation for } c\}$ . Define the mapping  $yld_t: C \rightarrow \mathcal{P}^*(L^{st})$ , called the yield of  $t$ , by  $yld_t(c) = \{yld(\gamma) \mid \gamma \in comp_t(c)\}$ .
- (v) Define the valuation function  $\phi_t: C \rightarrow \mathcal{P}^*(L^{st})$  of the transition system  $t$  as the least fixed point of  $\Phi_t$ , where  $\Phi_t \in Map(C \rightarrow \mathcal{P}^*(L^{st}))$ , the function space of mappings from  $C \rightarrow \mathcal{P}^*(L^{st})$  to  $C \rightarrow \mathcal{P}^*(L^{st})$ , is given by  $\Phi_t(\phi)(c) = \{\epsilon\}$  if  $c \in FC_t$ ,  $\Phi_t(\phi)(c) = \min(\cup\{\ell.\phi(c') \mid c \xrightarrow{\ell}_t c'\})$  if  $c \in IC_t$ .

**2.6. LEMMA.** *Let  $t \in TS(C, L)$ . Then  $yld_t = \phi_t$ .*

PROOF. Fix  $t \in TS(C, L)$ . Note that  $yld_t: C \rightarrow \mathcal{P}^*(L^{st})$  is well-defined. For if  $c \in C$  then the yield for  $c$  under  $t$ ,  $yld_t(c)$ , is flat, since  $yld_t(c) \subseteq L^* \cup L^\omega$ . Also, for all  $\bar{c}$  there exist only finitely many transitions  $\bar{c} \xrightarrow{\ell}_t \bar{c}'$ , hence  $yld_t(c)$  is bounded. Moreover, by König's Lemma it follows that  $yld_t(c)$  is closed. The well-definedness of  $\phi_t$  relies on the continuity of finite sums in  $\mathcal{P}^*(L^{st})$  and is left to the reader.

Put  $\phi_{t,0} = \lambda c. \{\perp\}$ ,  $\phi_{t,n+1} = \Phi_t(\phi_{t,n})$ . Define  $yld_{t,n} = \lambda c. yld_t(c)[n]$ . We prove by induction on  $n$ :  $yld_{t,n} = \phi_{t,n}$ . From this it follows by theorem 2.2 that  $yld_t = \text{lub}_n yld_{t,n} = \text{lub}_n \phi_{t,n} = \phi_t$ .

Clearly  $yld_{t,0} = \phi_{t,0}$ , for  $yld_t(c)$  is a non-empty set, since every configuration admits a finished or infinite computation. Suppose  $c \in C$ . To prove:  $yld_{t,n+1}(c) = \phi_{t,n+1}(c)$ . Without loss of generality we assume  $c \in IC_t$ . So we have  $yld_{t,n+1}(c) = \min\{yld(\gamma)[n+1] \mid \gamma \in comp_t(c)\} = \min\{\ell.yld_t(\gamma')[n] \mid c \xrightarrow{\ell}_t c', \gamma' \in comp_t(c')\} = \min(\cup\{\ell.yld_{t,n}(c') \mid c \xrightarrow{\ell}_t c'\}) = \min(\cup\{\ell.\phi_{t,n}(c') \mid c \xrightarrow{\ell}_t c'\}) = \phi_{t,n+1}(c)$ .  $\square$

**2.7. DEFINITION.**

- (i) Choose  $t \in TS(C, L)$ . Let  $C' \subseteq C$  be a subset of configurations. Define  $\phi'_t: C \rightarrow \mathcal{P}^*(L^{st})$  as the least fixed point of  $\Phi'_t \in Map(C \rightarrow \mathcal{P}^*(L^{st}))$ , where  $\Phi'_t$  is given by  $\Phi'_t(\phi)(c) = \{\epsilon\}$  if  $c \in FC_t \cap C'$ ,  $\Phi'_t(\phi)(c) = \min(\cup\{\ell.\phi(c') \mid c \xrightarrow{\ell}_t c'\})$  if  $c \in C' \cap IC_t$ ,  $\Phi'_t(\phi)(c) = \{\perp\}$  otherwise.  $\phi'_t$  is called the restriction of  $\phi_t$  to  $C'$ .
- (ii) Fix  $C' \subseteq C$ . Let  $\gamma = \langle \ell_i, c_i \rangle_{i \in I}$  be a  $t$ -computation. We say that  $\gamma$  is a  $t$ -computation for  $c$  in  $C' \Leftrightarrow \forall i \in I : c_{i-1} \in C'$  (where by convention  $c_0 = c$ ). We define for  $c \in C$  the set of maximal  $t$ -computations  $comp'_t(c)$  for  $c$  in  $C'$  by  $\gamma \in comp'_t(c) \Leftrightarrow (lgt(\gamma) < \infty \Rightarrow c_{lgt(\gamma)} \notin C \text{ or } \neg \exists \ell \in L \exists c' \in C : c_{lgt(\gamma)} \xrightarrow{\ell}_t c') \text{ for any } t\text{-computation } \gamma \text{ for } c$

in  $C'$ . Define  $yld'_t: C \rightarrow \mathcal{P}^*(L^{st})$ , the restriction of  $yld_t$  to  $C'$ , by  $yld'_t(c) = \min\{yld(\gamma) \mid \gamma \in comp'_t(c)\}$ .

### 2.8. LEMMA.

- (i) Let  $t \in TS(C, L)$  and  $C' \subseteq C$ . Let  $\phi'_t$  be the restriction of  $\phi_t$  to  $C'$ . Then we have  $\forall c \in C: \phi'_t(c) = yld'_t(c)$ .
- (ii) Let  $C_0 \subseteq C_1 \subseteq \dots$  be an infinite sequence of subsets of  $C$ . Put  $C_\infty = \bigcup_n C_n$ . Let  $\phi_{t,\alpha}: C \rightarrow \mathcal{P}^*(L^{st})$  be the restriction of  $\phi_t$  to  $C_\alpha$ , ( $\alpha \in \mathbb{N}_\infty$ ). Then we have that  $\langle \phi_{t,n} \rangle_n$  is a chain in  $C \rightarrow \mathcal{P}^*(L^{st})$  and  $\phi_{t,\infty} = lub_n \phi_{t,n}$ .

PROOF. Left to the reader. □

## 3. Operational Semantics

In this section we introduce the simple programming language *Prog* for which we shall illustrate our method of designing equivalent operational and denotational semantics. Also a (parametrised) transition system is given, that represents the computation steps of an abstract machine running the language.

### 3.1. DEFINITION.

- (i) Fix a set  $\mathcal{A}$  and a set  $\mathcal{X}$ , the elements of which are called actions and procedure variables, respectively.  $\mathcal{A}$  is ranged over by  $a$ ,  $\mathcal{X}$  is ranged over by  $x$ .
- (ii) The class of statements *Stat*, with typical element  $s$ , is given by  $s ::= a|x|(s_1; s_2) \mid (s_1 + s_2) \mid (s_1 \parallel s_2)$ .
- (iii) The class of declarations *Decl*, ranged over by  $d$ , has elements of the format  $x_1 \Leftarrow s_1 : \dots : x_n \Leftarrow s_n$  where  $n \in \mathbb{N}$ ,  $x_i \in \mathcal{X}$  all distinct and  $s_i \in \text{Stat}$  ( $i \in \{1, \dots, n\}$ ). We say that the procedure variable  $x_i$  is declared in  $d$  with body  $s_i$  and write  $x_i \Leftarrow s_i \in d$ , ( $i \in \{1, \dots, n\}$ ).
- (iv) The class of programs *Prog* consists of pairs  $d|s$ , where  $d \in \text{Decl}$  and  $s \in \text{Stat}$  such that each procedure variable  $x$  occurring in  $s$  and  $d$  is declared in  $d$ .

A statement is either an action  $a$  in  $\mathcal{A}$ , a procedure call  $x$  in  $\mathcal{X}$  or a sequential composition  $s_1; s_2$ , a nondeterministic choice  $s_1 + s_2$  or a concurrent execution  $s_1 \parallel s_2$ . We suppress parentheses if no confusion may arise.

Our main interest here is in the programming concepts embodied by the syntactical operators and by recursion. Therefore, the programming language *Prog* under consideration is kept uniform or schematic, since the elementary actions are left unspecified. (Cf. DE BAKKER ET AL [1986].)

The restriction to syntactically closed programs  $d|s$  in *Prog*—each variable in the program should be coupled with a body—facilitates an easy formulation

of the semantical definitions. However, the restriction could be dispensed with at the price of appropriate precautions and exceptions below.

**3.2. DEFINITION.** We distinguish  $E$  which is called the empty statement. We define  $Stat_E$ , with typical element  $\bar{s}$ , by  $Stat_E = Stat \cup \{E\}$ . We also distinguish  $\tau \notin \mathcal{A}$  and put  $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$ .  $\mathcal{A}_\tau$  is ranged over by  $\alpha$ .

The empty statement  $E$  is associated with successful termination, (cf. APT [1983]). It will be convenient below to allow the expressions  $\bar{s}; \bar{s}'$ ,  $\bar{s} + \bar{s}'$  and  $\bar{s} \parallel \bar{s}'$  for arbitrary  $\bar{s}, \bar{s}' \in Stat_E$ . Therefore we stipulate  $\bar{s} * E = E * \bar{s} = \bar{s}$  for  $\bar{s} \in Stat_E$  and  $*$   $\in \{;, +, \parallel\}$ .

The silent step  $\tau$ —which plays a predominant role in algebraic approaches to the semantics of concurrency as e.g., MILNER [1980], BERGSTRA and KLOP [1986]—will be used in definition 3.3 below to indicate body replacement. (As in DE BAKKER ET AL [1984].)

Next we give the definition of the transition system that we will associate with a declaration  $d$ . The definition illustrates the syntax-directedness which is typical for transition system based operational models. Such a Structural Operational Semantics (originating from HENNESSY and PLOTKIN [1979], PLOTKIN [1981]), can be considered a proof system for *proving* transitions and as a consequence the represented abstract machine can be considered as a theorem prover. (We refer to the recent developments BADOUEL [1987], GROOTE and VAANDRAGER [1989], RUTTEN [1989] for some interesting consequences of this point of view.)

**3.3. DEFINITION.** A declaration  $d \in Decl$  induces a transition system  $d \in TS(Stat_E, \mathcal{A}_\tau)$  which is the smallest subset of  $Stat_E \times \mathcal{A}_\tau \times Stat_E$  such that

$$a \xrightarrow{a}_d E, \quad (\text{Action})$$

$$x \xrightarrow{\tau}_d s \text{ where } x \Leftarrow s \in d, \quad (\text{Proc})$$

$$\text{if } s \xrightarrow{\alpha}_d \bar{s} \text{ then } s; s' \xrightarrow{\alpha}_d \bar{s}; s', \quad (\text{Seq})$$

$$\text{if } s \xrightarrow{\alpha}_d \bar{s} \text{ then } s + s' \xrightarrow{\alpha}_d \bar{s} \text{ and } s' + s \xrightarrow{\alpha}_d \bar{s}, \quad (\text{Choice})$$

$$\text{if } s \xrightarrow{\alpha}_d \bar{s} \text{ then } s \parallel s' \xrightarrow{\alpha}_d \bar{s} \parallel s' \text{ and } s' \parallel s \xrightarrow{\alpha}_d s' \parallel \bar{s}. \quad (\text{Par})$$

The axiom (Action) states that an action  $a \in \mathcal{A}$  can always be executed successfully after performing  $a$ . If a procedure call  $x$  is about to be executed, a silent step  $\tau$  is signaled and the computation continues with the body  $s$  of  $x$ , which is looked up in the declaration. (Since  $d$  is syntactically closed such a body is assumed to exist; since the declared procedure variables are pairwise

distinct the body is unique.) The three rules (Seq), (Choice) and (Par) can be used to unravel composed statements. Alternative and parallel compositions can perform the same actions (in  $\mathcal{A}_\tau$ ) as their constituting components do. For a sequential composition  $s; s'$ , however, this depends on the actions of its first component, for intuitively the execution of  $s'$  is started after the execution of  $s$  has finished.

**3.4. EXAMPLE.** Suppose  $x \Leftarrow a; x + b \in d$ .

- (i)  $x; c \parallel y \xrightarrow{a}_d (a; x + b); c \parallel y$ , for  $x \xrightarrow{a}_d a; x + b$  by (Proc), hence  $x; c \xrightarrow{a}_d (a; x + b); c$  by (Seq), so  $x; c \parallel y \xrightarrow{a}_d (a; x + b); c \parallel y$  by (Par).
- (ii)  $(a; x + b); c \parallel y \xrightarrow{b}_d c \parallel y$ , for  $b \xrightarrow{b}_d E$  by (Action), hence  $a; x + b \xrightarrow{b}_d E$  by (Choice), so  $(a; x + b); c \xrightarrow{b}_d c$  by (Seq) and therefore by (Par)  $(a; x + b); c \parallel y \xrightarrow{b}_d c \parallel y$ .

The operational semantics for a program  $d|s \in Prog$  can now be defined straightforwardly. The declaration  $d$  specifies the transition system; the statement  $s$  specifies the initial configuration in which the computation starts. The operational semantics will collect all the yields of the transition sequences in  $d$  starting from  $s$ .

**3.5. DEFINITION.** The operational semantics  $\mathcal{O}: Prog \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  is defined by  $\mathcal{O}(d|s) = yld_d(s)$ , where  $yld_d$  is the yield function of the transition system  $d \in TS(Stat_E, \mathcal{A}_\tau)$ .

This completes the construction of the operational semantics. The operational model will be compared to a denotational one to be constructed in the next section.

## 4. Denotational Semantics

In this section we present a denotational semantics for the language under consideration. It will assign to each program a *denotation*, i.e., an object representing its meaning, in a suitably chosen mathematical domain. Moreover, the semantics will be compositional or homomorphic: the meanings of a composed construct depends only upon the meaning of its constituting components. Finally, we will use a technique based on so-called environments and fixed points to handle recursion.

Programs will be given meaning in the domain  $\mathcal{P}^*(\mathcal{A}_\tau^{st})$ . So compact stream sets serve as denotations. (We have to resort to sets of streams, rather than streams, in order to cater for nondeterminism.) In order to deal with procedure calls we introduce the notion of an environment. Environments are used to store and retrieve the meaning of procedure variables, i.e., mappings from  $\mathcal{X}$  to  $\mathcal{P}^*(\mathcal{A}_\tau^{st})$ . For a program  $d|s$ , the statement  $s$  will be evaluated w.r.t. an

environment depending on the declaration  $d$ . So the definition of the denotational semantics amounts to specifying the evaluation of a statement and the environment corresponding to a given declaration.

The compositionality requirement for the evaluation of statements will be met by designing for each syntactical operator, viz.  $;$ ,  $+$  and  $\parallel$ , a semantical one, written as  $;\mathcal{D}$ ,  $+\mathcal{D}$  and  $\parallel\mathcal{D}$ , respectively. We will have  $\mathcal{D}(d|s_1 * s_2) = \mathcal{D}(d|s_1) *_{\mathcal{D}} \mathcal{D}(d|s_2)$  for  $*$   $\in \{;, +, \parallel\}$ . Each declaration induces an environment transformation. We will take the least fixed point of this transformation, say  $\eta_d$ , for the environment associated with the declaration. For this to work we need that this transformation is continuous. Since the transformation of an environment  $\eta$  is in essence body replacement—a procedure variable will be mapped on the denotation of its body w.r.t.  $\eta$ —we will need continuity of the semantical operators  $;\mathcal{D}$ ,  $+\mathcal{D}$  and  $\parallel\mathcal{D}$ . In its turn this will be guaranteed by the extension and lifting lemma, presented in §2, that we will use in the construction of these operators.

#### 4.1. DEFINITION.

- (i) We define  $;\mathcal{D}: \mathcal{P}^*(\mathcal{A}_\tau^{st}) \times \mathcal{P}^*(\mathcal{A}_\tau^{st}) \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  as the extended and lifted version of  $;\mathcal{D}: \mathcal{A}_\tau^f \times \mathcal{A}_\tau^f \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$ , that is given by  $\epsilon;_{\mathcal{D}} y = \{y\}$ ,  $\perp;_{\mathcal{D}} y = \{\perp\}$ ,  $\alpha x;_{\mathcal{D}} y = \alpha(x;_{\mathcal{D}} y)$ .
- (ii) We define  $+\mathcal{D}: \mathcal{P}^*(\mathcal{A}_\tau^{st}) \times \mathcal{P}^*(\mathcal{A}_\tau^{st}) \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  as the extended and lifted version of  $+\mathcal{D}: \mathcal{A}_\tau^f \times \mathcal{A}_\tau^f \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$ , that is given by  $x +_{\mathcal{D}} y = \min\{x, y\}$ .
- (iii)  $\parallel\mathcal{D}, \llbracket\mathcal{D}: \mathcal{A}_\tau^f \times \mathcal{A}_\tau^f \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  are given by  $\epsilon\llbracket\mathcal{D} y = \{y\}$ ,  $\perp\llbracket\mathcal{D} y = \{\perp\}$ ,  $\alpha x\llbracket\mathcal{D} y = \alpha(x\llbracket\mathcal{D} y)$ ,  $x\llbracket\mathcal{D} y = x\llbracket\mathcal{D} y +_{\mathcal{D}} y\llbracket\mathcal{D} x$ . We define  $\parallel\mathcal{D}: \mathcal{P}^*(\mathcal{A}_\tau^{st}) \times \mathcal{P}^*(\mathcal{A}_\tau^{st}) \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  as the extended and lifted version of  $\parallel\mathcal{D}$  on  $\mathcal{A}_\tau^f$ .

Having now available semantical interpretations for our syntactical operators we can proceed with the definition of the denotational semantics for *Prog*. It follows the same general scheme as in e.g. DE BAKKER and MEYER [1988], DE VINK [1988].

**4.2. DEFINITION.** Let  $Env = \mathcal{X} \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$ , with typical element  $\eta$ , denote the collection of environments.

- (i) We define the statement evaluator  $\mathcal{S}: Stat \rightarrow Env \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  by  $\mathcal{S}(a)(\eta) = \{a\}$ ,  $\mathcal{S}(x)(\eta) = \eta(x)$ ,  $\mathcal{S}(s_1 * s_2)(\eta) = \mathcal{S}(s_1)(\eta) *_{\mathcal{D}} \mathcal{S}(s_2)(\eta)$  for  $*$   $\in \{;, +, \parallel\}$ .
- (ii) For  $d \in Decl$  we define an environment transformation  $\Phi_d: Env \rightarrow Env$ ,  $\Phi_d(\eta)(x) = \tau\mathcal{S}(s)(\eta)$  where  $x \Leftarrow s \in d$ .
- (iii) The denotational semantics  $\mathcal{D}: Prog \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  is defined by  $\mathcal{D}(d|s) = \mathcal{S}(s)(\eta_d)$  where  $\eta_d$  is the least fixed point of  $\Phi_d$ .

The definition of  $\eta_d$  in clause (iii) above appeals to the Knaster-Tarski theorem 2.2. Therefore we have to check continuity of the transformation  $\Phi_d$  and hence of  $\mathcal{S}$ . This boils down to continuity of the semantical operators, which is guaranteed by the tools used for their construction.



**4.3. LEMMA.**

- (i)  $;\mathcal{D}, +_{\mathcal{D}}, \|\mathcal{D}$  are well-defined and continuous on  $\mathcal{P}^*(\mathcal{A}_{\tau}^{st})$ .
- (ii)  $\mathcal{S}$  and  $\Phi_d$  are continuous in  $\eta$ .
- (iii)  $\mathcal{D}$  is well-defined. □

Let  $d \in Decl$  and  $\eta_d$  be the least fixed point of  $\Phi_d$ . From the lemma we deduce the following. Define the environments  $\eta_{d,k}$  ( $k \in \mathbb{N}$ ) inductively by  $\eta_{d,0} = \lambda x. \{\perp\}$ ,  $\eta_{d,k+1} = \Phi_d(\eta_{d,k})$ . Then we have  $\eta_d = \text{lub}_n \eta_{d,n}$ . So the denotational semantics  $\mathcal{D}(d|s) = \mathcal{S}(s)(\eta_d)$  can be represented as the least upper bound of the chain  $\langle \mathcal{S}(s)(\eta_{d,i}) \rangle_i$ . This property will be exploited in the next section devoted to the equivalence of the operational and denotational definitions.

**5. Equivalence of  $\mathcal{O}$  and  $\mathcal{D}$** 

Having defined both an operational and a denotational semantics for the programs in *Prog* the question rises whether the denotational model  $\mathcal{D}$  is correct with respect to the computational intuition captured by the operational model  $\mathcal{O}$ . Therefore the concern of this section is to prove the next theorem.

**5.1. THEOREM.**  $\mathcal{O} = \mathcal{D}$ .

The strategy we shall exploit in proving theorem 5.1 is the following: First we extend the operational semantics obtaining a similar so-called intermediate semantics  $\mathcal{I}$ . (Similar in the sense that  $\mathcal{I}$  is also transition system based. Moreover it can almost directly be observed that  $\mathcal{O} = \mathcal{I}$ .) Both the intermediate and denotational semantics—unlike the operational one—can be represented as least upper bounds of chains of approximations as was indicated already for the denotational semantics at the end of §4. We shall apply a fixed point induction technique at the level of the approximations in order to establish the equality of the limits, i.e.,  $\mathcal{I} = \mathcal{D}$ .

The configurations of the intermediate transition systems are so-called generalised statements—statements augmented with indices indicating the number of nestings of procedure calls—together with the empty statement E.

**5.2. DEFINITION.**

- (i) Let  $n$  be a typical variable of  $\mathbb{N}$ . The collection of generalised statements  $GStat$  is given by  $G ::= (s, n) | G_1; G_2 | G_1 \cup G_2 | G_1 \| G_2$ . We put  $GStat_E = GStat \cup \{E\}$ .
- (ii) For  $G \in GStat$  we define  $ind(G)$ —the index of  $G$ —in  $\mathbb{N}$  as follows:  $ind(a, n) = 0$ ,  $ind(x, n) = n + 1$ ,  $ind(s_1 * s_2, n) = \max\{ind(s_1, n), ind(s_2, n)\}$ ,  $ind(G_1 * G_2) = \max\{ind(G_1), ind(G_2)\}$ .
- (iii) For  $k \in \mathbb{N}$  we put  $GStat_k = \{G \in GStat | ind(G) \leq k\}$ .

The intermediate transition system  $d \in TS(GStat_E, \mathcal{A}_\tau)$  below that we shall associate with a declaration  $d$  is much alike the induced transition system  $d \in TS(Stat_E, \mathcal{A}_\tau)$ . Note that in *Proc* we increment the index since the body  $s$  is considered to be of a deeper call level, (viz.  $n+1$ ), than the procedure call  $x$  itself, (of level  $n$ ). The auxiliary rule *Aux* is used to distribute operators inside out. It amounts to the identification of the generalised statements of the formats  $(s, n) * (s', n)$  and  $(s * s', n)$ , respectively.

### 5.3. DEFINITION.

- (i) Let  $d \in Decl$ . The declaration  $d$  induces a transition system  $d \in TS(GStat_E, \mathcal{A}_\tau)$  which is the smallest subset of  $(GStat_E, \mathcal{A}_\tau, GStat_E)$  such that

$$(a, n) \xrightarrow{a}_d E, \quad (\text{Action})$$

$$(x, n) \xrightarrow{\tau}_d (s, n+1), \text{ where } x \Leftarrow s \in d, \quad (\text{Proc})$$

$$\text{if } (s, n) * (s', n) \xrightarrow{\alpha}_d \overline{G} \quad (\text{Aux})$$

$$\text{then } (s * s', n) \xrightarrow{\alpha}_d \overline{G}, \text{ for } * \in \{;, +, \|\},$$

$$\text{if } G \xrightarrow{\alpha}_d \overline{G} \text{ then } G; G' \xrightarrow{\alpha}_d \overline{G}; G', \quad (\text{Seq})$$

$$\text{if } G \xrightarrow{\alpha}_d \overline{G} \text{ then } G + G' \xrightarrow{\alpha}_d \overline{G} \text{ and } G' + G \xrightarrow{\alpha}_d \overline{G}, \quad (\text{Choice})$$

$$\text{if } G \xrightarrow{\alpha}_d \overline{G} \text{ then } G \| G' \xrightarrow{\alpha}_d \overline{G} \| G' \text{ and } G' \| G \xrightarrow{\alpha}_d G' \| \overline{G}. \quad (\text{Par})$$

- (ii) The intermediate semantics  $\mathcal{I}: Prog \rightarrow \mathcal{P}^*(\mathcal{A}_\tau^{st})$  is defined by  $\mathcal{I}(d|s) = yld_d(s, 0)$  where  $yld_d$  is the yield function of the transition system  $d \in TS(GStat_E, \mathcal{A}_\tau)$ .

The intermediate semantics of a program  $d|s$  is defined as the yield function of the transition system  $d$  applied to the initial configuration for  $s$  in the intermediate model being  $(s, 0)$ . So  $\mathcal{I}(d|s)$  is a set of streams of labels. Therefore lemma 5.4 below will not come as a surprise since the respective transition systems are the same modulo configurations.

### 5.4. LEMMA. $\mathcal{O} = \mathcal{I}$ .

PROOF. By definition of  $\mathcal{O}$  and  $\mathcal{I}$  it suffices to show for all  $d \in Decl$ ,  $s \in Stat$  that  $yld_d(s) = yld_d(s, 0)$  where  $yld_d$  is taken with respect to  $d \in TS(Stat_E, \mathcal{A}_\tau)$  and  $d \in TS(GStat_E, \mathcal{A}_\tau)$ , respectively.

Define a projection  $\pi: GStat_E \rightarrow Stat_E$  as follows:  $\pi(E) = E$ ,  $\pi(s, n) = s$ ,  $\pi(G_1 * G_2) = \pi(G_1) * \pi(G_2)$ . By induction on the derivation for  $s \xrightarrow{\alpha}_d \overline{s}$ ,  $G \xrightarrow{\alpha}_d \overline{G}$  we can establish (i)  $s \xrightarrow{\alpha}_d \overline{s} \& \pi(G) = s \Rightarrow G \xrightarrow{\alpha}_d \overline{G} \& \pi(\overline{G}) = \overline{s}$  for some  $\overline{G}$ , and (ii)  $G \xrightarrow{\alpha}_d \overline{G} \Rightarrow \pi(G) \xrightarrow{\alpha}_d \pi(\overline{G})$ . From this we derive straightforwardly  $yld_d(s) = yld_d(s, 0)$ .  $\square$

We proceed with exploiting the extra information available in generalised statements. Intuitively it is clear that, for fixed  $k$ , computations in  $GStat_k$  are

finite. Infinite computations will eventually encounter a configuration of index exceeding  $k$ , for such computations involve unbounded nestings of recursive calls.

**5.5. THEOREM.** *Every  $d$ -computation in  $GStat_k$  is finite.*

PROOF. For  $s \in Stat$  we define a complexity measure  $compl(s)$  in  $\mathbb{N}_+$  as follows:  $compl(a) = compl(x) = 1$ ,  $compl(s_1 * s_2) = compl(s_1) + compl(s_2)$ . Let  $k \in \mathbb{N}$ . Associate with  $\overline{G} \in GStat_E$  a polynomial  $wgt_k(\overline{G})$  in  $\mathbb{N}[X]$ —called the weight of  $\overline{G}$ —as follows:  $wgt_k(s, n) = compl(s) \cdot X^{k+1-n}$  if  $ind(s, n) \leq k$ ,  $wgt_k(G_1 * G_2) = wgt_k(G_1) + wgt_k(G_2)$  if  $ind(G_1 * G_2) \leq k$ ,  $wgt_k(\overline{G}) = 0$  otherwise. Note, that we have  $wgt_k(\overline{G}) \neq 0 \Leftrightarrow \overline{G} \in GStat_k$ .

Define an ordering  $\leq$  on  $\mathbb{N}[X]$  by  $f \leq g \Leftrightarrow f(n) \leq g(n)$  for almost all  $n \in \mathbb{N}$ . Then it is the case that  $(\mathbb{N}[X], \leq)$  is a well-founded partial order, (i.e., a partial order in which every decreasing sequence is finite). Therefore it suffices to prove: If  $G \xrightarrow{\alpha}_d \overline{G}$  in  $GStat_k$ , then  $wgt_k(G) > wgt_k(\overline{G})$ . Cf. KLOP [1980], DERSHOWITH [1987].

Suppose  $G \xrightarrow{\alpha}_d \overline{G}$  with  $\overline{G} \notin GStat_k$ . From  $wgt_k(G) \neq 0$  in  $\mathbb{N}[X]$  and  $wgt_k(\overline{G}) = 0$  we infer  $wgt_k(G) > wgt_k(\overline{G})$ . So, assume  $G \xrightarrow{\alpha}_d \overline{G}$  with  $\overline{G} \in GStat_k$ . We show  $wgt_k(G) > wgt_k(\overline{G})$  by induction on the derivation of  $G \xrightarrow{\alpha}_d \overline{G}$ . (Action) Trivial. (Proc) Say  $G = (x, n)$ ,  $\overline{G} = (s, n+1)$  with  $n, n+1 \leq k$  and  $x \Leftarrow s \in d$ . Clearly  $wgt_k(G) = X^{k+1-n} > compl(s) \cdot X^{k-n} = wgt_k(\overline{G})$ , for  $k+1-n > k-n$ . (Aux) By induction hypothesis, for the weights of the left-hand sides of premise and conclusion are equal. (Seq) Say  $G = G''; G'$ ,  $\overline{G} = \overline{G}''; G'$ . By induction hypothesis or the above we have  $wgt_k(G'') > wgt_k(\overline{G}'')$ . So  $wgt_k(G) = wgt_k(G'') + wgt_k(G') > wgt_k(\overline{G}'') + wgt_k(G') = wgt_k(\overline{G})$ . (Choice) and (Par) Similar to the case (Seq).  $\square$

Next we establish a compositionality result for the approximations of the intermediate semantics. As a corollary to theorem 5.5 we have that this can be obtained using the principle of Noetherian induction. (See HUET[1980].) The collection of all computations in  $GStat_k$  starting from a fixed generalised statement can be represented by a *finitely branching* tree. Since by theorem 5.5 all paths in this tree are finite it follows by König's lemma that there exists a uniform bound on the length of these computations which justifies the appropriateness of the principle.

**5.6. LEMMA.** *Let  $yld_{d,k}$  be the yield function of the restriction of the transition system  $d$  to  $GStat_k$ . Suppose  $G_1, G_2 \in GStat$ . Then it holds that  $yld_{d,k}(G_1 * G_2) = yld_{d,k}(G_1) *_{\mathcal{D}} yld_{d,k}(G_2)$  for  $*$  in  $\{;, +, \parallel\}$ .*

PROOF. We only prove the case  $\parallel$ . It suffices to show

$$\{yld(\gamma) \mid \gamma \in comp_{d,k}(G_1 \parallel G_2)\} = \cup \{yld(\gamma_1) \parallel yld(\gamma_2) \mid \gamma_i \in comp_{d,k}(G_i)\}.$$

By strictness of  $\parallel$ , we can assume  $ind(G_1 \parallel G_2) \leq k$ .

(a) Let  $\gamma$  be a maximal  $d$ -computation for  $G_1 \parallel G_2$  in  $GStat_k$ . Say

$$\gamma: G_1 \parallel G_2 \xrightarrow{w}_d Z$$

with  $Z \in GStat_E \setminus GStat_k$ . Put  $\zeta = \epsilon$  if  $Z = E$ ,  $\zeta = \perp$  if  $Z \in GStat \setminus GStat_k$ . To prove  $w\zeta \in yld(\gamma_1) \parallel yld(\gamma_2)$  for some maximal  $d$ -computations  $\gamma_1, \gamma_2$  for  $G_1, G_2$  in  $GStat_k$ . Without loss of generality  $\gamma: G_1 \parallel G_2 \xrightarrow{\alpha}_d \overline{G}_1 \parallel G_2 \xrightarrow{w'} Z$  where  $G_1 \xrightarrow{\alpha}_d \overline{G}_1$  and  $\alpha w' = w$ . " $\overline{G}_1 = E$ " Then we have  $\gamma: G_1 \parallel G_2 \xrightarrow{\alpha}_d G_2 \xrightarrow{w'}_d Z$ . Put  $\gamma_1: G_1 \xrightarrow{\alpha}_d E$ ,  $\gamma_2: G_2 \xrightarrow{w'}_d Z$ . Notice  $|w'| \geq 1$ . So  $w\zeta = \alpha w' \zeta \in \alpha \parallel w' \zeta = yld(\gamma_1) \parallel yld(\gamma_2)$ . " $ind(\overline{G}_1) \leq k$ " By induction hypothesis there exist maximal  $d$ -computations  $\gamma'_1: \overline{G}_1 \xrightarrow{w'}_d Z_1$ ,  $\gamma_2: G_2 \xrightarrow{w_2}_d Z_2$  in  $GStat_k$  such that  $w' \zeta \in w'_1 \zeta_1 \parallel w_2 \zeta_2$  (with  $Z_i$  corresponding to  $\zeta_i$ ,  $i = 1, 2$ ). Put  $\gamma_1: G_1 \xrightarrow{\alpha}_d \overline{G}_1 \xrightarrow{w'_1}_d Z_1$ . Then  $\gamma_1, \gamma_2$  satisfy the conditions. " $ind(\overline{G}_1) > k$ " Then we have  $w\zeta = \alpha \perp$ . Put  $\gamma_1: G_1 \xrightarrow{\alpha}_d \overline{G}_1$  and choose an arbitrary (non-empty) maximal  $d$ -computation  $\gamma_2: G_2 \xrightarrow{w_2}_d Z$  in  $GStat_k$ . So it holds that  $w\zeta = \alpha \perp \in \alpha \perp \parallel w_2 \zeta_2 = yld(\gamma_1) \parallel yld(\gamma_2)$ .

(b) Let  $\gamma_1, \gamma_2$  be maximal  $d$ -computations in  $GStat_k$  for  $G_1, G_2$ . Say  $\gamma_1: G_1 \xrightarrow{w_1}_d Z_1$ ,  $G_2 \xrightarrow{w_2}_d Z_2$ . Choose  $w\zeta \in w_1 \zeta_1 \parallel w_2 \zeta_2$ . To prove  $w\zeta = yld(\gamma)$  for some maximal  $d$ -computation  $\gamma: G_1 \parallel G_2 \xrightarrow{w}_d Z$  in  $GStat_k$ . Without loss of generality  $\gamma_1: G_1 \xrightarrow{\alpha}_d \overline{G}_1 \xrightarrow{w'_1}_d Z_1$  and  $w\zeta \in w_1 \zeta_1 \parallel w_2 \zeta_2$ . So  $w = \alpha w'$  and  $w' \zeta \in w'_1 \zeta_1 \parallel w_2 \zeta_2$ . " $\overline{G}_1 = E$ " Then we have  $w\zeta \in \alpha \parallel w_2 \zeta_2 = \{\alpha w_2 \zeta_2\}$ . So  $w = \alpha w_2$ ,  $\zeta = \zeta_2$ . Put  $\gamma: G_1 \parallel G_2 \xrightarrow{\alpha}_d G_2 \xrightarrow{w_2}_d Z_2$ . Then  $\gamma \in comp_{d,k}(G_1 \parallel G_2)$  with  $yld(\gamma) = w\zeta$ . " $ind(\overline{G}_1) \leq k$ " By induction hypothesis there exists  $\gamma' \in comp_{d,k}(\overline{G}_1 \parallel G_2)$  such that  $\gamma': \overline{G}_1 \parallel G_2 \xrightarrow{w'}_d Z$ ,  $w' \zeta = yld(\gamma')$ . Put  $\gamma: G_1 \parallel G_2 \xrightarrow{\alpha}_d \overline{G}_1 \parallel G_2 \xrightarrow{w'}_d Z$ . Clearly,  $\gamma \in comp_{d,k}(G_1 \parallel G_2)$  with  $yld(\gamma) = w\zeta$ . " $ind(\overline{G}_1) > k$ " Then we have  $w\zeta = \alpha \perp$ . Put  $\gamma: G_1 \parallel G_2 \xrightarrow{\alpha}_d \overline{G}_1 \parallel G_2$ . So  $\gamma \in comp_{d,k}(G_1 \parallel G_2)$  with  $yld(\gamma) = \alpha \perp = w\zeta$ .  $\square$

In order to deal with procedure calls in the proof of the main theorem 5.1 we need a little lemma concerning body replacement, which is a direct consequence of the definition of (Proc) for the intermediate transition systems.

**5.7. LEMMA.** *Let  $d \in Decl$ . Suppose  $x \Leftarrow s \in d$ . Then  $yld_{d,k+1}(x, 0) = \tau.yld_{d,k}(s, 0)$ .*

PROOF. Since  $(x, 0) \xrightarrow{\tau}_d (s, 1)$  is the only  $d$ -transition for  $(x, 0)$  in  $GStat_{k+1}$ , it suffices to show  $\# : yld_{d,k+1}(s, 1) = yld_{d,k}(s, 0)$ .

Define for  $\overline{G} \in GStat_E$  the generalised statement  $\overline{G}+1$  as follows:  $E+1 = E$ ,  $(s, n) + 1 = (s, n+1)$ ,  $(G_1 * G_2) + 1 = (G_1 + 1) * (G_2 + 1)$ . Then  $\#$  follows from  $G_1 \xrightarrow{\alpha}_d \overline{G}_1$  in  $GStat_k \Rightarrow G_1 + 1 \xrightarrow{\alpha}_d \overline{G}_1 + 1$  in  $GStat_{k+1}$ , and  $G_1 \xrightarrow{\alpha}_d \overline{G}_1$  in  $GStat_{k+1} \& G_0 + 1 = G_1 \Rightarrow \exists \overline{G}_0: G_0 \xrightarrow{\alpha}_d \overline{G}_0$  in  $GStat_k \& \overline{G}_0 + 1 = \overline{G}_1$ , which can be proved by induction on the derivation for  $G_i \xrightarrow{\alpha}_d \overline{G}_i$ .  $\square$

Finally we have arrived at a position in which we can prove the operational and denotational semantics for *Prog* equivalent. The proof now is based of the compositionality of the intermediate semantics and on a continuity argument on the level of approximations.

PROOF. (of theorem 5.1) Let  $d \in Decl$ ,  $s \in Stat$ . By lemma 5.4 it suffices to show  $\mathcal{I} = \mathcal{D}$ . By theorem 2.2 and lemma 4.3 we have on the one hand  $\mathcal{I}(d|s) = yld_d(s, 0) = lub_k yld_{d,k}(s, 0)$  and on the other hand  $\mathcal{D}(d|s) = \mathcal{S}(s)(\eta_d) = lub_k \mathcal{S}(s)(\eta_{d,k})$ . Therefore it remains to prove  $\forall s \in Stat \forall k \in \mathbb{N}: yld_{d,k}(s, 0) = \mathcal{S}(s)(\eta_{d,k})$ . We prove this by induction on the pair  $(s, k)$ .

" $(a, k)$ " Consider the configuration  $(a, 0)$ . There is only one maximal  $d$ -computation for  $(a, 0)$  in  $GStat_k$ , viz.  $(a, 0) \xrightarrow{\alpha}_d E$ . So  $yld_{d,k}(a, 0) = \{a\}$ . By definition  $\mathcal{S}(a)(\eta_{d,k}) = \{a\}$ . " $(x, 0)$ " We have  $yld_{d,0}(x, 0) = \{\perp\}$ , for  $ind(x, 0) = 1 > 0$ . On the other hand  $\mathcal{S}(x)(\eta_{d,0}) = \eta_{d,0}(x) = \{\perp\}$ . " $(x, k+1)$ " By induction hypothesis and lemma 5.7 we have  $yld_{d,k+1}(x, 0) = \tau.yld_{d,k}(s, 0) = \tau.\mathcal{S}(s)(\eta_{d,k}) = \Phi_d(\eta_{d,k})(x) = \eta_{d,k+1}(x) = \mathcal{S}(x)(\eta_{d,k+1})$ . " $(s_1 * s_2, k)$ " By lemma 5.6 and the induction hypothesis we have  $yld_{d,k}(s_1 * s_2, 0) = yld_{d,k}(s_1, 0) * yld_{d,k}(s_2, 0) = \mathcal{S}(s_1)(\eta_{d,k}) * \mathcal{S}(s_2)(\eta_{d,k}) = \mathcal{S}(s_1 * s_2)(\eta_{d,k})$ .  $\square$

## 6. Conclusion and Open Problems

In this paper we have seen how we can relate denotational and operational semantics for languages with recursion via an intermediate semantics that keeps track of the recursion depth. We illustrated the method by considering a very simple programming language that includes a basic form of concurrency. We claim that the method presented here is universal in the sense that it can be employed for a wide range of languages with more complicated forms of concurrency, (cf. MEYER and DE VINK [1988, 1989b, 1989a], DE VINK [1988]), although we cannot expect the equality of the operational and denotational models ( $\mathcal{O} = \mathcal{D}$ ) to hold any more. In general when dealing with communication of synchronisation one can not hope for a transition system based operational semantics that is compositional as well. In these cases we should settle for  $\mathcal{O} = \alpha \circ \mathcal{D}$  for some abstraction operator  $\alpha$ . However, in these cases an intermediate semantics  $\mathcal{I}$  might be employed along the lines followed in this paper for which it can be established that  $\mathcal{I} = \mathcal{D}$  and  $\mathcal{O} = \alpha \circ \mathcal{I}$ .

Although the technique for relating operational and denotational semantics presented here can be extended to more realistic programming languages one can argue about the treatment of recursion. In the operational semantics we have the axiom  $x \xrightarrow{\tau}_d s$  for  $x \Leftarrow s$  in  $d$ . Hence we can observe—by means of the label  $\tau$ —the replacement of  $x$  by its body  $s$ . (This amounts to, e.g.,  $\mathcal{D}(d|x) = \tau \mathcal{D}(d|s)$  for  $x \Leftarrow s \in d$ .)

If occurrences of  $\tau$  are considered as undesirable observations one could model the procedure call by using a rule (instead of an axiom), viz.  $x \xrightarrow{\alpha}_d \bar{s}$

if  $s \xrightarrow{\alpha}_d \bar{s}$  for  $x \Leftarrow s$  in  $d$  where  $\alpha$  ranges over the collection of actions  $\mathcal{A}$  excluding  $\tau$ . In order to make this to work one has to impose a guardedness or Greibach condition on the bodies of the declared procedure variables, i.e., each occurrences of a procedure variable is preceded (guarded) by an occurrence of an action. This restriction guarantees that there exists a proper first step for each body  $s$  without spanning of an infinite digression of procedure calls. More formally, we can construct from the axioms and rules of the transition system induced by the declaration  $d$  a proof validating an transition  $s \xrightarrow{\alpha}_d \bar{s}$  with  $\alpha \in \mathcal{A}$ . (The notion of a first step is exploited fruitfully in the metric approach to denotational semantics as proposed by De Bakker et al that uses contracting functions on complete metric spaces with Banach's theorem playing a similar role as the Knaster-Tarski theorem in the cpo setting. Cf. DE BAKKER and ZUCKER [1982], DE BAKKER and KOK [1985], DE BAKKER ET AL [1986], KOK and RUTTEN [1988], DE BAKKER and MEYER [1988], DE BAKKER [1988], AMERICA ET AL [1989] and DE BRUIN and DE VINK [1989].)

Alternatively to the adoption of the guardedness condition is the interpretation of finite strings of  $\tau$ 's as a skip action, i.e.,  $\epsilon$ , and infinite ones as divergence, i.e.,  $\perp$ . This brings us to the problem of so-called unguarded recursion for which—to our knowledge—no satisfactory solution have been given yet. In the context of the present paper this problem can be formulated as follows:

**Problem.** *Does there exist a semantics  $\mathcal{D}'$  for the language Prog which is 548. ?*  
 (i) compositional, i.e.,  $\mathcal{D}'(d|s_1 * s_2) = \mathcal{D}'(d|s_1) *_{\mathcal{D}'} \mathcal{D}'(d|s_2)$  for every syntactic operator  $*$ , handles (ii) recursion by means of fixed point techniques, is (iii) correct with respect to the operational semantics, i.e.,  $\mathcal{D}' = \mathcal{O}$ , and which moreover satisfies (iv)  $\mathcal{D}'(d|x) = \mathcal{D}'(d|s)$  for each  $x \Leftarrow s \in d$ .

(Properties (i) through (iii) are satisfied by the denotational semantics  $\mathcal{D}$ , but property (iv) is not.)

From the operational point of view this interpretation of a procedure call is quit appropriate. A slight extension to the set up of the definition of the yield function of a transition system will take care of this. However, the fixed point characterisation of the yield does not hold any more. For it can very well be the case under this interpretation that operationally a program will have a non-flat and non-closed meaning. (E.g. intuitively in this setting one would like to have  $ba^* \cup \{\perp\}$  as the operational meaning of the program  $x \Leftarrow x; a + b|x$  in contrast to the treatment of section 3 where we would have  $\mathcal{O}(x \Leftarrow x; a + b|x) = \{\tau^n ba^n | n \in \mathbb{N}\} \cup \{\tau^\omega\}$ . So  $\mathcal{O}(x \Leftarrow x; a + b|x)$  is non-flat and does not contain the limit point  $a^\omega$ .) When designing a denotational semantics this should be considered a problem since we are forced to leave the realm of cpos including the Knaster-Tarski theorem. (A similar argument applies to the metric approach.) Therefore other mathematical structures, in

which fixed point equations can be solved, will have to be recognised as denotational domains before an appropriate answer to the question for a general approach for constructing and relating operational and denotational semantics for unguarded recursion can be given.

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## PART VI

### ALGEBRAIC AND GEOMETRIC TOPOLOGY

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## CHAPTER 25

### Problems on Topological Classification of Incomplete Metric Spaces

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## 1. Introduction

The aim of this note is to present the current state of affairs concerning the topological classification of incomplete metric spaces including linear spaces, convex sets, groups and special finite-dimensional open problems. The following are major goals in the theory of incomplete spaces:

- (I) To give a topological classification of metric linear spaces, their convex subsets, and of metric groups.
- (II) To recognize metric linear spaces, their convex subsets, and metric groups, respectively, that are homeomorphic to pre-Hilbert spaces, their convex subsets and additive subgroups, respectively.
- (III) To classify universal finite-dimensional spaces that occur naturally in classical topology.

The goals (I) and (II) go back to the program of FRÉCHET [1928] and BANACH [1932] to recognize normed linear spaces that are homeomorphic to pre-Hilbert spaces. Although this program has been realized quite satisfactory for complete spaces, the incomplete case is still in its initial stage. Goal (III) concerns for example the classical NÖBELING [1931] spaces and their complements in the appropriate euclidean spaces, and also GEOGHEGAN-SUMMERHILL [1974] pseudointeriors and pseudoboundaries. This goal is of interest for incomplete and complete spaces as well. The recent results of BESTVINA [1988] and DIJKSTRA ET AL [19∞] have created new hope that methods of infinite-dimensional topology can be used in the finite-dimensional case. There has been much recent interest in characterizing the Nöbeling spaces which are finite-dimensional analogues of  $\mathbb{R}^\infty$ , the countable product of lines (cf. DRANIŠNIKOV [1986], CHIGOGIDZE [19∞] and CHIGOGIDZE and VALOV [19∞]).

Almost all known results concerning (I), (II) and (III) were obtained by using variations of the method of *absorbing sets*. For this reason we start this article with a sketch of the theory of absorbing sets. The remaining part of this note consists of current open problems. It is likely that our list of problems is not complete and that some of the problems are inadequately or clumsy worded, are easy to answer or are already known. Except for the final remarks, questions about nonseparable spaces and manifolds modeled on incomplete spaces are not included in this text.

## 2. Absorbing sets: A Survey of Results

We start this section by recalling some necessary notions. A closed subset  $A$  of a metric space  $X$  is a *Z-set* (resp. a *strong Z-set*) if given an open cover  $\mathcal{U}$  of  $X$  there exists a  $\mathcal{U}$ -close to the identity map  $f: X \rightarrow X$  such that  $f(X)$  (resp. the closure of  $f(X)$ ) misses  $A$ . We recall the notion of *strong universality*. Let  $\mathcal{C}$  be a class of separable metric spaces which is

- (a) *topological* (i.e., for every  $C \in \mathcal{C}$  and every homeomorphism  $h: C \rightarrow D$  it follows that  $D \in \mathcal{C}$ );
- (b) *hereditary with respect to closed subsets* (i.e., every closed subset of any  $C \in \mathcal{C}$  belongs to  $\mathcal{C}$ );
- (c) *additive* (i.e., if  $C = C_1 \cup C_2$ , where  $C_1, C_2$  are elements of  $\mathcal{C}$  which are closed in  $C$ , then  $C \in \mathcal{C}$ );

A space  $X$  is *strongly  $\mathcal{C}$ -universal* if, for every map  $f: C \rightarrow X$  from a space  $C \in \mathcal{C}$  into  $X$ , for every closed subset  $D \subseteq C$  such that  $f|_D: D \rightarrow X$  is a  $Z$ -embedding (i.e.,  $f$  is an embedding and  $f(D)$  is a  $Z$ -set in  $X$ ) and for every open cover  $\mathcal{U}$  of  $X$ , there exists a  $Z$ -embedding  $h: C \rightarrow X$  such that  $h|_D = f|_D$  and  $h$  is  $\mathcal{U}$ -close to  $f$ . We say that  $X$  is  *$\mathcal{C}$ -universal* if  $\mathcal{C} \subseteq \mathcal{F}_0(X)$ , where  $\mathcal{F}_0(X)$  is the class of all spaces that are homeomorphic to a closed subset of  $X$ .

Let  $\mathcal{C}$  be a class. Then  $\mathcal{C}_\sigma$  is the class of separable metric spaces  $C$  such that  $C = \bigcup_{i=1}^\infty C_i$ , where  $C_i$  is closed in  $C$  and  $C_i \in \mathcal{C}$  for  $i = 1, 2, \dots$

Let us point out that if  $\mathcal{C}$  is a class which is topological and hereditary with respect to closed subsets, then every absolute retract  $X$  which is a countable union of strong  $Z$ -sets and is strongly  $\mathcal{C}$ -universal is also strongly universal for the class consisting of all spaces  $C$  of the form  $C = C_1 \cup C_2$ , where  $C_1, C_2$  are elements of  $\mathcal{C}$  which are closed in  $C$ .

Let us mention that the strong universality property is a local version of the universality properties which characterize the Hilbert cube  $I^\infty$  (TORUŃCZYK [1980]) and the Hilbert space  $\ell_2$  (TORUŃCZYK [1981]). Namely, an absolute retract  $X$  such that every  $Z$ -set in  $X$  is a strong  $Z$ -set is strongly  $\mathcal{C}$ -universal iff for every open subset  $U$  of  $X$  and every open subset  $V$  of  $C$ , where  $C \in \mathcal{C}$ , each map  $f: V \rightarrow U$  can be arbitrarily closely approximated by  $Z$ -embeddings into  $U$ .

Now we propose our version of the notion of absorbing set which seems to be both useful and general. We say that a space  $X$  is *infinite-dimensional* if  $X$  is not a finite-dimensional space and we also use the following notation:  $X \in AE(n)$  if  $X$  is an absolute extensor for all at most  $n$ -dimensional spaces; and  $X \in AE(\infty)$  if  $X$  is an absolute extensor for all metric spaces (equivalently,  $X$  is an absolute retract).

Fix  $n \in \mathbb{N} \cup \{\infty\}$ . An  $n$ -dimensional separable metric space  $X$  is a  *$\mathcal{C}$ -absorbing set* if:

- (abs<sub>1</sub>)  $X \in AE(n)$ ;
- (abs<sub>2</sub>)  $X$  is a countable union of strong  $Z$ -sets;
- (abs<sub>3</sub>)  $X \in \mathcal{C}_\sigma$ ;
- (abs<sub>4</sub>)  $X$  is strongly  $\mathcal{C}$ -universal.

Let  $k, n \in \mathbb{N} \cup \{\infty\}$ . We say that an  $n$ -dimensional  $\mathcal{C}$ -absorbing set  $X$  is *representable* in a  $k$ -dimensional space  $M \in AE(n)$  if  $X$  is homeomorphic to a subset  $X_0$  in  $M$  such that  $M \setminus X_0$  is *locally  $m$ -homotopy negligible* in  $M$  (i.e., for every open set  $U \subseteq M$  the inclusion  $U \cap X_0 \hookrightarrow U$  induces an isomorphism of  $i$ -dimensional homotopy groups for  $i - 1 < m$ , where  $m = \infty$  if  $k = \infty$  or  $m = n$  otherwise). The set  $X_0$  is called a *representation* of  $X$  in  $M$ . If  $X$  is itself a representation in  $M$  then we say that  $X$  is an absorbing set in  $M$ . Usually we will represent absorbing sets in either  $\mathbb{R}^k$ , where  $k \in \mathbb{N} \cup \{\infty\}$ , or in the Hilbert cube  $I^\infty$ .

The notion of  $\mathcal{C}$ -absorbing set in  $\mathbb{R}^\infty$  is taken from BESTVINA and MOGILSKI [1986] and generalizes concepts of ANDERSON [19 $\infty$ ], BESSAGA and PEŁCZYŃSKI [1970], TORUŃCZYK [1970b, 1970a] and WEST [1970].

We now quote some fundamental facts concerning absorbing sets (see BESTVINA and MOGILSKI [1986]).

- (A) **Uniqueness Theorem:** *Let  $X$  and  $Y$  be  $\mathcal{C}$ -absorbing sets in  $\mathbb{R}^\infty$ . Then for every open cover  $\mathcal{U}$  of  $\mathbb{R}^\infty$  there exists a homeomorphism  $h: X \rightarrow Y$  that is  $\mathcal{U}$ -close to the inclusion  $X \hookrightarrow \mathbb{R}^\infty$ . If, in addition,  $X$  and  $Y$  are countable union of  $Z$ -sets in  $\mathbb{R}^\infty$ , then we can achieve that the homeomorphism  $h$  extends to the whole space.*
- (B) **Characterization Theorem:** *Let  $X$  be an infinite-dimensional  $\mathcal{C}$ -absorbing set in  $\mathbb{R}^\infty$ . Then every  $\mathcal{C}$ -absorbing set  $Y$  is homeomorphic to  $X$ .*
- (C) **Z-Set Unknotting Theorem:** *Let  $X$  be a  $\mathcal{C}$ -absorbing set in  $\mathbb{R}^\infty$ . Let  $\mathcal{U}$  be an open cover of  $\mathbb{R}^\infty$  and suppose that  $h: A \rightarrow B$  is a homeomorphism between  $Z$ -sets  $A$  and  $B$  in  $X$ . If  $h$  is  $\mathcal{U}$ -homotopic to the inclusion  $A \hookrightarrow X$  and if  $\mathcal{B}$  is an open cover of  $X$ , then there is a homeomorphism  $H: X \rightarrow X$  such that  $H|_A = h$  and  $H$  is  $st(\mathcal{U}, \mathcal{B})$ -close to the identity.*
- (D) **Factor Theorem:** *Let a class  $\mathcal{C}$  have the property that  $C_1, C_2 \in \mathcal{C}$  implies  $C_1 \times C_2 \in \mathcal{C}$  and let  $Y$  be an infinite-dimensional  $\mathcal{C}$ -absorbing set in  $\mathbb{R}^\infty$ . Then  $X \times Y$  is homeomorphic to  $Y$  for every retract  $X$  of  $Y$ .*
- (E) **Countable Union Theorem:** *Every  $\mathcal{C}$ -absorbing set is also a  $\mathcal{C}_\sigma$ -absorbing set.*

Here is a list of known absorbing sets that are representable in  $\mathbb{R}^\infty$ .

- (1) Every countable dense subset of  $\mathbb{R}^\infty$  is an absorbing set for the class of all finite sets (see BESSAGA and PEŁCZYŃSKI [1975]).

- (2) The space  $\sigma = \mathbb{R}_f^\infty = \{(x_i) \in \mathbb{R}^\infty : x_i = 0 \text{ for all but finitely many } i\}$  is an  $\mathcal{R}_\omega$ -absorbing set, where  $\mathcal{R}_\omega$  is the class of all finite-dimensional compacta (ANDERSON [19 $\infty$ ], BESSAGA and PEŁCZYŃSKI [1970]) (cf. also BESSAGA and PEŁCZYŃSKI [1975] and MOGILSKI [1984]).
- (3) There are  $\mathcal{R}_n$ -absorbing sets  $\sigma_n$  in  $\mathbb{R}^\infty$  and  $\sigma_n^k$  in  $\mathbb{R}^k$ , where  $\mathcal{R}_n$  is the class of all at most  $n$ -dimensional compacta and  $k \geq 2n + 1$  (DIJKSTRA [1985] and GEOGHEGAN and SUMMERHILL [1974]).
- (4) There are  $\mathcal{R}_{nm}$ -absorbing sets  $\sigma_{nm}$  in  $\mathbb{R}^\infty$  and  $\sigma_{nm}^k$  in  $\mathbb{R}^k$ , where  $\mathcal{R}_{nm}$  is the class of all at most  $n$ -dimensional compacta embeddable in  $\mathbb{R}^m$ , where  $n = 0, 1, \dots$  and  $n \leq m \leq 2n + 1 \leq k$  (DIJKSTRA and MOGILSKI [19 $\infty$ ]).
- (5) The space  $\Sigma = \mathbb{R}_{bd}^\infty$  consisting of all bounded sequences of  $\mathbb{R}^\infty$  is an  $\mathcal{R}$ -absorbing set, where  $\mathcal{R}$  is the class of all compacta (ANDERSON [19 $\infty$ ], BESSAGA and PEŁCZYŃSKI [1970]) (cf. also BESSAGA and PEŁCZYŃSKI [1975] and MOGILSKI [1984]).
- (6) For every countable ordinal  $\beta$  there exist an ordinal  $\alpha \geq \beta$  and an  $\mathcal{R}_\alpha$ -absorbing set  $\sigma_\alpha$  in  $\ell_2$ , where  $\mathcal{R}_\alpha$  is the class of all compacta with transfinite dimension less than  $\alpha$  (DOBROWOLSKI and MOGILSKI [19 $\infty$ a]).
- (7) The space  $\sigma \times \mathbb{R}^\infty$  is a  $\mathcal{G}_\delta$ -absorbing set, where  $\mathcal{G}_\delta$  is the class of all topologically complete separable spaces (see BESSAGA and PEŁCZYŃSKI [1975]; see also BESTVINA and MOGILSKI [1986]).
- (8) For every countable ordinal  $\alpha \geq 1$  there exists a  $\mathcal{U}_\alpha$ -absorbing set  $\Lambda_\alpha$  in  $\mathbb{R}^\infty$ , where  $\mathcal{U}_\alpha$  is the class of all absolute Borel sets of the additive class  $\alpha$  (BESTVINA and MOGILSKI [1986]).
- (9) For every countable ordinal  $\alpha \geq 2$  there exists an  $\mathcal{M}_\alpha$ -absorbing set  $\Omega_\alpha$  in  $\mathbb{R}^\infty$ , where  $\mathcal{M}_\alpha$  is the class of all absolute Borel sets of the multiplicative class  $\alpha$  (BESTVINA and MOGILSKI [1986]).
- (10) Every infinite-dimensional,  $\sigma$ -compact locally convex metric linear space  $E$  is a  $\mathcal{C}(E)$ -absorbing set, where  $\mathcal{C}(E)$  is the class of all compacta embeddable in  $E$  (DOBROWOLSKI [1989]).
- (11) If  $X$  is a separable absolute retract and  $* \in X$ , then

$$X_f^\infty = \{(x_i) \in X^\infty : x_i = * \text{ for all but finitely many } i\}$$

is a  $\bigcup_{n=1}^\infty \mathcal{F}_0(X^n)$ -absorbing set (BESTVINA and MOGILSKI [1986]).

- (12) If  $X$  is a separable absolute retract which is a countable union of  $Z$ -sets, then  $X^\infty$  is an  $\mathcal{F}_0(X^\infty)$ -absorbing set (BESTVINA and MOGILSKI [1986]).

The method of absorbing sets mentioned in the introduction applies (A) and (B) to reduce the problem of homeomorphy of the spaces under consideration

to the problem of recognizing whether they are absorbing sets for the same class. The most general results in this way are included in the next theorem.

**Classification Theorem:**

- (i) Every infinite-dimensional metric linear space  $E$  such that  $E \in (\mathcal{R}_\omega)_\sigma$  is homeomorphic to  $\sigma$  (CURTIS ET AL [1984]; see also ANDERSON [19 $\infty$ ], BESSAGA and PEŁCZYŃSKI [1970, 1975], TORUŃCZYK [1970b] and WEST [1970]).
- (ii) Let  $X$  and  $Y$  be locally convex metric linear spaces such that  $X = \bigcup_{i=1}^\infty X_i$  and  $Y = \bigcup_{i=1}^\infty Y_i$ , where all  $X_i$  and  $Y_i$  are compacta. Then  $X$  and  $Y$  are homeomorphic iff  $X_i \in \mathcal{F}_0(Y)$  and  $Y_i \in \mathcal{F}_0(X)$  for all  $i$  (DOBROWOLSKI [1989]). In particular, a  $\sigma$ -compact locally convex metric linear space  $E$  is homeomorphic to  $\Sigma$  (resp. to  $\sigma_\alpha$ ) iff  $E$  is  $\mathcal{R}$ -universal (resp.  $E \in (\mathcal{R}_\alpha)_\sigma$  and  $E$  is  $\mathcal{R}_\alpha$ -universal) (DOBROWOLSKI and MOGILSKI [1982]) (resp. DOBROWOLSKI and MOGILSKI [19 $\infty$ a]).
- (iii) If  $X$  is a nondiscrete countable metric space, then  $C_p(X)$  is homeomorphic to  $\Omega_2$ , where  $C_p(X)$  is the space of all continuous functions on  $X$  endowed with the point-wise convergence topology (DOBROWOLSKI ET AL [1990] and CAUTY [19 $\infty$ ]) (see also VAN MILL [1987b], BAARS ET AL [1986] and BAARS ET AL [1989]).
- (iv) The spaces  $\sigma_n$  and  $\sigma_n^k$  are homeomorphic for  $n = 0, 1, \dots$  and  $k \geq 2n+1$  (DIJKSTRA ET AL [19 $\infty$ ]).
- (v) The spaces  $\sigma_{nm}$  and  $\sigma_{nm}^k$  are homeomorphic for  $n = 0, 1, \dots$  and  $k \geq 2n+1 \geq m \geq n$  (DIJKSTRA and MOGILSKI [19 $\infty$ ]).
- (vi) If  $k, l \geq 2n+1$  then the Geoghegan-Summerhill pseudointeriors  $s_n^k = \mathbb{R}^k \setminus \sigma_n^k$  and  $s_n^l = \mathbb{R}^l \setminus \sigma_n^l$  are homeomorphic to  $s_n^k$  (DIJKSTRA ET AL [19 $\infty$ ]).
- (vii) If  $X$  is a countable nondiscrete completely regular space such that  $C_p(X)$  is an absolute  $F_{\sigma\delta}$ -set, then  $C_p(X)$  is homeomorphic to  $\Omega_2$  (DOBROWOLSKI ET AL [19 $\infty$ ]; see also DIJKSTRA ET AL [1990]).

### 3. General Problems about Absorbing Sets

In our opinion the research effort in the theory of absorbing sets should concentrate either on enlarging the list of absorbing sets, improving and generalizing the basic theorems (A) – (E) or recognizing absorbing sets listed in §2 among objects naturally appearing in functional analysis or classical topology. In this section we pose problems concerning the abstract theory of absorbing sets.

**Question 3.1.** Find more absorbing sets.

549. ?

In particular:



- ? 550. **Question 3.2.** Find countable ordinals  $\alpha$  for which there are  $\mathcal{R}_\alpha$ -absorbing sets.
- ? 551. **Question 3.3.** Let  $\mathcal{U}_\alpha(n)$  (resp.  $\mathcal{M}_\alpha(n)$ ) denote the class of all at most  $n$ -dimensional absolute Borel sets of the additive (resp. multiplicative) class  $\alpha$ . Are there absorbing sets for the classes  $\mathcal{U}_\alpha(n)$  or  $\mathcal{M}_\alpha(n)$ , respectively?
- ? 552. **Question 3.4.** Let  $\mathcal{U}_\alpha(\omega)$  (resp.  $\mathcal{M}_\alpha(\omega)$ ) denote the class of all finite-dimensional absolute Borel sets of the additive (resp. multiplicative) class  $\alpha$ . Are there absorbing sets for the class  $\mathcal{U}_\alpha(\omega)$  or  $\mathcal{M}_\alpha(\omega)$ , respectively?
- ? 553. **Question 3.5.** Let  $X$  be a separable absolute retract which is a countable union of strong  $Z$ -sets. Under what conditions is  $X$  an  $\mathcal{F}_0(X)$ -absorbing set?

The next four questions are related to the Characterization Theorem (B).

- ? 554. **Question 3.6.** Is every  $\mathcal{C}$ -absorbing set representable in  $\mathbb{R}^\infty$ ?
- ? 555. **Question 3.7.** Is every finite-dimensional  $\mathcal{C}$ -absorbing set representable in a finite-dimensional Euclidean space?
- ? 556. **Question 3.8.** Let  $X$  be a finite-dimensional  $\mathcal{C}$ -absorbing set in  $\mathbb{R}^k$ . Is every  $\mathcal{C}$ -absorbing set  $Y$  homeomorphic to  $X$ ?

The answer is probably positive if  $Y$  is representable in a finite-dimensional euclidean space.

- ? 557. **Question 3.9.** Find a stronger version of the  $Z$ -set Unknotting Theorem (C). More precisely, is it possible to weaken the homotopy hypothesis in (C) such that it can be applied in the process of solving 3.7?
- ? 558. **Question 3.10.** Let  $X$  and  $Y$  be  $\mathcal{C}$ -absorbing sets in an absolute retract  $M$ . Under what conditions does there exist an arbitrarily close to the identity homeomorphism of  $M$  sending  $X$  onto  $Y$  (especially interesting for  $M = \mathbb{R}^\infty$  or  $I^\infty$ )?

#### 4. Problems about $\lambda$ -convex Absorbing Sets

This section is devoted to the question of recognizing absorbing sets among spaces equipped with an algebraic structure including metric linear spaces, their convex subsets and also contractible metric groups. All such spaces  $X$  have an *equiconnecting structure*, i.e., there exists a continuous map  $\lambda: X \times X \times I \rightarrow X$  satisfying  $\lambda(x, y, 0) = x$ ,  $\lambda(x, y, 1) = y$  and  $\lambda(x, x, t) = x$  for all

$x, y \in X$  and  $t \in I$ . In general, let  $X$  be a contractible metric group with a contracting homotopy  $\phi$  of  $X$  to its unit element  $e$ ; then the map

$$\lambda(x, y, t) = [\phi_t(e)]^{-1} \phi_t(x y^{-1}) y \quad (x, y \in X)$$

defines an equiconnecting structure on  $X$ . Every subset  $Y$  of a group  $X$  satisfying  $\lambda(Y \times Y \times I) = Y$  is said to be  $\lambda$ -convex. In particular, if  $X$  is a metric linear space then the  $\lambda$ -equiconnected structure of  $X$  defined in the above way for  $\phi_t(x) = tx$  is just the convex structure of  $X$  and each  $\lambda$ -convex subset  $Y$  of  $X$  is a convex subset of  $X$ .

We start with the following general question.

**Question 4.1.** *When is a separable incomplete  $\lambda$ -convex set  $Y$  an  $\mathcal{F}_0(Y)$ -absorbing set?* **559. ?**

At the very beginning we face the most outstanding question in infinite-dimensional topology:

**Question 4.2.** *Is every  $\lambda$ -convex set an absolute retract? (It is even unknown whether every metric linear space is an absolute retract.)* **560. ?**

Let us mention here that the problem of the topological classification of all closed convex subsets of separable complete metric linear spaces and of separable metric groups has been reduced to the problem of recognizing the absolute retract property (DOBROWOLSKI and TORUŃCZYK [1979], [1981]). The last problem seems to be very difficult: in general it has been solved positively only for convex subsets of locally convex linear spaces (see BESSAGA and PEŁCZYŃSKI [1975]) and contractible groups which are countable unions of finite-dimensional subsets (HAVER [1973]). The topological classification of incomplete metric linear spaces creates a lot of new difficulties; therefore we will concentrate on the problem of establishing the properties (abs<sub>1</sub>) – (abs<sub>4</sub>) for  $\lambda$ -convex sets which are absolute retracts.

The next three questions concern the condition (abs<sub>2</sub>). Obviously this condition implies that the space in question is of first category.

**Question 4.3.** *Let  $Y$  be an absolute retract  $\lambda$ -convex set. Is  $Y$  a countable union of  $Z$ -sets provided it is of first category?* **561. ?**

The answer to this question even for metric linear spaces would be very interesting.

It is known that Borelian incomplete metric groups are of first category (BANACH [1931]). Thus we ask:

**Question 4.4.** *Let  $E$  be an incomplete pre-Hilbert space which is an absolute* **562. ?**

*Borel set. Is  $E$  a countable union of  $Z$ -sets?*

- ? **563. Question 4.5.** *Let  $Y$  be an absolute retract  $\lambda$ -convex set. Is every  $Z$ -set in  $Y$  a strong  $Z$ -set?*

Let us mention that each  $Z$ -set is a strong  $Z$ -set in a separable absolute retract  $X$ , where  $X$  is either a metric group or  $X$  is a convex subset of a complete metric linear space such that its closure  $\overline{X}$  is either an absolute retract or is nonlocally compact (this follows from BESTVINA and MOGILSKI [1986, Proposition 1.7] and the results of DOBROWOLSKI and TORUŃCZYK [1981]). Question 4.5 is interesting for compact convex subsets of nonlocally convex linear spaces (see ROBERTS [1976, 1977]).

- ? **564. Question 4.6.** *Let  $Y$  be an infinite-dimensional  $\lambda$ -convex set which is an absolute retract. Does the  $\mathcal{C}$ -universality property imply the strong  $\mathcal{C}$ -universality property of  $Y$ ?*

It is even unknown whether  $\mathcal{F}_0(Y)$  is additive, where  $Y$  is a pre-Hilbert space which is an absolute Borel set.

- ? **565. Question 4.7.** *Let  $Y$  be an infinite-dimensional  $\lambda$ -convex set which is an absolute retract. Does every homeomorphism between  $Z$ -sets of  $Y$  extend to a homeomorphism of the whole space  $Y$ ?*

A general answer is unknown, even for Borelian pre-Hilbert spaces.

- ? **566. Question 4.8.** *Does every homeomorphism between compacta of a nonlocally convex metric linear space  $E$  extend to a homeomorphism of  $E$ ?*

By theorem (C'), each infinite-dimensional absorbing set in  $\mathbb{R}^\infty$  is a homogeneous absolute retract and all infinite-dimensional absorbing sets listed in §2 have representations in  $\mathbb{R}^\infty$  as linear subspaces. Therefore it is interesting whether absorbing sets admit algebraic structures and how nice these structures could be.

- ? **567. Question 4.9.** *Find an infinite-dimensional absorbing set in  $\mathbb{R}^\infty$  which does not admit a group structure.*

- ? **568. Question 4.10.** *Let  $Y$  be a  $\lambda$ -convex absorbing set. Can  $Y$  be represented as a convex subset of a metric linear space?*

- ? **569. Question 4.11.** *Let  $Y$  be an absorbing set which is a metric linear space (resp. a convex subset of a metric linear space). Can  $Y$  be represented as a linear subspace of  $\mathbb{R}^\infty$  or  $\ell_2$  (resp. a convex subset of  $\mathbb{R}^\infty$  or  $\ell_2$ )?*

**Question 4.12.** *Let  $Y$  be an absorbing set which is represented as a linear subspace of  $\mathbb{R}^\infty$ . Can  $Y$  be represented as a linear subspace of  $\ell_2$ ?* **570. ?**

Of the absorbing sets of (2), (5)–(9), (11) and (12) in §2 we know that they can be represented in  $\ell_2$  as linear subspaces. It is not clear whether every absorbing set of (10) in §2 also admits such a representation.

## 5. Problems about $\sigma$ -Compact Spaces

We start with questions concerning possible generalizations of the statements (i) and (ii) of the Classification Theorem (F) to nonlocally convex spaces.

**Question 5.1.** *Let  $W \in (\mathcal{R}_\omega)_\sigma$  be an infinite-dimensional convex subset of a complete metric linear space. Is  $W$  homeomorphic to  $\sigma$ ?* **571. ?**

**Question 5.2.** *Let  $E$  be an  $\mathcal{R}$ -universal  $\sigma$ -compact metric linear space that is an absolute retract. Is  $E$  homeomorphic to  $\Sigma$ ?* **572. ?**

The answer is positive if  $E$  contains an infinite-dimensional compact convex set (CURTIS ET AL [1984]).

**Question 5.3.** *Let  $E$  be a  $\sigma$ -compact metric linear space that is an absolute retract. Is  $E$  a  $\mathcal{C}(E)$ -absorbing set?* **573. ?**

Is it enough to check the strong  $\mathcal{C}(E)$ -universality property of  $E$  (see DOBROWOLSKI [1986b]).

To answer 5.1 in the affirmative, it is enough to show that  $W$  is a countable union of strong  $Z$ -sets (CURTIS ET AL [1984]). In general, it is known that  $W$  is a countable union of  $Z$ -sets (DOBROWOLSKI [1986a]). It can be checked that every  $Z$ -set in  $W$  is a strong  $Z$ -set if, additionally, the closure  $\overline{W}$  of  $W$  is either an absolute retract or nonlocally compact (cf. DOBROWOLSKI [1986a]). Here are the two most intriguing special cases:

**Question 5.4.** *Let  $W = \text{conv}\{x_i\}_{i=1}^\infty$  be the convex hull of countably many vectors  $x_i$  of a nonlocally convex metric linear space, so that  $W$  is infinite-dimensional. Is each  $\text{conv}\{x_i\}_{i=1}^n$  a strong  $Z$ -set in  $W$ ?* **574. ?**

**Question 5.5.** *Let  $W \in (\mathcal{R}_\omega)_\sigma$  be a dense convex subset of Roberts' compact convex set in ROBERTS [1976, 1977]. Is every  $Z$ -set in  $W$  a strong  $Z$ -set?* **575. ?**

Let us note that there are examples of (everywhere nonlocally compact)  $\sigma$ -compact convex sets  $W$  in  $\ell_2$  such that not all compacta are  $Z$ -sets in  $W$  (CURTIS ET AL [1984]). Thus a suitable analogue of 5.2 for convex sets should be:

- ? 576. **Question 5.6.** *Let  $W$  be an  $\mathcal{R}$ -universal  $\sigma$ -compact absolute retract convex subset of a complete metric linear space such that all compacta are  $Z$ -sets in  $W$ . Is  $W$  homeomorphic to  $\Sigma$ ?*

An answer, except for the case where  $\overline{W}$  is nonlocally compact (see CURTIS ET AL [1984]), is unknown even for  $W$  contained in  $\ell_2$ .

We now ask questions concerning a generalization of the Classification Theorem (F) to metric groups.

- ? 577. **Question 5.7.** *Let  $G \in (\mathcal{R}_\omega)_\sigma$  be an infinite-dimensional contractible metric group. Is  $G$  homeomorphic to  $\sigma$ ? (Equivalently, is  $G$  strongly  $\mathcal{R}_\omega$ -universal (DOBROWOLSKI [1986b])?)*

In particular we ask:

- ? 578. **Question 5.8.** *Is a group  $G$  such as in 5.7  $\mathcal{R}_\omega$ -universal? It is even unknown whether  $G$  contains a disk.*

An interesting special case of 5.7 is:

- ? 579. **Question 5.9.** *Let  $G$  be the (additive) group generated by a linearly independent arc in  $\ell_2$ . Assume moreover that  $G$  is contractible. Is  $G$  homeomorphic to  $\sigma$ ?*

- ? 580. **Question 5.10.** *Let  $H$  be a nonlocally compact separable complete metric group that is an absolute retract. Is there a subgroup  $G$  of  $H$  which is a  $\mathcal{R}_\omega$ -absorbing set in  $H$ ?*

It is known that  $H$  is homeomorphic to  $\mathbb{R}^\infty$  (DOBROWOLSKI and TORUŃCZYK [1981]). Moreover,  $H$  contains a subgroup which is an  $\mathcal{R}$ -absorbing set in  $H$ .

- ? 581. **Question 5.11.** *Let  $G$  be an  $\mathcal{R}$ -universal  $\sigma$ -compact absolute retract metric group. Is  $G$  homeomorphic to  $\Sigma$ ?*

The following are particular cases of 5.11:

- ? 582. **Question 5.12.** *Let  $G$  be the (additive) group generated by a linearly independent copy of the Hilbert cube in  $\ell_2$ . Assume moreover that  $G$  is contractible. Is  $G$  homeomorphic to  $\Sigma$ ?*

- ? 583. **Question 5.13.** *Let  $LIP_\partial(I^n)$  be the group of Lipschitz homeomorphisms of the  $n$ -dimensional cube  $I^n$  that fix the boundary. Is  $LIP_\partial(I^n)$  homeomorphic to  $\Sigma$ ?*

It is known that there are countably many different topological types of  $\sigma$ -compact pre-Hilbert spaces (see BESSAGA and PEŁCZYŃSKI [1975]). Therefore it is reasonable to ask:

**Question 5.14.** *Are there continuum many topologically different  $\sigma$ -compact pre-Hilbert spaces?* **584. ?**

**Question 5.15.** *For a  $\sigma$ -compact space  $X$ , let  $\gamma(X)$  be the infimum of ordinals  $\alpha$  such that  $X$  is the union of a countable family of subcompacta all having transfinite dimension less than  $\alpha$ . Does the equality  $\gamma(E) = \gamma(F)$ , where  $E$  and  $F$  are pre-Hilbert spaces, imply that  $E$  and  $F$  are homeomorphic?* **585. ?**

Recall that the well-known “product” questions whether for every infinite-dimensional pre-Hilbert space  $E$ , the products  $E \times \mathbb{R}$  and  $E \times E$  are homeomorphic to  $E$ , were solved in the negative by VAN MILL [1987a] and POL [1984]. Since the counterexamples are far from being  $\sigma$ -compact, let us ask:

**Question 5.16.** *Let  $E$  be a  $\sigma$ -compact pre-Hilbert space. Is it true that  $\{C \times I : C \in \mathcal{C}(E)\} \subseteq \mathcal{C}(E)$  or, respectively,  $\{C \times D : C, D \in \mathcal{C}(E)\} \subseteq \mathcal{C}(E)$ ?* **586. ?**

If the answer is positive then  $E \times \mathbb{R}$  or, respectively,  $E \times E$  are homeomorphic to  $E$  (DOBROWOLSKI [1989]).

A very interesting special case of 4.11 is:

**Question 5.17.** *Is every  $\sigma$ -compact linear subspace  $E$  of  $\mathbb{R}^\infty$  homeomorphic to a pre-Hilbert space  $V$ ?* **587. ?**

To get an affirmative answer, it is enough to find a one-to-one map of  $E$  onto  $V$  (DOBROWOLSKI [1989]).

The  $\sigma$ -compact absorbing sets described in (2) and (6) are countable-dimensional. It suggests the following question.

**Question 5.18.** *Find interesting (different from  $\Sigma$  and from that of DOBROWOLSKI and MOGILSKI [1989a, Ex. 4.4])  $\sigma$ -compact absorbing sets which are not countable-dimensional.* **588. ?**

The last question is a specification of 4.7.

**Question 5.19.** *Does every homeomorphism between finite-dimensional compacta of a nonlocally convex  $\sigma$ -compact metric linear space  $E$  extend to a homeomorphism of the whole space  $E$ ?* **589. ?**

## 6. Problems about Absolute Borel Sets

In the previous section we have discussed  $\sigma$ -compact absorbing sets. In the Borel hierarchy they represent the first additive class  $\mathcal{U}_1$ . The condition  $(\text{abs}_2)$  implies that there are no  $\mathcal{M}_1$ -absorbing sets; however there are absorbing sets in all higher classes (see (8) and (9) in §2). Let  $K$  be a linearly independent compactum in  $\ell_2$ . Then for every  $A \subseteq K$  with  $A \in \mathcal{U}_\alpha$  or  $A \in \mathcal{U}_\alpha \setminus \mathcal{M}_\alpha$  for  $\alpha \geq 1$  (respectively,  $A \in \mathcal{M}_\alpha$  or  $A \in \mathcal{M}_\alpha \setminus \mathcal{U}_\alpha$  for  $\alpha \geq 2$ ),  $\text{span}(A) \in \mathcal{U}_\alpha$  or  $\text{span}(A) \in \mathcal{U}_\alpha \setminus \mathcal{M}_\alpha$  (respectively,  $\text{span}(A) \in \mathcal{M}_\alpha$  or  $\text{span}(A) \in \mathcal{M}_\alpha \setminus \mathcal{U}_\alpha$ ). It shows that all classes of absolute Borel sets are representable among pre-Hilbert spaces (moreover taking the Cantor set as  $K$  we get countable-dimensional pre-Hilbert spaces; see BESSAGA and PEŁCZYŃSKI [1975]. In a similar way we obtain a linear representation of the absorbing sets  $\Lambda_\alpha$  and  $\Omega_\alpha$  in  $\ell_2$ .

? 590. **Question 6.1.** Let  $\Lambda_\alpha$  and  $\Omega_\alpha$  be subsets of a linearly independent compactum in  $\ell_2$ . Are  $\text{span}(\Lambda_\alpha)$  and  $\text{span}(\Omega_\alpha)$  strongly universal for the classes  $\mathcal{U}_\alpha$  and  $\mathcal{M}_\alpha$ , respectively?

? 591. **Question 6.2.** Let  $A \in \mathcal{M}_\alpha \setminus \mathcal{U}_\alpha$ ,  $\alpha \geq 2$ , be a subset of a linearly independent Cantor set in  $\ell_2$ . Is  $(\text{span}(A))^\infty$  homeomorphic to  $\Omega_\alpha$ ? Equivalently, is  $(\text{span}(A))^\infty$  universal for  $\mathcal{M}_\alpha$ ?

More generally, we ask:

? 592. **Question 6.3.** Let  $X \in (\mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}) \cup (\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha)$ ,  $\alpha \geq 1$ , be an absolute retract which is a countable union of  $Z$ -sets. Is  $X^\infty$  homeomorphic to  $\Omega_{\alpha+1}$ ?

The answer is positive for  $\alpha = 1$  (DOBROWOLSKI and MOGILSKI [1990b]).

? 593. **Question 6.4.** Are there uncountably (continuum) many topologically different pre-Hilbert spaces in each of the classes  $\mathcal{M}_\alpha \setminus \mathcal{U}_\alpha$  and  $\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha$ ,  $\alpha \geq 2$ ?

? 594. **Question 6.5.** Let  $\mathcal{C} = \mathcal{M}_\alpha$  or  $\mathcal{U}_\alpha$ ,  $\alpha \geq 2$ , and let  $E$  be a pre-Hilbert space which is  $\mathcal{C}$ -universal. Is  $E$  strongly  $\mathcal{C}$ -universal?

At the beginning of this section we have described a way of obtaining pre-Hilbert spaces of arbitrarily high Borel complexity. Now, we will present a way of constructing Borelian linear subspaces of  $\mathbb{R}^\infty$ . Let  $\mathcal{F}$  be a filter on the set of natural numbers  $\mathbb{N}$  (i.e.,  $\mathcal{F}$  is a nonempty family of subsets of  $\mathbb{N}$  such that  $\emptyset \notin \mathcal{F}$ ,  $S_1 \cap S_2 \in \mathcal{F}$  provided  $S_1, S_2 \in \mathcal{F}$ , and  $T \in \mathcal{F}$  provided  $S \subseteq T$  for some  $S \in \mathcal{F}$ ). Then the space

$$c_{\mathcal{F}} = \{(x_i) \in \mathbb{R}^\infty : \forall \epsilon > 0 \exists S \in \mathcal{F} \forall i \in S |x_i| < \epsilon\}$$

is a linear subspace of  $\mathbb{R}^\infty$ . If  $\mathcal{F}$ , as a subset of the Cantor set  $2^\mathbb{N}$ , is an absolute Borel set of the class  $\mathcal{M}_{\alpha+1}$  or  $\mathcal{U}_\alpha$ , then  $c_{\mathcal{F}} \in \mathcal{M}_{\alpha+1}$ ,  $\alpha \geq 1$ , and also  $c_{\mathcal{F}}$  is a countable union of strong  $Z$ -sets. Moreover, if  $\mathcal{F} \in (\mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}) \cup (\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha)$ ,  $\alpha \geq 1$ , then  $c_{\mathcal{F}} \in \mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}$  (see CALBRIX [1985, 1988], cf. LUTZER ET AL [1985] and DOBROWOLSKI ET AL [19 $\infty$ ]).

**Question 6.6.** *Classify topologically the spaces  $c_{\mathcal{F}}$  for Borelian filters  $\mathcal{F}$ .* **595. ?**

**Question 6.7.** *Let  $\mathcal{F}$  be a Borelian filter. Is  $c_{\mathcal{F}}$  an  $\mathcal{F}_0(c_{\mathcal{F}})$ -absorbing set?* **596. ?**

**Question 6.8.** *Let  $\mathcal{F} \in (\mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}) \cup (\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha)$ ,  $\alpha \geq 1$ , be a filter. Is  $c_{\mathcal{F}}$  homeomorphic to  $\Omega_{\alpha+1}$ ?* **597. ?**

It is even unknown whether  $c_{\mathcal{F}}$  is  $\mathcal{M}_{\alpha+1}$ -universal. The answer is yes for  $\alpha = 1$  (DOBROWOLSKI ET AL [19 $\infty$ ]).

We say that a filter  $\mathcal{F}$  on  $\mathbb{N}$  is *decomposable* if  $\mathcal{F}$  contains all cofinite subsets of  $\mathbb{N}$  and there exists infinite disjoint sets  $\mathbb{N}_1$  and  $\mathbb{N}_2$  such that  $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$  and the family  $\mathcal{F}_i = \{S \cap \mathbb{N}_i : S \in \mathcal{F}\}$  is a filter equivalent to  $\mathcal{F}$ ,  $i = 1, 2$ . It can be shown that for a decomposable filter  $\mathcal{F}$  the space  $c_{\mathcal{F}}$  is strongly  $\mathcal{F}_0(c_{\mathcal{F}})$ -universal (more generally, for such a Borelian filter the spaces  $c_{\mathcal{F}}$  and  $c_{\mathcal{F}}^\infty$  are homeomorphic) (DOBROWOLSKI ET AL [19 $\infty$ ]).

**Question 6.9.** *Let  $\mathcal{F} \in (\mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}) \cup (\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha)$ ,  $\alpha \geq 1$ , be a decomposable filter. Is  $\mathcal{F}_0(c_{\mathcal{F}}) = \mathcal{M}_{\alpha+1}$ ?* **598. ?**

A positive answer to this question yields homeomorphy of  $c_{\mathcal{F}}$  and  $\Omega_{\alpha+1}$ .

**Question 6.10.** *Let  $\mathcal{F}_1, \mathcal{F}_2 \in (\mathcal{M}_{\alpha+1} \setminus \mathcal{U}_{\alpha+1}) \cup (\mathcal{U}_\alpha \setminus \mathcal{M}_\alpha)$ ,  $\alpha \geq 1$ , be decomposable filters. Are the spaces  $c_{\mathcal{F}_1}$  and  $c_{\mathcal{F}_2}$  homeomorphic?* **599. ?**

**Question 6.11.** *Let  $\mathcal{F}_0$  be the filter consisting of all cofinite subsets of  $\mathbb{N}$ . Does there exist a homeomorphism  $h: \mathbb{R}^\infty \rightarrow (\mathbb{R}^\infty)^\infty$  such that  $h(c_{\mathcal{F}_0}) = (\mathbb{R}_f^\infty)^\infty = \Omega_2$ ?* **600. ?**

It is known that  $c_{\mathcal{F}_0}$  and  $\Omega_2$  are homeomorphic (DOBROWOLSKI ET AL [1990] and CAUTY [19 $\infty$ ]).

Some of the above questions are also interesting for filters which are not Borel. Since, in general, the  $c_{\mathcal{F}}$  need not be of the first category (LUTZER ET AL [1985]) we can not expect that  $c_{\mathcal{F}}$  is an absorbing set. Anyway, we ask:

**Question 6.12.** *Let  $\mathcal{F}$  be a filter on  $\mathbb{N}$ . Is  $c_{\mathcal{F}}$  strongly  $\mathcal{F}_0(c_{\mathcal{F}})$ -universal? Is  $c_{\mathcal{F}}$  homeomorphic to  $c_{\mathcal{F}}^\infty$ ?* **601. ?**

**Question 6.13.** *Let  $\mathcal{F}_1, \mathcal{F}_2$  be filters on  $\mathbb{N}$  such that there are maps* **602. ?**



$f_1, f_2: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  with  $f_1^{-1}(\mathcal{F}_1) = \mathcal{F}_2$  and  $f_2^{-1}(\mathcal{F}_2) = \mathcal{F}_1$ . Are the spaces  $c_{\mathcal{F}_1}$  and  $c_{\mathcal{F}_2}$  homeomorphic?

We finish this section with some “product” questions.

? **603. Question 6.14.** *Let  $X$  be a retract of an infinite-dimensional Borelian pre-Hilbert space  $E$ . Is  $X \times E$  homeomorphic to  $E$ ?*

In particular, we ask:

? **604. Question 6.15.** *Is  $E \times \mathbb{R}$ ,  $E \times E$  or  $E_f^\infty$  homeomorphic to  $E$ , for every infinite-dimensional pre-Hilbert Borelian space  $E$ ?*

If  $E$  is a countable union of  $Z$ -sets (cf. 4.4) and  $E \times E$  is homeomorphic to  $E$ , then also  $E_f^\infty$  (and hence  $E \times \mathbb{R}$ ) and  $E$  are homeomorphic.

? **605. Question 6.16.** *Let  $X$  be a retract of an infinite-dimensional Borelian pre-Hilbert space  $E$ . Is  $X \times E^\infty$  homeomorphic to  $E^\infty$ ?*

To get a positive answer it is enough to show that  $E$  is a countable union of  $Z$ -sets.

? **606. Question 6.17.** *Let  $X$  be a retract of a pre-Hilbert space of first category. Is  $X \times E^\infty$  homeomorphic to  $E^\infty$ ?*

There exists a pre-Hilbert space  $E$  such that  $E$  is of the second category while moreover the space  $\sigma$  is a retract of  $E$ . Thus  $E^\infty$  is of second category while  $\sigma \times E^\infty$  is of the first category (POL [19∞]).

## 7. Problems about Finite-Dimensional Spaces

All questions in this section concern the problem of finding a topological characterization of the spaces  $\sigma_n$ ,  $\sigma_{nm}$ ,  $s_n^k$  and  $N_n^k$ .

? **607. Question 7.1.** *Characterize topologically the spaces  $\sigma_n$  and  $\sigma_{nm}$ . In particular, is every  $\mathcal{R}_n$ -absorbing set and  $\mathcal{R}_{nm}$ -absorbing set homeomorphic to  $\sigma_n$  and  $\sigma_{nm}$ , respectively?*

? **608. Question 7.2.** *Characterize topologically the Nöbeling spaces  $N_n^k$ , where  $k \geq 2n+1$ . In particular, is a space  $X$  homeomorphic to  $N_n^k$  provided that  $X$  is a separable,  $n$ -dimensional, complete metrizable space such that  $X \in AE(n)$  and for every separable,  $n$ -dimensional, complete metrizable space  $M$  each map of  $M$  into  $X$  can be approximated arbitrarily closely by  $Z$ -embeddings?*

**Question 7.3.** Let  $X$  be a separable,  $n$ -dimensional, complete metrizable space such that  $X \in AE(n)$  and for every separable  $n$ -dimensional, complete metrizable space  $M$  each map of  $M$  into  $X$  can be arbitrarily closely approximated by  $Z$ -embeddings. Is  $\sigma_n$  representable in  $X$ ? **609. ?**

**Question 7.4.** Are  $\sigma_n$  and  $\sigma_{nm}$  representable in the  $n$ -dimensional Menger cube  $M_n^k$ , where  $k \geq 2n + 1 \geq m \geq n$ ? **610. ?**

**Question 7.5.** Let  $M_n^k$  be the Menger cube, where  $k \geq 2n + 1$ . Is it true that  $M_n^k = X \cup Y$ , where  $X$  is homeomorphic to  $\sigma_n^k$ ,  $Y$  is homeomorphic to  $N_n^k$  and both  $X$  and  $Y$  are locally  $n$ -homotopy negligible in  $M_n^k$ ? **611. ?**

**Question 7.6.** Let, for  $k \geq 2n + 1$ ,  $f_n: M_n^k \rightarrow I^\infty$  be the  $(n - 1)$ -soft map of DRANIŠNIKOV [1986, Theorem 1]. Are  $f_n^{-1}((-1, 1)^\infty)$  and  $f_n^{-1}(I^\infty \setminus (-1, 1)^\infty)$  homeomorphic to  $N_n^k$  and  $\sigma_n^k$ , respectively? **612. ?**

**Question 7.7.** Let  $X = \sigma_n^k$ ,  $\sigma_{nm}^k$  or  $N_n^k$ , where  $k \geq 2n + 1 \geq m \geq n$ , and  $f: X \rightarrow Y$  be a  $UV^{n-1}$ -map of  $X$  onto  $Y \in AE(n)$ . Prove that for every open cover  $\mathcal{U}$  of  $Y$  there exists an open cover  $\mathcal{B}$  of  $Y$  such that for every homeomorphism  $h: A \rightarrow B$  between two  $Z$ -sets in  $X$ , with  $f \circ h$   $\mathcal{B}$ -close to  $f|_A$ , there exists a homeomorphism  $H: X \rightarrow X$ , with  $H|_A = h$  and  $f \circ H$   $\mathcal{U}$ -close to  $f$ . **613. ?**

## 8. Final Remarks

**8.1. Nonseparable absorbing sets.** Formally, the definition of absorbing set does not require separability. Natural spaces to represent nonseparable absorbing sets in are the Hilbert spaces  $\ell_2(A)$  for uncountable  $A$ . The following examples corresponds to (2) and (7) of our list of separable absorbing sets in §2:

**a. Example.** The linear subspace

$$\ell_2^f(A) = \{(x_\alpha) \in \ell_2(A) : x_\alpha \neq 0 \text{ for finitely many } \alpha \in A\}$$

is an absorbing set for the class of all finite dimensional metric complexes with no more than  $\text{card}(A)$  vertices (see WEST [1970]).

**b. Example.** The linear subspace  $(\ell_2(A))_f^\infty$  is an absorbing set for the class of all complete metrizable spaces with density at most  $\text{card}(A)$  (see TORUŃCZYK [1970a]).

Almost all questions of the Sections 3, 4 and 6 can be repeated for nonseparable spaces. Since two  $\mathcal{C}$ -absorbing sets in  $\ell_2(A)$  are homeomorphic,

answers to these questions could throw light on the topological classification of nonseparable incomplete metric linear spaces.

**8.2. Noncontractible absorbing sets.** The notion of absorbing set can be naturally extended to absolute neighborhood retracts represented in  $\mathbb{R}^\infty$ -manifolds. Some of the questions of Sections 3, 4 and 5 make sense for non-contractible absorbing sets. Especially, questions 5.7–5.12, formulated in the absolute neighborhood setting, seem to be very interesting.

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## CHAPTER 26

### Problems about Finite-Dimensional Manifolds

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What follows amounts, by and large, to an annotated combination of several lists I have been hoarding, expanding, polishing the last few years. It is highly personalized—the title topic is far too extensive to allow treatment of all its various components, so I have not even tried. Instead, the combination identifies questions mainly in the areas of manifold structure theory, decomposition theory, and embedding theory. The more significant issues, and the one I prefer, tend to occur where at least two of these intersect, but admittedly several of the problems presented are light-hearted, localized, outside any overlap.

Before launching out into those areas named above, however, and mindful that the effort undoubtedly will invite disputation, I cannot resist stealing the opportunity to restate some of the oldest, most famous problems of this subject. Occasional reiteration spreads awareness, and this occasion seems timely, which is justification enough. Accordingly, well-versed readers should not expect to discover new material in the opening list of “Venerable Conjectures”; either they should skip it entirely or they can scan it critically for glaring omissions or whatever. Any other readers will benefit, I trust, by finding such a list in one convenient place.

The bibliography is intended as another convenience. Extensive but by no means complete, it is devised mainly to offer recent entry points to the literature.

At inception this project involved a host of mathematicians. Late one Oregon summer night during the 1987 Western Workshop in Geometric Topology, several people, including Mladen Bestvina, Phil Bowers, Bob Edwards, Fred Tinsley, David Wright (their names would have been protected if they were truly innocent), set out to construct a list of lesser known, intriguing problems deserving of wider publicity. They all made suggestions, and I kept the record. The evening’s discussion led directly to a number of the problems presented here, which at one time constituted a separate list, but which in my tinkering I eventually grouped under topic headings. (No one else deserves any blame for my rearrangements.) If a question had strong support that evening for inclusion in the collection of “not-famous-enough problems”, or if it just had marginal support with no major opposition, it shows up here preceded by an asterisk.

Other problems sets about finite dimensional manifolds published within the past decade should be mentioned. Here are a few. The most famous is KIRBY’s list(s) of low dimensional problems [1978, 1984]; the first installment is a bit old, but the second, put together after the 1982 conference on four-manifolds, includes a thorough update. THURSTON [1982] has set forth some fundamental open problems about 3-manifolds and Kleinian groups. Much to my surprise, I could find no major collection focused on knot theory questions<sup>2</sup>, although many such appear in Kirby’s lists, and information

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<sup>2</sup>Remark by the editors: see however the paper by Kauffman in this volume.



arrived at press time about an extensive collection of braid theory problems edited by MORTON [1988]. DONALDSON [1987] has raised some key 4-dimensional matters. In a more algebraic vein, HSIANG [1984] has surveyed geometric applications of  $K$ -theory.

Finally, an acknowledgement of indebtedness to Mladen Bestvina, Marshall Cohen, Jim Henderson, Larry Husch, Dale Rolfsen, and Tom Thickstun for helpful comments and suggestions.

## 1. Venerable Conjectures

? 614. **V1.** *Poincaré Conjecture.*

? 615. **V2.** (Thurston's Geometrization Conjecture) *The interior of every compact 3-manifold has a canonical decomposition into pieces with geometric structure, in other words, into pieces with structure determined by a complete, locally homogeneous Riemannian metric.*

See THURSTON [1982]. Of relatively recent vintage, this conjecture probably does not qualify as “venerable”; nevertheless, its boldness and large-scale repercussions have endowed it with stature clearly sufficient to support its inclusion on any list of important topological problems. It fits here in part by virtue of being stronger than the Poincaré Conjecture. A closely related formulation posits that every closed orientable 3-manifold can be expressed as a connected sum of pieces which are either hyperbolic, Seifert fibered, or Haken (i.e., contains some incompressible surface and each  $PL$  2-sphere bounds a 3-ball there).

? 616. **\*V3.** (Hilbert-Smith Conjecture) *No  $p$ -adic group can act effectively on a manifold. Equivalently, no compact manifold  $M$  admits a self-homeomorphism  $h$  such that*

- (i) *each orbit  $\{h^n(x)\}$  has small diameter in  $M$  and*
- (ii)  *$\{h^n | n \in \mathbb{Z}\}$  is a relatively compact subgroup of the group of all homeomorphism  $M \rightarrow M$ .*

? 617. **V4.** ( $PL$  Schoenflies Conjecture) *Every  $PL$  embedding of the  $(n-1)$ -sphere in  $\mathbb{R}^n$  is  $PL$  standard, or equivalently, has image bounding a  $PL$   $n$ -ball.*

The difficulty is 4-dimensional: if true for  $n = 4$  then the conjecture is true for all  $n$ .

? 618. **V5.** *There is no topologically standard but smoothly exotic 4-sphere.*

This is the 4-dimensional Poincaré Conjecture in the smooth category, and an affirmative answer implies the truth of V4. In broader terms DONALDSON [1987] has asked which homotopy types of closed 1-connected 4-manifolds contain distinct smooth structures; specifically, do there exist homotopy equivalent but smoothly inequivalent manifolds of this type such that the positive part of the intersection form on 2-dimensional homology is even-dimensional?

**V6.** (A problem of Hopf) *Given a closed, orientable manifold  $M$ , is every (absolute) degree one map  $f: M \rightarrow M$  a homotopy equivalence?* **619. ?**

HAUSMANN [1987] has split this problem into component questions and has provided strong partial results:

- (1) must  $f$  induce fundamental group isomorphisms? and if so,
- (2) must  $f$  induce isomorphisms of  $H_*(M; \mathbb{Z}\pi)$ ?

Hopf's problem led to the concept of Hopfian group, namely, a group in which every self-epimorphism is 1-1.

**V6'.** *Does every compact 3-manifold have Hopfian fundamental group?* **620. ?**

Yes, if Thurston's Geometrization Conjecture is valid (HEMPEL [1987]).

**V7.** (Whitehead Conjecture; WHITEHEAD [1941]) *Every subcomplex of an aspherical 2-complex is itself aspherical.* **621. ?**

**V7'.** *If  $K$  is a subcomplex of a contractible 2-complex, is  $\pi_1(K)$  locally indicable (i.e., every nontrivial, finitely generated subgroup admits a surjective homomorphism to  $\mathbb{Z}$ ).* **622. ?**

Groups with this property are sometimes called locally  $\mathbb{Z}$ -representible. As mentioned in Howie's useful survey HOWIE [1987], an affirmative answer implies the Whitehead conjecture.

**V8.** (Borel Rigidity Conjecture) *Every homotopy equivalence  $N \rightarrow M$  between closed, aspherical manifolds is homotopic to a homeomorphism.* **623. ?**

Evidence in favor of this rigidity has been accumulating ; see for example the work of FARRELL and HSIANG [1983] and FARRELL and JONES [1986]. More generally, FERRY, ROSENBERG and WEINBERGER [1988] conjecture: every homotopy equivalence between aspherical manifolds which is a homeomorphism over a neighborhood of  $\infty$  is homotopic to a homeomorphism.

**V9.** (Zeeman Conjecture; ZEEMAN [1964]) *If  $X$  is a contractible finite 2-complex, then  $X \times I$  is collapsible.* **624. ?**

This is viewed as unlikely, because it is stronger than the Poincaré conjecture. Indeed, when restricted to special spines (where all vertex links are circles with either 0, 2 or 3 additional radii) of homology 3-cells, it is equivalent to the Poincaré Conjecture (GILLMAN and ROLFSEN [1983]). COHEN [1975] introduced a related notion, saying that a complex  $X$  is  $q$ -collapsible provided  $X \times I^q$  is collapsible, and he showed (among other things) that all contractible  $n$ -complexes  $X$  are  $2n$ -collapsible. Best possible results concerning  $q$ -collapsibility are yet to be achieved, but BERNSTEIN, COHEN and CONNELLY [1978] have examples in all but very low dimensions (suspensions of nonsimply connected homology cells) for which the minimal  $q$  is approximately that of the complex.

- ? 625. **V10.** (Codimension 1 manifold factor problem (generalized Moore problem)) *If  $X \times Y$  is a manifold, is  $X \times \mathbb{R}^1$  a manifold?*

The earliest formulations of this problem, calling for  $X$  to be the image of  $S^3$  under a cell-like map (see the decomposition section for a definition), date back at least to the early 1960s; see DAVERMAN [1980] for a partial chronology. In the presence of the manifold hypothesis on  $X \times Y$ , Quinn's obstruction theory QUINN [1987] ensures the existence of a cell-like map from some manifold onto  $X \times \mathbb{R}^1$ . When  $X \times \mathbb{R}^1$  has dimension at least 5, the question is just whether it has the following Disjoint Disks Property: any two maps of  $B^2$  into  $X \times \mathbb{R}^1$  can be approximated, arbitrarily closely, by maps having disjoint images. No comparably simple test detects whether a 4-dimensional  $X \times \mathbb{R}^1$  is a manifold. Since  $X \times \mathbb{R}^2$  does have the Disjoint Disks Property mentioned above, Edwards' Cell-like Approximation Theorem (EDWARDS [1980]) attests it is a manifold.

- ? 626. **\*V11.** (Resolution Problem) *Does every generalized  $n$ -manifold  $X$ ,  $n \geq 4$ , admit a cell-like resolution? That is, does there exist a cell-like map of some  $n$ -manifold  $M$  onto  $X$ ?*

In one sense this has been answered—QUINN [1987] showed such a resolution exists iff a certain integer-valued obstruction  $i(X) = 1$ —but in another sense it remains unsettled because no one knows whether  $i(X)$  ever assumes a different value. A large measure of its significance is attached to the consequent characterization of topological manifolds: a metric space  $X$  is an  $n$ -manifold ( $n \geq 5$ ) iff  $X$  is finite dimensional, locally contractible,  $H_*(X, X-x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n-0)$  for all  $x \in X$  (i.e.,  $X$  is a generalized  $n$ -manifold),  $X$  has the Disjoint Disks Property, and  $i(X) = 1$ . Is the final condition necessary?

- ? 627. **V12.** (Kervaire Conjecture (also known as the Kervaire-Laudenbach Conjecture)) *If  $A$  is a group for which the normalizer of some element  $r$  in the free product  $A * \mathbb{Z}$  is  $A * \mathbb{Z}$  itself, then  $A$  is trivial.*

The main difficulty occurs in the case of an infinite simple group  $A$ . See Howie's survey HOWIE [1987] again for connections to other more obviously topological problems.

## 2. Manifold and Generalized Manifold Structure Problems

A *generalized  $n$ -manifold* is a finite dimensional, locally compact, locally contractible metric space  $X$  with  $H_*(X, X - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$  for all  $x \in X$ . As Problems V10 and \*V11 suggest, the central problems are

- (1) whether every generalized manifold  $X$  is a factor of some manifold  $X \times Y$  and
- (2) whether  $X \times \mathbb{R}^1$  is always a manifold.

Implications of homogeneity have not been fully determined, neither for distinguishing generalized manifolds from genuine ones nor for distinguishing locally flat embeddings of codimension one manifolds from wild embeddings.

**M1.** *Does there exist a homogeneous compact absolute retract of dimension  $2 < n < \infty$ ?* **628. ?**

BING and BORSUK [1965] shows that every homogeneous compact **ANR** (= absolute neighborhood retract) of dimension  $n < 3$  is a topological manifold.

**M2.** (Homogeneous **ENRs** versus generalized manifolds) *If  $X$  is a homogeneous, locally compact **ENR** (= finite dimensional **ANR**), is  $X$  a generalized manifold?* **629. ?**

According to BREDON [1970] (see alternatively BRYANT [1987]), it is provided

$$H_*(X, X - x; \mathbb{Z})$$

is finitely generated for some (and, hence, for every) point  $x \in X$ .

**M2'.** *Does every compact **ENR**  $X$  contain a point  $x_0$  such that  $H_*(X, X - x_0)$  is finitely generated?* **630. ?**

**M3.** *Is every homogeneous generalized manifold necessarily a genuine manifold?* **631. ?**

No if the 3-dimensional Poincaré Conjecture is false (JAKOBSCHKE [1980]), but otherwise unknown.

- ? 632. **M4.** *Do all finite dimensional  $H$ -spaces have the homotopy type of a closed manifold?*

CAPPELL and WEINBERGER [1988], who attribute the original question to Browder, have recent results.

- ? 633. **M5.** *If  $M$  is a compact manifold, is the group  $\text{Homeo}(M)$  of all self-homeomorphisms an **ANR**?*

FERRY [1977] proved  $\text{Homeo}(M)$  is an **ANR** when  $M$  is a compact Hilbert cube manifold.

- ? 634. **M6.** *Is every closed, aspherical 3-manifold virtually Haken (have a finite-sheeted cover by a Haken manifold)? Even stronger, does it have a finite sheeted cover by a manifold with infinite first homology?*

- ? 635. **\*M7.** *Is every contractible 3-manifold  $W$  that covers a closed 3-manifold necessarily homeomorphic to  $\mathbb{R}^3$ ?*

Here one should presume  $W$  contains no fake 3-cells (i.e., no compact, contractible 3-manifolds other than 3-cells). Elementary cardinality arguments indicate some contractible 3-manifolds cannot be universal covers of any compact one, and Myers has identified specific examples, including Whitehead's contractible 3-manifold, that cannot do so. Davis's higher dimensional examples DAVIS [1983], by contrast, indicate this is a uniquely 3-dimensional problem.

*Local connectedness of limit sets of conformal actions on  $S^3$ .* A group  $G$  of homeomorphisms of the 2-sphere is called a discrete convergence group if every sequence of distinct elements from  $G$  has a subsequence  $g_j$  for which there are points  $x, y \in S^2$  with  $g_j \rightarrow x$  uniformly on compact subsets of  $S^2 - \{x\}$  (or, equivalently,  $G$  acts properly discontinuously on the subset of  $S^2 \times S^2 \times S^2$  consisting of triples of distinct points of  $S^2$ ). Its limit set  $L(G)$  is the closure of the set of all such points  $x$ .

- ? 636. **\*M8.** *If  $L(G)$  is connected, must it be locally connected?*
- ? 637. **M9.** (Freedman and Skora) *Must a  $K(G, 1)$ -manifold  $M$ , where  $G$  is finitely generated, have only a finite number of ends? What if  $M$  is covered by  $\mathbb{R}^n$ ?*
- ? 638. **M10.** (M. Davis) *Must the Euler characteristic (when nonvanishing) of a closed, aspherical  $2n$ -manifold have the same sign as  $(-1)^n$ ?*
- ? 639. **M11.** *Under what conditions does a closed manifold cover itself? cover itself*

both regularly and cyclically? Are the two classes different?

**M12.** Does there exist an aspherical homology sphere of dimension at least 4? **640.** ?

**\*M13.** (Simplified free surface problem in high dimensions—see also E1) **641.** ?  
Suppose  $W$  is a contractible  $n$ -manifold such that, for every compact  $C \subseteq W$ , there exists an essential map  $S^{n-1} \rightarrow W - C$ . Is  $W$  topologically equivalent to  $\mathbb{R}^n$ ?

The *Lusternik-Schnirelman category* of a polyhedron  $P$ , written  $\text{cat}(P)$ , is the least integer  $k$  for which  $P$  can be covered by  $k$  open sets, each contractible in  $P$ . See Montejano's survey MONTEJANO [1986] [19∞] for a splendid array of problems on this and related topics. Here are two eye-catching ones.

**\*M14.** Does  $\text{cat}(M \times S^r) = \text{cat}(M) + 1$ ? **642.** ?

SINGHOF [1979] has answered this affirmatively for closed  $PL$  manifolds where  $\text{cat}(M)$  is fairly large compared to  $\dim(M)$ .

**M15.** If  $M$  is a closed  $PL$  manifold, does  $\text{cat}(M - \text{point}) = \text{cat}(M) - 1$ ? **643.** ?

**M16.** (Ulam-problem #68 in *The Scottish Book* MAULDIN [1981]) If  $M$  is a compact manifold with boundary in  $\mathbb{R}^n$  for which every  $(n-1)$ -dimensional hyperplane  $H$  that meets  $M$  in more than a point has  $H \cap \partial M$  an  $(n-1)$ -sphere, is  $M$  convex?<sup>3</sup> **644.** ?

**M17.** (Borsuk) Can every bounded  $S \subseteq \mathbb{R}^n$  be partitioned into  $(n+1)$ -subsets  $S_i$  such that  $\text{diam}(S_i) < \text{diam } S$ ? What about for finite  $S$ ? **645.** ?

**M18.** If  $X$  is a compact,  $n$ -dimensional space having a strongly convex metric without ramifications, is  $X$  an  $n$ -cell? What if  $X$  is a generalized manifold with boundary? In that case, is  $X - \partial X$  homogeneous? **646.** ?

For definitions see ROLFSEN [1968], which solves the case  $n = 3$ .

**M19.** Is there a complex dominated by a 2-complex but not homotopy equivalent to a 2-complex? **647.** ?

**M20.** Is every finitely presented perfect group (perfect = trivial abelianization) the normal closure of a single element? **648.** ?

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<sup>3</sup>L. Montejano has a preprint in which he obtains an affirmative answer.

### 3. Decomposition Problems

A decomposition  $G$  of a space  $X$  is a partition of  $X$ ; it is *upper semicontinuous* if each  $g \in G$  is compact and for every open set  $U \supseteq g$  there exists another open set  $V \supseteq g$  such that all  $g' \in G$  intersecting  $V$  are contained in  $U$ . Associated with  $G$  is an obvious decomposition map  $\pi: X \rightarrow X/G$  sending  $x \in X$  to the unique  $g \in G$  containing  $x$ ; here  $X/G$  has the quotient topology.

The study of upper semicontinuous decompositions of a space  $X$  coincides with the study of proper closed mappings defined on  $X$ , but the emphasis is much different. Decomposition theory stresses, or aims to achieve, understanding of the image spaces through information about the decomposition elements.

All decompositions mentioned in this part are understood to be upper semicontinuous.

A compact subset  $C$  of an **ANR** is *cell-like* if it contracts in every pre-assigned neighborhood of itself, a property invariant under embeddings in **ANRs**; equivalently,  $C$  is cell-like if it has the shape of a point. A decomposition (a map) is *cell-like* if each of its elements (point inverses) is cell-like. A decomposition  $G$  of a compact metric space  $X$  is *shrinkable* iff for each  $\epsilon > 0$  there exists a homeomorphism  $H: X \rightarrow X$  such that  $\text{diam} H(g) < \epsilon$  for all  $g \in G$  and  $d(\pi, \pi H) < \epsilon$ , where  $d$  is a metric on  $X/G$ ; a convenient phrasing stems from the theorem (cf. DAVERMAN[1986, p. 23]) showing  $G$  to be shrinkable iff  $\pi: X \rightarrow X/G$  can be approximated, arbitrarily closely, by homeomorphisms. All elements in a shrinkable decomposition of an  $n$ -manifold are both cell-like and, better, cellular (i.e., can be expressed as the intersection of a decreasing sequence of  $n$ -cells).

The initial questions concern conditions precluding a decomposition (or a map) from raising dimension.

? 649. **D1.** *The cell-like dimension-raising map problem for  $n = 4, 5, 6$ .*

DRANISHNIKOV [1988] has described a cell-like map defined on a 3-dimensional metric compactum and having infinite dimensional image; this example automatically gives rise to another such map defined on  $S^7$ . On the other hand, KOZŁOWSKI and WALSH [1983] showed no such map can be defined on any 3-manifold. What can happen between these bounds is still open, although MITCHELL, REPOVŠ and SHCHEPIN [19∞] have characterized the finite dimensional cell-like images of 4-manifolds in terms of a disjoint homological disk triples property. See also the surveys by DRANISHNIKOV and SHCHEPIN [1986] and, more recently, MITCHELL and REPOVŠ [1988].

? 650. **D2.** *Can a cell-like map defined on  $\mathbb{R}^n$  have infinite dimensional image if all point-inverses are contractible? absolute retracts? cells? starlike sets? 1-dimensional compacta?*

**D3.** If  $G$  is a usc decomposition of a compact space  $X$  into simple closed curves, is  $\dim(X/G) \leq \dim X$ ? **651. ?**

**D4.** Could there be a decomposition  $G$  of an  $n$ -manifold  $M$  into closed connected manifolds (of varying dimensions) with  $\dim(M/G) > n$ ? **652. ?**

**D5.** (Edwards) Can an open map  $M \rightarrow X$  defined on a compact manifold having 1-dimensional solenoids as point inverses ever raise dimension? **653. ?**

**D6.** (The resolution problem for generalized 3-manifolds) Assuming the truth of the 3-dimensional Poincaré Conjecture, does every generalized 3-manifold  $X$  have a cell-like resolution? Does  $X \times \mathbb{R}^1$  have such a resolution? **654. ?**

Independent of the Poincaré Conjecture, is  $X$  the cell-like image of a “Jakobsche” 3-manifold (i.e., an inverse limit of a sequence of 3-manifolds connected by cell-like bonding maps, as in JAKOBSCH [1980]). THICKSTUN [1987] verified this for  $X$  having 0-dimensional nonmanifold set.

**D6.** (Thickstun’s Full Blow-up Conjecture THICKSTUN [1987]) A compact homology  $n$ -manifold  $X$  is the conservative strongly acyclic, hereditarily  $\pi_1$ -injective image of a compact  $n$ -manifold if for each  $x \in X$  there exists a compact, orientable  $n$ -manifold  $M_x$  and a map  $(M_x, \partial M_x) \rightarrow (X, X - \{x\})$  inducing an isomorphism on  $n$ -dimensional Čech homology. **655. ?**

(Terminology: a *homology  $n$ -manifold* is a finite-dimensional, locally compact metric space for which  $H_*(X, X - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$ : by way of contrast, a generalized  $n$ -manifold is an **ANR** homology  $n$ -manifold. A map is *conservative* if its restriction to the preimage of the manifold set is an embedding; it is *hereditarily  $\pi_1$ -injective* if its restriction to the preimage of any connected open set induces an injection of fundamental groups; it is strongly acyclic if for each neighborhood  $U$  of a point preimage  $f^{-1}(x)$  there exists another neighborhood  $V$  of  $f^{-1}(x)$  such that inclusion induces the trivial homomorphism  $H_*(V) \rightarrow H_*(U)$ .) Thickstun avers (THICKSTUN [1987]) this may be an overly optimistic conjecture, since it implies the resolution conjecture for generalized  $n$ -manifolds and, therefore, the 3-dimensional Poincaré Conjecture as well. He adds that according to M. H. Freedman the 4-dimensional case implies 4-dimensional topological surgery can be done in the same sense it is done in higher dimensions.

**D7.** (The Approximation Problem for 3- and 4-manifolds) Which cell-like maps  $p: M \rightarrow X$  from a manifold onto a finite dimensional space can be approximated by homeomorphisms? Is it sufficient to know that, given any two disjoint, tame 2-cells  $B_1$ , and  $B_2 \subseteq M$ , there are maps  $\mu_i: B_i \rightarrow X$  approximating  $p|_{B_i}$  with  $\mu_1(B_1) \cap \mu_2(B_2) = \emptyset$ ? **656. ?**



The question carries a degree of credibility because for  $n \geq 5$  the condition is equivalent to  $X$  having the Disjoint Disks Property, which yields an affirmative answer, see EDWARDS [1980].

Next, some problems about shrinkability of cellular decompositions of manifolds. The 3-dimensional version of each has been solved, all but D12 affirmatively.

- ? 657. **\*D8.** *Is each decomposition of  $\mathbb{R}^n$  involving countably many starlike-equivalent sets shrinkable?*

A compact set  $X \subseteq \mathbb{R}^n$  is *starlike* if it contains a point  $x_0$  such that every linear ray emanating from  $x_0$  meets  $X$  in an interval, and  $X$  is *starlike-equivalent* if it can be transformed to a starlike set via an ambient homeomorphism. DENMAN and STARBIRD [1983] have established shrinkability for  $n = 3$ .

- ? 658. **D9.** *Let  $f: S^n \rightarrow X$  be a map such that if  $f^{-1}f(x) \neq x$ , then  $f^{-1}f(x)$  is a standardly embedded  $n$ -cell. Can  $f$  be approximated by homeomorphisms?*

Same question when there are countably many nondegenerate  $f^{-1}f(x)$ , all standardly embedded  $(n-2)$ -cells. Although closely related, these are not formally equivalent. See EVERETT [1979] and STARBIRD and WOODRUFF [1979] concerning  $n = 3$ .

- ? 659. **D10.** *Suppose  $G$  is a usc decomposition of  $n$ -space such that each  $g \in G$  has arbitrarily small neighborhoods whose frontiers are  $(n-1)$ -spheres missing the nondegenerate elements of  $G$ ? Is  $G$  shrinkable? What if the neighborhoods are Euclidean patches?*

WOODRUFF [1977] developed the low dimensional result.

- ? 660. **D11.** *Suppose  $A \subseteq \mathbb{R}^n$  is an  $n$ -dimensional annulus. Is there a parametrization of  $A$  as a product  $S^{n-1} \times I$  for which the associated decomposition into points and the fiber arcs is shrinkable?*

DAVERMAN and EATON [1969] did this when  $n = 3$ ; work by ANCEL and McMILLAN [1976] and CANNON and DAVERMAN [1981] combines with QUINN's [1982] homotopy-theoretic characterization of locally flat 3-spheres in  $\mathbb{R}^4$  to take care of  $A \subseteq \mathbb{R}^4$  as well.

- ? 661. **D12.** *Is a countable, cell-like decomposition  $G$  of  $\mathbb{R}^n$  shrinkable if every nondegenerate  $g \in G$  lies in some affine  $(n-1)$ -hyperplane?*

In the case that all nondegenerate elements live in one of two predetermined hyperplanes, BING [1962] produced a remarkable 3-dimensional counterexample and WRIGHT [1982] established shrinkability for  $n \geq 5$ , but the matter is unsolved for  $n = 4$ .

The rich variety of nonshrinkable decompositions of  $\mathbb{R}^3$  is not matched in higher dimensions; a plausible explanation is that descriptions of unusual 3-dimensional examples rely in unreproducible fashion on real world visualization experience. The next two problems point to 3-dimensional constructions lacking higher dimensional analogs.

**D13.** *Consider any sequence  $\{C(i)\}$  of nondegenerate cellular subsets of  $\mathbb{R}^{n \geq 4}$ . Does there exist a nonshrinkable, cellular decomposition of  $\mathbb{R}^n$  whose nondegenerate elements form a null sequence  $\{g(i)\}$  with  $g(i)$  homeomorphic to  $C(i)$ ?* **662. ?**

STARBIRD's [1981] 3-dimensional construction prompts the question.

**D14.** *Is there a nonshrinkable decomposition of  $n$ -space into points and straight line segments? Into convex sets?* **663. ?**

ARMENTROUT [1970] provided a 3-dimensional example, and later EATON [1975] demonstrated the nonshrinkability of an older example developed by Bing.

Presented next are some uniquely 4-dimensional issues. Most are relatively unpredictable in that, like the second half of D12, higher/lower dimensional analogs transmit conflicting information.

**D15.** *If  $X$  is the cell-like image of a 3-manifold  $M$ , does  $X$  embed in  $M \times \mathbb{R}$ ?* **664. ?**

More technically, if  $G$  is a cell-like decomposition of  $\mathbb{R}^3$ , regarded as  $\mathbb{R}^3 \times 0 \subseteq \mathbb{R}^4$ , and if  $G^*$  denotes the trivial extension of  $G$  (i.e.,  $G^*$  consists of the elements from  $G$  and the singletons from  $\mathbb{R}^4 - (\mathbb{R}^3 \times 0)$ ), is  $\mathbb{R}^4/G^*$  topologically  $\mathbb{R}^4$ ? This must be true if V10 has an affirmative answer.

**D16.** *If  $X$  is a cellular subset of 4-space and  $G$  is a cell-like decomposition of  $X$  such that  $\dim(X/G) \leq 1$ , is the trivial extension of  $G$  over 4-space shrinkable? What if  $X$  is an arc?* **665. ?**

No to the latter when  $n = 3$  by ROW and WALSH [1985] and yes for the former when  $n \geq 5$  by DAVERMAN [1979b].

**D17.** *Is each simple decomposition of  $\mathbb{R}^4$  shrinkable?* **666. ?**

Here one starts with a collection  $\{N_i\}$  of compact  $n$ -manifolds with boundary in  $\mathbb{R}^n$ , with  $N_{i+1} \subseteq \text{Int } N_i$ , and studies the decomposition consisting of singletons and the components of  $\bigcap N_i$ . It is called *simple* if each component  $C_i$  of each  $N_i$  contains a pair of  $n$ -cells  $B_1, B_2$  such that every component  $C'$  of  $N_{i+1}$  lies in either  $B_1$  or  $B_2$ . The remarkable nonshrinkable decomposition of BING [1962] mentioned in D12 is simple, whereas the Cell-like Approximation Theorem of Edwards quickly reveals shrinkability when  $n > 4$  DAVERMAN [1986, p. 185].

- ? 667. **D18.** *If  $f: S^4 \rightarrow S^4$  is a map which is 1-1 over the complement of some Cantor set  $K \subseteq S^4$ , is  $f$  cell-like? What if  $f$  is 1-1 over the complement of a noncompact 0-dimensional set?*

Yes by work of MCMILLAN [1969] for  $n = 3$ , but counterexamples exist for  $n > 4$  DAVERMAN [1979a].

- ? 668. **D19.** *Can every cellular map  $\theta: P \rightarrow Q$  between finite 4-complexes be approximated by homeomorphisms?*

HENDERSON [1982a, 1982b] produced approximations in the 3-dimensional case and counterexamples in higher dimensions.

Finite dimensional compact metric spaces  $X, Y$  are *CE-equivalent* if they are related through a finite sequence

$$X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_m = Y,$$

where “ $X_i \leftrightarrow X_{i+1}$ ” requires the existence of a cell-like surjection of one of the spaces onto the other. In short, the definition is satisfied iff some compactum  $Z$  admits a cell-like, surjective mappings onto both  $X$  and  $Y$ . FERRY [1980] shattered a suspicion that *CE* equivalences might behave like simple homotopy equivalences; he also made repeated remarks suggesting a closer connection if one restricts to  $LC^1$  spaces—see D22 below.

- ? 669. **D20.** *If  $X$  and  $Y$  are  $n$ -dimensional,  $LC^{n-1}$  compacta that are shape equivalent, are they *CE-equivalent*?*

DAVERMAN and VENEMA [1987a] have taken care of the always-difficult  $n = 1$  case.

- ? 670. **D21.** (Ferry) *If  $X$  and  $Y$  are shape equivalent  $LC^k$  compacta, are they  $UV^k$ -equivalent?*

Here one seeks a compactum  $Z$  as a source for surjective  $UV^k$  mappings onto  $X$  and  $Y$ , where “ $UV^k$ ” means that each point preimage has the shape of an  $i$ -connected object,  $i \in \{0, 1, \dots, k\}$ .

**D22.** If  $X$  and  $Y$  are  $CE$ -equivalent,  $LC^1$  compacta, are they related through a finite sequence as in the definition of  $CE$  equivalence above where, in addition, all intermediate spaces  $X_i$  are  $LC^1$ ? What happens for homotopy equivalent but simple homotopy inequivalent polyhedra  $X, Y$ ? **671. ?**

The relationship does hold for  $LC^0$  spaces, see DAVERMAN and VENEMA [1987b].

**D23.** (Kozłowski) Suppose  $X$  is the inverse limit of a sequence of homotopy equivalences  $S^2 \leftarrow S^2$ . Is  $X$   $CE$ -equivalent to  $S^2$ ? **672. ?**

**D24.** Let  $K \subseteq \mathbb{R}^n$  denote a  $k$ -cell. Under what conditions can  $K$  be squeezed to a  $(k-1)$ -cell, in the sense that there is a map  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  from which  $f|K$  is conjugate to the “vertical” projection  $B^k \rightarrow B^{k-1}$  while  $f|(\mathbb{R}^n - K)$  is a homeomorphism onto  $\mathbb{R}^n - f(K)$ . What if  $K$  is cellular? What if each Cantor set in  $K$  is tame? **673. ?**

BASS [1980] provides a useful sufficient condition and raises several other appealing questions.

**D25.** Given a cell-like map  $f: M \rightarrow X$  of an  $n$ -manifold onto a finite dimensional space, can  $f$  be approximated by a new cell-like map  $F: M \rightarrow X$  such that each  $F^{-1}(x)$  is 1-dimensional? Specifically, can this be done when  $n \in \{3, 4, 5\}$ ? **674. ?**

**D26.** Is there a decomposition of  $\mathbb{R}^n$  into  $k$ -cells ( $k > 0$ ) into copies of some fixed compact absolute retract ( $\neq$  point)? **675. ?**

See JONES [1968] and WALSH and WILSON [1981].

**D27.** Is there a decomposition of  $B^n$  into simple closed curves? of a compact contractible space? of a cell-like set? **676. ?**

**D28.** (Bestvina-Edwards) Does there exist a cell-like, noncontractible compactum whose suspension is contractible? **677. ?**

Standard Notation:  $M$  is an  $(n+k)$ -manifold;  $G$  is a usc decomposition of  $M$  into closed connected  $n$ -manifolds;  $B$  is the decomposition space  $M/G$ ; and  $p: M \rightarrow B$  is the decomposition map. For convenience assume both  $M$  and all the elements of  $G$  are orientable.

Due to similarities imposed on the set of point preimages, one can regard the study of these maps  $p: M \rightarrow B$  as somewhat comparable to the study of cell-like maps. At another level, when all point preimages are topologically

the same, one can strive for much more regular sorts of conclusions suggested by the theory of fibrations and/or locally trivial bundle maps.

- ? 678. **D29.** *Is  $B$  an ANR? What if the elements of  $G$  are pairwise homeomorphic?*
- ? 679. **D30.** *Is  $B$  finite-dimensional?*

(It deserves emphasis here that if the elements of  $G$  are not required to be genuine manifolds but merely to be of that shape, a fairly common hypothesis in this topic, the product of  $S^n$  with a Dranishnikov dimension-raising cell-like decomposition of  $S^k$  quickly provides negative solutions.) What if the elements of  $G$  are simple closed curves?

- ? 680. **D31.** *For which integers  $n$  and  $k$  is there a usc decomposition of  $S^{n+k}$  into  $n$ -spheres? into  $n$ -tori? into fixed products of spheres? Into closed  $n$ -manifolds? Does  $\mathbb{R}^{n+k}$  ever admit a decomposition into closed  $n$ -manifolds ( $n > 0$ )?*
- ? 681. **D32.** *When  $n$  and  $k$  are both odd, does every closed  $(n+k)$ -manifold  $M$  admitting a decomposition into closed  $n$ -manifolds have Euler characteristic zero?*
- ? 682. **D33.** *If  $G$  is a usc decomposition of an  $(n+k)$ -manifold  $M$  into  $n$ -spheres, where  $2 < n+1 < k < 2n+2$ , is  $M/G$  a generalized  $k$ -manifold? What if into homology  $n$ -spheres?*

Investigations when  $k < n+1$  and  $k = n+1$  are detailed in DAVERMAN and WALSH [1987] and SNYDER [1988], respectively.

- ? 683. **D34.** *In case  $k = 3$ , is the set of points at which  $B$  fails to be a generalized 3-manifold locally finite?*
- ? 684. **D35.** *If  $k = 3$ ,  $n = 1$ , and the degeneracy set  $K(B)$  of local 1-winding functions is empty (i.e., the 1-dimensional cohomology sheaf of  $p: M \rightarrow B$  is Hausdorff), is  $B$  a generalized 3-manifold?*
- ? 685. **D36.** *If  $k = 1$  and all elements of  $G$  are 2-sided in  $M$ , must  $M$  have the homotopy type of a closed  $n$ -manifold?*
- ? 686. **D37.** *If  $W$  is a compact  $(n+1)$ -dimensional manifold with  $\partial W \neq \emptyset$  and the inclusion  $N \rightarrow W$  of some component  $N$  of  $\partial W$  is a homotopy equivalence, does  $W$  admit a decomposition into closed  $n$ -manifolds? What if the kernel of the induced  $\pi_1$ -homomorphism is simple (but contains no finitely generated perfect group)?*

**D38.** When  $n = 3$  and  $k = 1$  does there exist a decomposition  $G$  of a 687. ?  
connected  $M$  containing homotopy inequivalent elements?

Information from DAVERMAN [1985] surrounds this 4-dimensional matter, comparable to D15–D19.

**D39.** Does there exist a compact 5-manifold  $W$  having boundary components 688. ?  
 $M_0$  and  $M_1$ , where  $\pi_1(M_0) \cong 1$  and  $\pi_1(M_1) \cong A_5$ , the alternating group on 5 symbols, such that  $W$  admits a decomposition  $G$  into closed 4-manifolds (with  $M_0, M_1 \in G$ )? closed 4-manifolds (with  $M_0, M_1 \in G$ )?

DAVERMAN and TINSLEY [19∞] locate  $W$  when  $H_*(M_1) \cong H_*(S^4)$  but not when  $\pi_1(M_1)$  is an arbitrary finitely presented perfect group.

**D40.** Given a closed manifold  $N$ , does some  $(n + k)$ -manifold  $M$  admit a 689. ?  
decomposition into copies of  $N$  such that  $p: M \rightarrow B$  is not an approximate fibration? Are there other examples besides those with homology sphere factors and those that regularly, cyclically cover themselves? Is there a 2-manifold example  $N$  with negative Euler characteristic?

**D41.** For which  $n$ -manifolds  $N$  and integers  $k$  does the hypothesis that all 690. ?  
elements of  $G$  are copies of  $N$  imply  $p: M \rightarrow B$  is an approximate fibration?

What if  $\pi_1(N)$  is finite and  $k = 2$ ? What if  $N$  is covered by the  $n$ -sphere? What if  $N$  is hyperbolic? What if all  $g \in G$  are required to be locally flat in  $M$ ?

**D42.** If  $k = 2m$ ,  $n = 2m + 1$ , and  $p: M \rightarrow B$  is a PL map from a PL  $(n + k)$ - 691. ?  
manifold  $M$  to a simplicial complex  $B$  such that  $H_j(p^{-1}(b)) \cong 0$  whenever  $0 < j < n$ , is  $B$  a generalized manifold?

#### 4. Embedding Questions

**E1.** (Free surface problem) Suppose  $\Sigma$  is an  $(n - 1)$ -dimensional-sphere topo- 692. ?  
logically embedded in  $\mathbb{R}^n$  and for every  $\epsilon > 0$  there exists a map  $f: \Sigma \rightarrow \text{Int } \Sigma$  ( $\text{Int } \Sigma = \text{bounded component of } \mathbb{R}^n - \Sigma$ ) moving points less than  $\epsilon$ . Is the closure of  $\text{Int } \Sigma$  an  $n$ -cell?

The 3-dimensional case has withstood attack for over 25 years, and talented people have mounted attacks. Any particular counterexample, under multiple suspension, would give counterexamples in all larger dimensions.

**E2.** (Burgess's "locally spanned in" problem) An  $(n - 1)$ -sphere  $\Sigma \subseteq \mathbb{R}^n$  is 693. ?  
said to be locally spanned in  $\text{Int } \Sigma$  if corresponding to each  $p \in \Sigma$  and each

$\epsilon > 0$  is an  $(n-1)$ -cell  $D$  with  $p \in \text{Int } D \subseteq \Sigma$  such that for every  $\gamma > 0$  there exists an  $(n-1)$ -cell  $E \subseteq \text{Int } \Sigma$ , where  $E$  has diameter less than  $\epsilon$ , and also a homeomorphism  $\partial D \rightarrow \partial E$  moving the points less than  $\gamma$ . Must  $\Sigma$  bound a topological  $n$ -cell if it is locally spanned in  $\text{Int } \Sigma$ ?

This is unknown even for  $n = 3$ , although BURGESS [1965] gave an affirmative answer then provided  $\Sigma$  can be uniformly locally spanned in  $\text{Int } \Sigma$  (where there exists  $\delta > 0$  such that for all  $(n-1)$ -cells  $D \subseteq \Sigma$  of diameter less than  $\delta$  and for all  $\gamma > 0$  one has an  $(n-1)$ -cell  $E$  as above). However, for  $n > 3$  it is still unsettled whether ever this uniform property implies  $\Sigma$  bounds an  $n$ -cell.

? 694. **\*E3.** (Uniform tangent balls) Suppose  $\Sigma$  is an  $(n-1)$ -sphere topologically embedded in  $\mathbb{R}^n$  and there exists a fixed  $\delta > 0$  such that for each  $p \in \Sigma$  there exists a round  $n$ -cell  $B_p$  of radius  $\delta$  satisfying

- (i)  $p \in B_p$  and
- (ii)  $\text{Int } B_p \subseteq \text{Int } \Sigma$

Is the closure of  $\text{Int } \Sigma$  an  $n$ -cell?

The answer is affirmative when  $n = 3$  (DAVERMAN and LOVELAND [1981]).

? 695. **E4.** (0-dimensional homotopy taming sets) Given a map  $\psi: B^2 \rightarrow C$ , where  $C$  denotes the closure of an  $(n-1)$ -sphere complement in  $\mathbb{R}^n$ , can one approximate  $\psi$  arbitrarily closely by a map  $\psi: B^2 \rightarrow C$  such that  $\dim(\psi(B^2) \cap \text{Fr } C) \leq 0$ ?

? 696. **E5.** (Homogeneity versus wildness for codimension 1 embeddings) If  $\Sigma$  is an  $(n-1)$ -sphere topologically embedded in  $\mathbb{R}^n$  such that for any points  $p, q \in \Sigma$  there is a self-homeomorphism  $h$  of  $\mathbb{R}^n$  with  $h(\Sigma) = \Sigma$  and  $h(p) = q$ , must  $\Sigma$  be flat?

What is  $\Sigma$  is strongly homogeneous? A subset of  $\mathbb{R}^n$  is said to be *strongly homogeneous* (or, better, *strongly homogeneously embedded*) if every self-homeomorphism extends to a space homeomorphism.

? 697. **E6.** Is every strongly homogeneous Cantor set in  $\mathbb{R}^3$  ( $\mathbb{R}^4$ ) tame?

Wild Cantor sets in  $\mathbb{R}^{n \geq 5}$  can be strongly homogeneously embedded DAVERMAN [1979a]; the familiar Antoine necklace is homogeneously embedded but not strongly so in  $\mathbb{R}^3$ .

? 698. **E7.** Is there a wild Cantor set  $K$  in  $\mathbb{R}^4$  defined by contractible objects?

Specifically,  $K$  is to be expressed as  $K = \bigcap M_i$ , where each  $M_i$  is a compact, contractible manifold (with boundary) and  $\text{Int } M_i \supseteq M_{i+1}$ .

**\*E8.** (Sticky Cantor Set) *If  $X$  is an arbitrary wild Cantor set in  $S^n$ , do there exist arbitrarily small homeomorphisms  $h$  of  $S^n$  to itself with  $X \cap h(X) = \emptyset$ ?* **699. ?**

A sufficiently strong counterexample would provide a negative solution to D11.

Despite the difference in context, several people believe the following is a closely connected question.

**E8'.** (Wright) *Does the Mazur 4-manifold  $M^4$ —a compact, contractible manifold with non-simply connected boundary—have a pair of disjoint spines?* **700. ?**

A *spine* of  $M^4$  is a polyhedron to which  $M^4$  collapses.

**E9.** *Are objects  $X \subseteq \mathbb{R}^n$  that can be instantaneously pushed off themselves geometrically tame?* **701. ?**

The central issue is whether codimension 3 objects  $X$  are 1-*LCC* embedded (i.e., can maps  $B^2 \rightarrow \mathbb{R}^3$  be approximated by maps into  $\mathbb{R}^n - X$ ?). Here the hypothesis calls for an ambient isotopy  $\theta_t$ , starting at the identity, such that  $X \cap \theta_t(X) = \emptyset$  for all  $t > 0$ . WRIGHT [1976] has shown that Cantor sets with this property are tame.

**E10.** (“Approximating” compacta by Cantor sets) *Let  $P$  be a compact subset of  $\mathbb{R}^n$  and  $U \supseteq P$  an open set. Must  $U$  contain a Cantor set  $C$  such that all loops in  $\mathbb{R}^n - U$  which are contractible in  $\mathbb{R}^n - C$  are also contractible in  $\mathbb{R}^n - P$ ?* **702. ?**

For  $n = 3$  this is not at all difficult, for  $n = 4$  it is harder (Daverman and Lay have an unpublished construction), and otherwise it is still open.

**E11.** *Let  $\lambda: X \rightarrow M$  denote a closed embedding of a generalized  $n$ -manifold  $X$  in a genuine  $(n + 1)$ -manifold  $M$ . Can  $\lambda$  be approximated by 1-*LCC* embeddings?* **703. ?**

Yes for  $n \geq 4$  (see DAVERMAN [1985, p. 283]—key ideas are due to CANON, BRYANT and LACHER [1979]); what about for  $n = 3$ ? What if  $X$  is a generalized  $n$ -manifold with boundary? Ancel discusses this and related problems in ANCEL [1986].

**E12.** *Which homology  $n$ -spheres  $K$  bound acyclic  $(n + 1)$ -manifolds  $N$  such* **704. ?**



that  $\pi_1(K) \rightarrow \pi_1(N)$  is an isomorphism? Is there a homology 4-sphere example?

? 705. E13. Let  $X$  be a cell-like subset of  $\mathbb{R}^n$ . Does  $\mathbb{R}^n$  contain an arc  $\alpha$  with  $\mathbb{R}^n - \alpha$  homeomorphic to  $\mathbb{R} - X$ ?

For  $n \geq 6$ ,  $\mathbb{R}^n$  has a 1-dimensional compact subset  $A$  with  $\mathbb{R}^n - A \approx \mathbb{R}^n - X$  (NEVAJDIĆ [19∞]).

? 706. E14. Can there be a codimension 3 cell  $D$  in  $\mathbb{R}^n$  ( $n > 5$ ) such that all 2-cells in  $D$  are wildly embedded in  $\mathbb{R}^n$  but each arc (each Cantor set) there is tame?

This question calls for new embedding technology, since existing examples (DAVERMAN [1975]) in which all 2-cells are wild essentially exploit the presence there of Cantor sets wildly embedded in the ambient manifold.

? 707. E15. Can every  $n$ -dimensional compact absolute retract be embedded in  $\mathbb{R}^{2n}$ ?

? 708. E16. Can every  $S^n$ -like continuum be embedded in  $\mathbb{R}^{2n}$ ?

A metric space  $X$  is  $S^k$ -like if there exist  $\epsilon$ -maps  $X \rightarrow S^k$  for every  $\epsilon > 0$ .

? 709. E17. Does  $S^4$  contain a 2-sphere  $\Sigma$ , possibly wildly embedded, such that  $S^4 - \Sigma$  is topologically  $S^1 \times \mathbb{R}^3$  but not smoothly so?

? 710. E18. (M. Brown) If a wedge  $A \vee B \subseteq \mathbb{R}^3$  is cellular, is  $A$  cellular?<sup>4</sup>

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## CHAPTER 27

### A List of Open Problems in Shape Theory

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Here we consider four types of problems for compact metric spaces which are of current interest in shape theory. Earlier lists of problems include BALL [1976], BORSUK [1975], DYDAK, KADLOF and NOWAK [1981], OGHEGAN [1979], KRASINKIEWICZ [1981], and MARDEŠIĆ and SEGAL [1987].

## 1. Cohomological and shape dimensions

**Problem 1.1.** Suppose a metrizable and separable space  $X$  is the union of two of its subsets  $A$  and  $B$  whose cohomological dimensions  $\text{cdim}_{\mathbb{Z}} A$  and  $\text{cdim}_{\mathbb{Z}} B$  are finite. Is the cohomological dimension of  $X$  finite? If  $\text{cdim}_{\mathbb{Z}} X$  is finite does the inequality 711. ?

$$\text{cdim}_{\mathbb{Z}} X \leq \text{cdim}_{\mathbb{Z}} A + \text{cdim}_{\mathbb{Z}} B + 1$$

hold?

Yes, if  $\dim A$  is finite. The second part of the above problem was asked by V. I. KUZ'MINOV in [1968] for any group  $G$ .

**Problem 1.2.** Suppose  $f: X \rightarrow Y$  is a closed map of metrizable spaces such that  $\text{cdim}_{\mathbb{Z}} X \leq n$  and there exist natural numbers  $m$  and  $k$  such that  $\text{rank}(H^*(f^{-1}(y))) \leq m$  and  $\text{card}(\text{Tor } H^*(f^{-1}(y))) \leq k$  for all  $y \in Y$ . Is the cohomological dimension of  $Y$  finite? 712. ?

**Problem 1.3.** Is there a compact ANR-space  $X$  of infinite dimension such that its cohomological dimension  $\text{cdim}_G X$  with respect to some non-trivial group  $G$  is finite? 713. ?

Recently, in [1988], A. N. DRANISHNIKOV found an infinite-dimensional compactum  $X$  of cohomological dimension 3. It is known that  $\text{cdim}_{\mathbb{Z}} X \leq 1$  implies  $\dim X = 1$ .

**Problem 1.4.** Is there an infinite-dimensional compactum  $X$  of cohomological dimension 2? 714. ?

Let  $PS^n$  be the Postnikov  $n$ -sphere. It is the inverse limit of the Postnikov system (see SPANIER [1966, p.444])  $\{E_i, p_i^{i+1}\}$  of  $S^n$  (i.e.  $\pi_k(E_i) = 0$  for  $k \geq i$ , each bonding map  $p_i^{i+1}$  is a fibre bundle with the fibre being a  $K(\pi_i, m_i)$ ) and there is a weak homotopy equivalence  $h: S^n \rightarrow PS^n$ . In [1988] A. N. DRANISHNIKOV proved that any continuous map  $f: A \rightarrow PS^n$ ,  $A$  being a closed subset of a compactum  $X$  with  $\text{cdim}_{\mathbb{Z}} X \leq n$ , extends over  $X$ . He posed the following problem:

**Problem 1.5.** Suppose for any closed subset  $A$  of a given compactum  $X$  and 715. ?



for any map  $f: A \rightarrow PS^n$  there is an extension  $f'$  over  $X$ . Is the cohomological dimension of  $X$  at most  $n$ ?

? **716. Problem 1.6.** Suppose  $f: M^n \rightarrow X$  is a map from a closed  $n$ -manifold onto an **ANR** of finite dimension such that all the fibers are of the same shape. Is it true that the Čech cohomology of the fibers is finitely generated?

? **717. Problem 1.7.** (D. R. McMillan) Given an Euclidean space  $\mathbb{E}^n$  is there a finite polyhedron  $P_n \subset \mathbb{E}^n$  such that a compactum  $X \subset \mathbb{E}^n$  is of trivial shape iff all the maps  $f: X \rightarrow P_n$  are null-homotopic?

Yes, if  $n \leq 3$ .

? **718. Problem 1.8.** Given a natural number  $n$  is there a finite polyhedron  $P_n$  such that a compactum  $X$  of shape dimension at most  $n$  is of trivial shape iff all the maps  $f: X \rightarrow P_n$  are null-homotopic?

Yes, if  $n \leq 2$ .

? **719. Problem 1.9.** Is every finite dimensional compactum  $X$   $CE$ -equivalent to a compactum  $Y$  with  $\dim Y = \text{Sd } X$ ?

? **720. Problem 1.10.** Is the shape dimension of  $X \times S^1$  equal to 3 for every movable  $X$  with  $\text{Sd } X = 2$ ?

## 2. Movability and polyhedral shape

? **721. Problem 2.1.** (J. KRASINKIEWICZ [1981], D. R. McMILLAN [1975]) Is a movable continuum pointed movable?

? **722. Problem 2.2.** If the wedge  $X \vee Y$  of two continua is movable, is  $X$  pointed movable?

Yes, if  $\text{pro-}\pi_1 Y$  is not trivial.

As defined by S. MARDEŠIĆ in [1981] a compactum  $X$  is an *approximate polyhedron* (AP) if for each  $\varepsilon > 0$  there is a polyhedron  $P$  and maps  $f: X \rightarrow P$ ,  $g: P \rightarrow X$  such that the distance  $d(gf(x), x)$  is less than  $\varepsilon$  for all  $x$  in  $X$ .

? **723. Problem 2.3.** (DYDAK and SEGAL [1981a]) Is  $X \in AP$  pointed 1-movable?

Yes, if  $X$  has the fixed point property or its Euler characteristic (defined using the Čech homology groups) does not equal 0.

A compactum is called *regularly movable* provided it is the inverse limit of an inverse sequence of **ANR**'s with bonding maps being homotopy dominations.

**Problem 2.4.** (DYDAK and SEGAL [1981a]) *Is  $X \in AP$  regularly movable?* **724.** ?

**Problem 2.5.** *Is every movable subcontinuum of  $\mathbb{E}^3$  regularly movable?* **725.** ?

**Problem 2.6.** *Is a movable continuum  $X$  regularly movable if  $\text{pro-}\pi_1 X$  is trivial?* **726.** ?

**Problem 2.7.** (J. KRASINKIEWICZ [1981]) *Does every non-movable continuum  $X$  contain a non-movable curve?* **727.** ?

**Problem 2.8.** (D. R. McMILLAN [1975]) *Suppose  $A \times B$  is embeddable in a 3-manifold and each continuum  $A$  and  $B$  is non-degenerate. Is  $A$  movable?* **728.** ?

**Problem 2.9.** *If  $\text{Sd } X = 2$  and  $X$  is an **ANR**, is there a 2-dimensional polyhedron of the same shape as  $X$ ?* **729.** ?

**Problem 2.10.** *Suppose a compactum  $X \subset \mathbb{R}^4$  is shape equivalent to a polyhedron. Is it shape equivalent to a finite polyhedron?* **730.** ?

**Problem 2.11.** (K. BORSUK [1975]) *If  $X$  is movable and  $\text{Sd } X \leq n$ , is there a compactum  $Y \subset \mathbb{E}^{2n}$  of the same shape as  $X$ ?* **731.** ?

Yes, if  $\text{pro-}\pi_1 X$  is equivalent to a group.

A compactum  $X$  is called an **ANR-divisor** provided  $Q/X \in \mathbf{ANR}$ , where  $Q$  is the Hilbert cube.

**Problem 2.12.** *Suppose  $X$  is an **ANR-divisor** and  $\text{pro-}\pi_1 X$  is trivial. Is there a polyhedron of the same shape as  $X$ ?* **732.** ?

Yes, if (a)  $\text{Sd } X$  is finite, (b)  $X$  is movable, or (c)  $X$  is acyclic.

**Problem 2.13.** *If  $X$  and  $Y$  are **ANR-divisors** is  $X \times Y$  an **ANR-divisor**?* **733.** ?

**Problem 2.14.** *If  $X$  is shape dominated by an **ANR-divisor** is it an **ANR-divisor**?* **734.** ?

Yes, if  $\text{Sd } X$  is finite.

**Problem 2.15.** *Suppose  $f: M^n \rightarrow X$  is a proper surjection such that all its fibers are closed  $k$ -manifolds and  $\dim X$  is finite. Is  $X$  an **ANR**?* **735.** ?

The above problem is related to work of DAVERMAN and WALSH [1987] (which was the culmination of the previous efforts of Coram-Duvall and Liem) on upper semicontinuous decompositions of a manifold  $M^n$  into closed, connected  $k$ -manifolds. The primary question in that area is: Under what conditions is the decomposition space a generalized  $(n - k)$ -manifold?

? **736. Problem 2.16.** *A map  $f: X \rightarrow Y$  of compacta has the property that for each  $n > 0$  there is a map  $f_n: Y \rightarrow X$  with  $\text{dist}(f \circ f_n, \text{id}_Y) < 1/n$ . Is  $Y$  an **ANR** if  $X$  is an **ANR**?*

Yes, if  $Y \in LC^1$ .

? **737. Problem 2.17.** *Let  $X$  be a homogeneous **ANR** of finite dimension. Is it a generalized manifold?*

? **738. Problem 2.18.** *Is there a homogeneous contractible **ANR** of finite dimension?*

? **739. Problem 2.19.** (J. BRYANT [19 $\infty$ ]) *Suppose  $X$  is an **ANR** of finite dimension and  $k \geq 1$ . Is there a point  $x$  in  $X$  such that  $H_k(X, X - \{x\}; \mathbb{Z})$  is finitely generated?*

? **740. Problem 2.20.** *Suppose  $X$  is an **ANR** such that for some integer  $n$ ,  $H_k(X, X - \{x\})$  is trivial for  $k \neq n$  and  $H_n(X, X - \{x\}) \approx \mathbb{Z}$  for all  $x$  in  $X$ . Is the dimension of  $X$  finite?*

### 3. Shape and strong shape equivalences

An inclusion  $i: A \rightarrow X$  is called a *shape equivalence* if every map  $g$  from  $A$  to a  $CW$  complex  $K$  extends uniquely over  $X$  up to homotopy.

An inclusion  $i$  from  $A$  to  $X$  is called a *strong shape equivalence* if both  $i$  and the inclusion from  $A$  to the double mapping cylinder  $DM(i)$  of  $i$  are shape equivalences.

A map  $f: X \rightarrow Y$  is called a *shape equivalence* (*strong shape equivalence*) if the inclusion from  $X$  to the mapping cylinder  $M(f)$  of  $f$  is a shape equivalence (*strong shape equivalence*).

? **741. Problem 3.1.** *Is every shape equivalence a strong shape equivalence?*

? **742. Problem 3.2.** (J. DYDAK and J. SEGAL [1981b]) *Suppose  $A$  is a closed subset of a compact space  $X$  and  $f$  and  $g$  are maps from  $X$  to  $K$  coinciding on  $A$ , where  $K$  is a  $CW$  complex. If the inclusion from  $A$  to  $X$  is a shape equivalence and  $f|_A = g|_A$ , is  $f$  homotopic to  $g$  rel.  $A$ ?*

**Problem 3.3.** (J. DYDAK and J. SEGAL [1981b]) Suppose  $A$  is a closed subset of a compact space  $X$  and  $f: A \rightarrow Y$  is a shape equivalence. Is the natural projection  $p: X \rightarrow Y \cup_f X$  a shape equivalence? **743. ?**

**Problem 3.4.** Suppose a map  $f: X \rightarrow Y$  is a shape equivalence and  $x$  is a point in  $X$ . Is the map  $f$  from  $(X, x)$  to  $(Y, f(x))$  a pointed shape equivalence? **744. ?**

See DYDAK and GEOGHEGAN [1982, 1986] for partial answers to this problem.

**Problem 3.5.** Let the shape dimension of a compactum  $X$  be finite. Is there a shape equivalence  $f: X \rightarrow Y$  ( $f$  is a map) where the dimension of  $Y$  is finite? **745. ?**

**Problem 3.6.** Suppose  $f: M \rightarrow X$  is a shape domination (i.e.  $fg = id_X$  for some shape morphism  $g: X \rightarrow M$ ) and  $Sd X \geq n$ . Is  $f$  a shape isomorphism? **746. ?**

**Problem 3.7.** Given a space  $X$ , is there a strong shape equivalence  $f$  from  $X$  to  $Y$  such that for any space  $Z$  the natural function from  $[Z, Y]$  (= the set of homotopy classes of maps from  $Z$  to  $Y$ ) to  $SSh(Z, Y)$  (= the set of strong shape morphisms from  $Z$  to  $Y$ ) is a bijection? **747. ?**

The above problem is closely related to a problem of FERRY [1980] on improving compacta. A compactum  $X$  is *improved* (resp. *n-improved*) iff for all compacta  $Z$  (resp. of dimension  $\leq n$ )  $[Z, X] \rightarrow SSh(Z, X)$  is a bijection. In [1980] S. FERRY has given conditions on  $X$  which imply it is shape equivalent to an  $n$ -improved compactum for each  $n$ . S. ZDRAVKOVSKA has shown in [1981] that the wedge of  $S^1$  and the 2-dimensional Hawaiian earring is not shape equivalent to a 1-improved compactum. We do not know if this space is equivalent to an improved space. Finally, Geoghegan and Krasinkiewicz (unpublished) have shown that the Case-Chamberlain continuum is not equivalent to a 0-improved compactum. In case the answer to it is positive one would get that the strong shape category  $SSH$  can be obtained by localizing the homotopy category  $HTOP$  at strong shape equivalences (see DYDAK and NOWAK [19∞] for a more detailed discussion).

**Problem 3.8.** Suppose  $i: X \rightarrow Y$  is the inclusion of continua such that both  $X$  and  $Y/X$  are of polyhedral shape. Is  $i$  a shape equivalence if it induces isomorphisms of all homotopy pro-groups? **748. ?**

**Problem 3.9.** Suppose  $f: X \rightarrow Y$  is a cell-like map and  $\{A_n\}_{n=1}^\infty$  is a sequence of compacta in  $Y$  satisfying the following properties: **749. ?**

- (a)  $f|f^{-1}(A): f^{-1}(A) \rightarrow A$  is a shape equivalence whenever  $A$  is contained in one of the  $A_n$ 's.

(b)  $f^{-1}(y)$  is a one-point set for every  $y \in Y - \bigcup_{n=1}^{\infty} A_n$ .  
Is  $f$  a shape equivalence?

Yes, if  $Y = \bigcup_{n=1}^{\infty} A_n$ .

#### 4. $P$ -like continua and shape classifications

A mapping  $f: X \rightarrow Y$  of a compactum  $X$  onto a compactum  $Y$  is said to be an  $\varepsilon$ -mapping if the sets  $f^{-1}(y)$  have diameters less than  $\varepsilon$  for all  $y$  in  $Y$ . A compactum  $X$  is said to be  $Y$ -like if for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -mapping of  $X$  onto  $Y$ . In [1963] and [1967] S. MARDEŠIĆ and J. SEGAL investigated the notion of  $P$ -like continua where  $P$  is a polyhedron or, in particular, a manifold. The following  $P$ -like continua have been classified up to shape:

- (1) for  $P = S^n$ , MARDEŠIĆ and SEGAL [1971],
- (2) for  $P = P^n$ , real projective  $n$ -space, HANDEL and SEGAL [1973b],
- (3) for  $P$  a finite wedge of Moore spaces of type  $(\mathbf{Z}^p, m)$ , with  $m \geq 3$  and  $p$  an odd prime, HANDEL and SEGAL [1973a],
- (4) for  $P = T^n$ , the  $n$ -dimensional torus, KEESLING [1973] and EBERHART, GORHD and MACK [1974], and
- (5) for  $P = CP^n$ ,  $n$ -dimensional complex projective space, WATANABE [1974].

In contrast to the above cases where the cardinality of the shape classes is infinite, D. HANDEL and J. SEGAL, in [1973a], show that for each positive integer  $n$  there is a polyhedron  $P_{n-1}$  such that the number of shape classes of  $P_{n-1}$ -like continua is exactly  $n$ .

In addition, A. TRYBULEC [1973] classified the movable curves up to shape. L. S. HUSCH [1983] enlarged the class  $B$  of iterated 1-bouquets of 2-manifolds to the class  $B_f$  of all finite connected coverings of elements of  $B$  and showed that some  $B_f$ -like continua have simple shape. A. KADLOF [1977] obtained several results concerning the shape groups, Čech homology and cohomology groups of  $P$ -like compacta.

In [1973a] and [1974] HANDEL and SEGAL show that for a given polyhedron  $P$ , the numeration of the shapes of  $P$ -like continua depends only on the algebraic structure of the semigroup  $[P, P]$  of homotopy classes of maps of  $P$  into itself under composition. So what is required is an algebraic description of shape classification of sequences in semigroups patterned after the ANR-system formulation of shape as in MARDEŠIĆ and SEGAL [1971].

SEGAL [1973] asked the following the question concerning manifold-like continua (as well as problem 4.2 which is related).

**? 750. Problem 4.1.** *Is the cardinality of the set of shape classes of continua which are like a closed manifold  $M$  necessarily uncountable?*

**Problem 4.2.** For a closed manifold  $M$  must the set  $[M, M]$  of homotopy classes of maps of  $M$  into itself be infinite? **751. ?**

KADLOF [1977] raised the following two questions concerning  $P$ -like compacta.

**Problem 4.3.** If  $(X, x_0)$  is a movable  $P$ -like compactum with finitely generated first shape group  $\pi_1(X, x_0)$ , then is  $X$  an FANR? **752. ?**

**Problem 4.4.** If  $X$  is  $P$ -like and the first shape group  $\pi_1(X, x_0)$  is finitely generated is the  $n$ th shape group  $\pi_n(X, x_0)$  countable for  $n = 1, 2, \dots$ ? **753. ?**

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## CHAPTER 28

### Problems on Algebraic Topology<sup>1</sup>

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## 1. Introduction

The following list of problems was collected at the 1986 Arcata Conference on algebraic topology, preceding the International Congress of Mathematicians in Berkeley. The period leading up to this conference was a particularly exciting one in algebraic topology, and it is hoped that these problems, collected by J. F. Adams, W. Browder and myself, will stimulate further advances in the area. I wish to thank all the contributors to the list.

## 2. Problem Session for Homotopy Theory: J. F. Adams

1. (A. Adem) *Let  $G$  be a finite  $p$ -group. If  $H^n(G; \mathbb{Z})$  has an element of order  $p^r$ , for some value of  $n$ , does the same follow for infinitely many  $n$ ?* **754. ?**

Recall that a group  $P$  is *perfect*  $P = [P, P]$ . Let  $PG$  be the maximal perfect subgroup of  $G$ .

2. (J. Berrick) *Consider fibrations  $F \rightarrow E \rightarrow B$  of connected spaces, where  $P\pi_1 E = 1$  and  $F$  is of finite type. If  $H_*(F; \mathbb{Z}) \rightarrow E^+ \rightarrow B^+$  is iso, does it follow that  $\pi_1 B = 1$ ?* **755. ?**

This question arises from the consideration of “plus-constructive fibrations”, i.e., those for which  $F^+ \rightarrow E^+ \rightarrow B^+$  is also a fibration.

3. (F. Cohen) *Let  $[2]: S^n \rightarrow S^n$  be a map of degree 2. When is  $\Omega^q[2]: \Omega^q S^n \rightarrow \Omega^q S^n$  homotopic to the  $H$ -like squaring map  $2: \Omega^q S^n \rightarrow \Omega^q S^n$ ?* **756. ?**

If  $q = 1$ , it is so iff  $n = 1, 3, 7$ . If  $q = 2$ , it can only be so if  $n = 2^k - 1$ ; what happens in this case? What happens for  $q = 3, n = 5$ ?

Call a  $p$ -group  $P$  “*necessary*” if there is a non-zero class  $x \in H^*(P; \mathbb{Z}/(p))$  which restricts to zero on all proper subgroups of  $P$ .

4. (M. Feshbach) *Can one give a useful alternative description of the necessary  $p$ -groups?* **757. ?**

Conjecture: for  $p = 2$ ,  $P$  is necessary iff every element of order 2 is central, i.e., iff  $P$  contains a unique maximal elementary abelian 2-subgroup and this subgroup is contained in the centre of  $P$ . (The obvious generalisation of this conjecture to  $p$  odd is false.)

5. (B. Gray) *For which values  $n, r$  is the fibre  $W_{n,r}$  of the iterated suspension map  $S^n \rightarrow \Omega^r S^{n+r}$ , localised at  $p = 2$ , an  $H$ -space?* **758. ?**

For  $p$  odd one should assume  $n$  even and replace  $S^n$  by the subspace  $S_{p-1}^n$  of the James construction; one then asks if the fibre is a double loop space.

? 759. 6. (B. Gray) Suppose  $\Phi: \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  satisfies

(a)  $\Phi|S^{2np-1}$  has degree  $p$  ( $> 2$ ).

(b) The composition

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H_p} \Omega^2 S^{2np+1} \xrightarrow{\Phi} S^{2np-1}$$

is nullhomotopic.

Is it then true that the composition

$$\Omega^3 S^{2np+1} \xrightarrow{\Omega \Phi} \Omega S^{2np-1} \xrightarrow{\Sigma^3} \Omega^3 S^{2np+1}$$

is the triple loops on the degree  $p$  map?

? 760. 7. (K. Ishiguro) Let  $\pi$  be a finite group and  $G$  a compact Lie group. Is  $[B\pi, BG]$  a finite set? Let  $\pi_p$  run over the Sylow  $p$ -subgroups of  $\pi$ . Is the map

$$[B\pi, BG] \rightarrow \prod_p [B\pi_p, BG]$$

injective?

? 761. 8. (N. Kuhn) Find conditions on  $X$  and  $Y$  such that

$$X_{K(n)} \approx Y_{K(n)} \forall n \geq 1 \text{ implies } X \approx Y.$$

(Here  $K(n)$  is the  $n$ th Morava  $K$ -theory.)

? 762. 9. (N. Kuhn after D.C. Ravenel) Are all suspension spectra  $E$  harmonic (meaning that  $E$  is  $\bigvee_{n \geq 0} K(n)$ -local)?

? 763. 10. (N. Kuhn) Describe the equivariant cobordism  $MU_G^*(pt)$ . Find an equivariant version of Landweber's "exact functor" theorem.

? 764. 11. (N. Kuhn) Set up equivariant  $K(n)^*$ ,  $E(n)^*$  and prove a "completion theorem" for these theories.

? 765. 12. (N. Kuhn) Find good models for the infinite loop spaces representing complex-oriented cohomology theories as  $K(n)^*$ ,  $E(n)^*$ .

Work of Barratt, Mahowald and Jones (The Kervaire invariant and the Hopf invariant, to appear in the Proceedings of the emphasis year in Seattle) leaves the following conjecture open:

**13. Conjecture** (M.E. Mahowald). *If  $\alpha \in \pi_{2^j-2}$  represents a framed manifold with Kervaire invariant one, then the composite* **766. ?**

$$QS^0 \xrightarrow{S} QP \rightarrow \Omega^\infty P \wedge J$$

where  $S$  is the Snaith map, induces a nonzero map on  $\alpha$ .

The conjecture is true if  $\alpha$  has order 2. Also, there are other Kervaire invariant questions in the Proceedings of the Northwestern 1982, Contemporary Mathematics AMS vol. 19.

In numerical analysis it is often very useful to consider differences in a sequence to gain understanding of how the sequence is put together. In topology, differences might be constructed to be fiber of a map which is an isomorphism in the first non-zero dimension of both spaces. Thus, the EHP sequence tells us that, at the prime 2, the difference between  $S^n$  and  $\Omega S^{n+1}$  is  $\Omega S^{2n+1}$ . Let  $W(n)$  be the difference between  $S^{2n-1}$  and  $\Omega^2 S^{2n+1}$ . Work in MAHOWALD [1975] shows that each of the spaces  $W(n)$  has a resolution by  $K(\mathbb{Z}/2, n)$ 's which is a good approximation to the Adams resolution for the stable  $\mathbb{Z}/2$  Moore space. Using the above language, the difference between  $W(n)$  and  $\Omega^4 W(n+1)$ , which we call  $X(n)$ , is constructed and shown to have a resolution which has many properties of the stable resolution of a spectrum whose cohomology is free on one generator over the subalgebra,  $A_1$ , of the Steenrod algebra generated by  $Sq^1$  and  $Sq^2$ . It would be interesting to construct a map from  $X(n)$  to the omega spectrum having as its stable cohomology an appropriate  $A_1$ . To do this at the chain level in the  $\Lambda$ -algebra sense, as was done in Mahowald's paper above, would already be very interesting.

**14. Conjecture** (M. E. Mahowald). *The difference between the  $X(n)$ 's approximates  $A_2$  and the differences between these differences approximates  $A_3$ , etc.* **767. ?**

**15.** (M. E. Mahowald) *Let  $C_n$  be the fiber of  $\Omega^{2n+1} S^{2n+1} \rightarrow \Omega^\infty P^{2n} \wedge J$ . Find a stable space  $B_n$  and a map  $C_n \rightarrow QB_n$  which is a  $v_2$  equivalence.* **768. ?**

$B_n$  could be a subspace of the space  $L(2)$  of Mitchell and Priddy.

**16.** (M. E. Mahowald) *The work of Devinatz, Hopkins and Smith suggests that the Freyd generating hypothesis is now approachable.* **769. ?**

Let  $G$  be a finite  $p$ -group,  $X$  a finite  $G$ -complex,  $SX$  its singular subspace. If  $G$  has order  $p^e$ , then

$$\Sigma(p^e - 1)^i \dim H^i(SX, \mathbb{Z}_p) \leq \Sigma(p^e - 1)^i \dim H^i(X, \mathbb{Z}_p).$$

? 770. 17. (J .P. May) *Are there other such generalizations of Smith theory?*

The point is that the homotopical structure of finite  $G$ -complexes is much more restricted than Smith theory alone dictates.

? 771. 18. (J. P. May) *Give an algebraic analysis of the rationalized stable category of  $G$ -spaces.*

Rational  $G$ -spectra fail to split as products of Eilenberg-MacLane  $G$ -spectra for general compact Lie groups  $G$ , although this does not hold for finite  $G$ .

? 772. 19. (J. P. May) *Let  $H \subseteq J \subseteq K \subseteq G$ , where  $G$  is a compact Lie group. Suppose  $q(H, p) = q(K, p)$ . Is  $q(J, p) = q(H, p)$ ?*

This is one of many questions involving the complexity of the lattice of closed subgroups of  $G$ .

Let  $B_G\Pi$  be the classifying  $G$ -space for principle  $(G, \Pi)$ -bundles. For a  $G$ -cohomology theory  $h_G^*$ ,  $h_G^*(B_G\Pi)$  gives all  $h_G^*$ -characteristic classes for  $(G, \Pi)$ -bundles.

? 773. 20. (J. P. May) *Calculate these groups in interesting cases.*

When  $h_G^*$  is Borel cohomology, complete information is easily obtained and is quite unilluminating. When  $h_G^*$  is stable cohomology or  $K$ -theory, a complete theoretical answer has been obtained (Adams, Haeberly, Jackowski, May, 1985) via generalizations of the Segal conjecture and the Atiyah-Segal completion theorem. When  $\Pi$  is Abelian,  $B_G\Pi \approx B\Pi \times K(R, 0)$ ,  $R(G/H) = \text{Hom}(H, \Pi)$ , and the Bredon cohomology of  $B_G\Pi$  is computable. This casts doubt on methods based on reduction to a maximal torus.

The Borel construction on  $(G, \Pi)$ -bundles corresponds to a  $G$ -map

$$\alpha: B_G\Pi \rightarrow \text{Map}(EG, B\Pi).$$

When  $G$  and  $\Pi$  are discrete or when  $G$  is compact Lie and  $\Pi$  is Abelian compact Lie,  $\alpha$  is a  $G$ -homotopy equivalence. When  $G$  is a finite  $p$ -group and  $\Pi$  is a compact Lie group, Dwyer's results show, essentially, that  $\alpha$  is a  $p$ -adic equivalence.

? 774. 21. (J. P. May) *What can be said when  $G$  is a general finite group?*

? 775. 22. (F. Quinn) *Get information about the space of based maps*

$$\text{Map}_0(B_\pi, B_{G(X)}),$$

where  $\pi$  is a finite group and  $G(X)$  is the monoid of self-equivalences of the finite complex  $X$ .

**23.** (F. Quinn) *Get information about*

**776.** ?

$$\mathrm{Map}(B_\pi, E_{G(X)} \times X/G(X))$$

(For background and discussion of (22), (23) see QUINN [19∞].)

Let  $A$  be the ring of integers in a finite extension of the  $p$ -adic numbers  $\mathbb{Q}_p$ . A *formal  $A$ -module*  $F$  over an  $A$ -algebra  $R$  is a formal group law over  $R$  equipped with power series  $[a](x)$  for each  $a \in A$  satisfying

- (i)  $[1](x) = x$ ;
- (ii)  $[a_1 + a_2](x) = F([a_1](x), [a_2](x))$ ;
- (iii)  $[a](x) \equiv ax \pmod{x^2}$ .

There is a well developed theory of such sets beginning with Lubin-Tate's work on local class field theory. In particular there is a universal  $p$ -typical formal  $A$ -module defined over an  $A$ -algebra  $V_A$  which is explicitly known and which is a  $BP*$ -module.

**24.** (D. C. Ravenel) *Is there a spectrum  $S_A$  such that  $BP*(S_A) = V_A$ ?*

**777.** ?

If the answer is yes then the Novikov  $E_2$ -term for  $S_A$  has many interesting properties.  $S_A$  is known to be an Eilenberg-MacLane spectrum when the field is algebraically closed. The spectrum  $S_A \wedge BP$  has been announced in many cases by A. Pearlman.

$T(n)$  is a spectrum with  $BP*T(n) = BP*[t_1, t_2 \cdots t_n]$ . It figures in the proof of the nilpotence theorem.

It is not difficult to compute the Adams-Novikov  $E_2$ -term for  $T(n)$  through dimension  $2(p^2 + 1)(p^{n+1} - 1)$ . It is roughly this dimension that the first possible nontrivial differential occurs. In the case  $n = 0$  this is the Toda differential which kills  $\alpha_1\beta^p$ , for  $p > 2$ .

**25.** (D. C. Ravenel) *For  $n > 0$ , is this differential nontrivial?*

**778.** ?

The usual extended power constructions do not settle the question.

For  $p > 2$ ,  $\beta_1 \in \pi_{2p^2-2p-2}^S$  is the first even dimensional stable homotopy element in positive dimensions.

**26.** (D. C. Ravenel) *Find the smallest  $k$  such that  $\beta_1^k = 0$ .*

**779.** ?

Toda showed (< 1970?) that  $k \leq p^2 - p + 1$  for all  $p > 2$ .

For  $p = 3$ ,  $k = 6$ , (*Toda*)

for  $p = 5$ ,  $k = 18$ .

The relevant ANSS differentials

$$\begin{aligned} \text{are } d_9(\alpha\beta_4) &= \beta_1^6 & (p=3), \\ \text{and } d_{33}(\gamma_{33}) &= \beta_1^{18} & (p=5). \end{aligned}$$

Conjecture:

$$\begin{aligned} \text{For } p=7, \quad \beta_{139} &= \gamma_3\gamma_2, \\ p=11, \quad \beta_1^{105} &\in \langle \gamma_3, \gamma_2, \gamma_2, \gamma_2 \rangle \end{aligned}$$

etc., and these lead to  $\beta^{p^2-p} = 0$  for  $p > 5$ .

For a finite group  $G$ , the Segal Conjecture tells us that the stable cohomotopy group  $[BG, S^0]$  is the completed Burnside ring of  $G$ ,  $A(G)^\wedge$ . Let  $L_n S^0$  denote the Bousfield localization of  $S^0$  with respect to  $\nu_n^{-1}BP$ . Let  $F_n A(G)^\wedge$  denote the kernel of the map

$$A(G)^\wedge = [BG, S^0] \rightarrow [BG, L_n S^0].$$

The problem is to determine the ideals  $F_n$ .

? **780. 27. Conjecture** (D. C. Ravenel). *A virtual finite  $G$ -set  $X$  is in  $F_n$  if  $\#(X^H) = 0$  for all subgroups  $H \subseteq G$  generated by  $\leq n$  elements.*

? **781. 28.** (D. C. Ravenel) *Find a way to compute  $K(n) * \Omega^k \Sigma^k X$ , either as a functor of  $K(n) * X$  or by showing that a suitable Eilenberg-Moore spectral sequence converges.*

McClure has done this for  $n=1$  and  $k=\infty$ . For  $n=1$  and  $k<\infty$  one could then prove that the Smith map

$$\Omega_0^{2n+1} S^{sn+1} \rightarrow QRP^{2n}$$

is a  $K(1)_*$ -equivalence.

? **782. 29.** (P. Shick) *Relate the  $v_n$ -torsion or  $v_n$ -periodic behaviour of  $\alpha \in [X, S^0]_j$  to that of its root invariant  $R(\alpha) \in [X, S^0]_{j+N}$  in the sense of “Implications of Lin’s theorem in stable and unstable homotopy theory”, M. E. Mahowald and D. C. Ravenel, to appear.*

(M. E. Mahowald and P. Shick have made some progress with this since the Arcata conference.)

### 3. $H$ -spaces

#### 3.1. $N$ . $Iwase$

? **783.** A) *Determine the higher associativity of the pull-back of a sphere extension of a Lie group by the degree  $k$  mapping.*

B) Find the example of a space which admits an  $A_n$ -structure but no  $A_n$ -primitive  $A_n$ -structure. **784. ?**

This is unsolved even if  $n = 2$ .

C) Is there a three-connected homotopy associative  $H$ -space? **785. ?**

D) When does the Bar construction functor induce a weak equivalence from the space of homeomorphisms between compact Lie groups to the mapping space between their classifying spaces? **786. ?**

This is not always true and not always false.

E) Justify the equivariant theory (homotopy theory, simple homotopy theory, algebraic  $K$ -theory, etc.) for non-compact Lie groups, or make clear the essential obstruction. **787. ?**

3.2. *J. P. Lin*

A) Prove a 14-connected finite  $H$ -space is acyclic. **788. ?**

B) Suppose  $f: Y \rightarrow Z$  factors as **789. ?**

$$Y \xrightarrow{(\overline{\Delta} \wedge 1) \overline{\Delta}} Y \wedge Y \wedge Y \longrightarrow Z,$$

where  $Y$  is an  $H$ -space. If  $X$  is the fibre of  $f$ , does  $X$  split as  $\Omega Y \times \Omega^1 Z$  as homotopy commutative  $H$ -spaces?

C) Are there any finite loop spaces whose mod 2 cohomology is not the mod 2 cohomology of a Lie group? **790. ?**

C) Suppose  $X$  is a 1-connected finite  $H$ -space and  $A = H^*(X; \mathbb{Z}_2)$  is the corresponding cohomology Hopf algebra over  $A(2)$ . Are there “irreducible” Hopf algebras over  $A(2)$  such that  $A$  splits as the tensor product of irreducible Hopf algebras over  $A(2)$ ? **791. ?**

E) Can a finite loop space  $\Omega B$  have **792. ?**

$$H^*(B; \mathbb{Q}) = Q[X_{n_1}, \dots, X_{n_r}]$$

where  $[n_1, \dots, n_r] = [4, 4, 48, 8, 812, 12, 16, 16, 20, 24, 24, 28]$ ?

(This is an example of Adams-Wilkerson.)

F) Given  $X$  a 1-connected finite  $H$ -space, what can be said about the action **793. ?**



of  $A(2)$  on  $PH^*(\Omega X; \mathbb{Z}_2)$ ?

? 794. G) Is a 6-connected finite  $H$ -space a product of seven spheres?

? 795. H) If  $X$  is a finite loop space is  $H^*(A; \mathbb{Z}) = H^*(\text{Lie grp}; \mathbb{Z})$ ?

#### 4. $K$ and $L$ -theory

4.1. *J. Milgram, L. Taylor, B. Williams*

The Quinn-Ranicki assembly map has been attacked by a factorization which we can define on bordism as follows. There are pairings

$$L_*(\mathbb{Z}[s]\pi) \otimes C \rightarrow L_{*+1\mp 1}(\mathbb{Z}\pi) \quad (*)$$

where  $C^\pm$  is the knot group of  $S^{4k\mp 1} \rightarrow S^{4k\pm 1}$  ( $k \geq 2$ ) and the involution on  $\mathbb{Z}[s]$  is  $s \rightarrow 1 - s$ . Moreover, there is a “symmetric signature”  $\Omega_*(B\pi) \rightarrow L_*(\mathbb{Z}[s]\pi)$  such that the composite of the “symmetric signature”  $\otimes \{\text{trefoil knot}\}$  is the obstruction to  $M \times$  (Kervaire problem). The  $E_8$  knot  $S^3 \rightarrow S^5$  gives the obstruction to  $M \times$  (Milnor problem).

? 796. A) Explain the presence of the knot group in the pairings  $(*)$ .

In dimension 3 surgery can be done on homology though we don’t have control of  $\pi_1$  of the resulting manifold. For example, results of Madson and Milgram on the spaceform problem for the groups  $Q(8p, q, 1)$ ,  $(p, q) = 1$ ,  $p, q$  prime, show that for certain pairs such as  $(17, 133)$ , the surgery obstruction is trivial. Hence there exists a *homology 3-sphere*  $M^3$  with a free action of  $Q(136, 113, 1)$ .

? 797. B) Find explicit examples of such actions. In particular what kinds of  $\pi_1$  occur for  $M^3$ ?

If  $\pi$  is finite, then the reduced surgery obstruction map

$$\bar{\sigma}: \Omega_n(B\pi \times G/\text{TOP}) \rightarrow L'_n(\mathbb{Z}\pi)/L_n(\mathbb{Z})$$

factors through  $\bigoplus_{i=1}^4 H_i(\pi, \mathbb{Z}/2)$ . Here “ $'$ ” denotes

$$\ker(Wh_1(\mathbb{Z}\pi) \rightarrow Wh_1(\mathbb{Q}\pi)).$$

? 798. C(a) Does  $\bar{\sigma}$  factor through  $\bigoplus_{i=1}^3 H_i(\pi; \mathbb{Z}/2)$ ?

? 799. C(b) Determine the corresponding  $L^s$ -results.

? 800. D) Understand the relationship between  $L$ -theory and Hermitian  $K$ -theory (in the sense of Quillen).

**Conjecture:** For any ring with involution  $(R, \alpha)$  and  $\epsilon = \pm 1$ , there exists a homotopy fibration

$$\Omega^\infty(S^\infty_{+\mathbb{Z}/2} K(R)) \xrightarrow{\tilde{H}} K \operatorname{Herm}(R, \alpha, \epsilon) \rightarrow \underline{L}^*(R, \alpha, \epsilon),$$

where  $\widehat{\mathbb{Z}/2}$  denotes  $\mathbb{Z}/2\mathbb{Z}$ -homotopy orbits and  $\underline{L}$  denotes symmetric  $L$ -theory. Karoubi periodicity implies this conjecture is true when  $\frac{1}{2} \in R$ . The hyperbolic map  $H: \underline{K}(R) \rightarrow \underline{K} \operatorname{Herm}(R, \alpha, \epsilon)$  factors through  $\tilde{H}$ .

#### 4.2. V. Snaith

A finite dimensional representation of a finite Galois group,  $G(L/K)$ , where  $L/K$  is a local field extension is called a *Galois representation*. The Deligne-Langlands local constants are homomorphisms  $W_K: R(G(L/K)) \rightarrow S^1$ . (See Tate: Proceedings of the Durham conference (1977), editor A. Frölich.)

I have a general formula for  $W_K(\rho)$  which is partially topological, partially number theoretic, and very complicated. However, if  $\rho: G(L/K) \rightarrow U_n(\mathbb{C})$  is the complexification of  $\rho': G(L/K) \rightarrow O_n(\mathbb{R})$ , DELIGNE [1976] showed that

$$W_K(\rho) = \operatorname{SW}_2[\rho'] \cdot W_K(\det \rho) \in \{\pm 1, \pm \sqrt{-1}\}$$

where  $\operatorname{SW}_2[\rho] \in H^2(K; \mathbb{Z}/2) \cong \{\pm 1\}$ . Hence on  $\operatorname{RSO}(G(L/K))$ ,  $W_K$  is  $\operatorname{ISO}(G(L/K))$ -adically continuous, in fact  $W_K(\operatorname{ISO}(G(L/K)))^3 = \{1\}$ . On  $\operatorname{RSp}(G(L/K))$ , the symplectic representation ring,  $W_K$  is  $\{\pm 1\}$ -valued.

**Question.** Is  $W_K$  trivial on some  $\operatorname{IO}(G(L/K))^N \cdot \operatorname{RSp}(G(L/K))$ ?

801. ?

## 5. Manifolds & Bordism

#### 5.1. P. Gilkey

The eta invariant of Atiyah-Patodi-Singer defines maps

$$\eta: \tilde{K}(S^{2k-1}/G) \oplus \tilde{K}(S^{2n-1}/G) \rightarrow \mathbb{Q}/\mathbb{Z}$$

$$\eta: \tilde{\Omega}_*^U(BG) \oplus R_0(G) \oplus R(U) \rightarrow \mathbb{Q}/\mathbb{Z}$$

where  $G$  is a spherical space form group,  $R_0(G)$  is the augmentation ideal of the representation ring of  $G$ , and  $R(U)$  is the representation ring of the unitary group. The first is a perfect pairing and the second is non-singular in the first factor—i.e., the eta invariant completely detects the  $K$ -theory of spherical space forms and equivariant unitary bordisms of spherical space form groups.

**Question 1.** What is the situation for constant curvature 0 or constant **802. ?**

curvature  $-1$ ? Why is the eta invariant so successful in this setting?

The second map can also be interpreted as giving a map

$$\eta: \tilde{\Omega}_*^U(BG) \otimes R(U) \rightarrow [bu_*(BG)]$$

For  $G = \mathbb{Z}_p$  ( $p$ -prime) it is well-known

$$\tilde{\Omega}_*^U(B\mathbb{Z}_p) \simeq bu_*(B\mathbb{Z}_p) \oplus \mathbb{Z}[X_4, X_6, \dots].$$

We conjecture for spherical space form groups an additive splitting.

? **803. Question 2.**  $\tilde{\Omega}_*^U(BG) = bU_*(BG) \oplus \mathbb{Z}[X_4, X_6, \dots]$  and have proved it for  $G = \mathbb{Z}_4$ ,  $G = \{\pm 1, \pm i, \pm j, \pm h\}$ . The proof of this additive isomorphism is analytic; one wants a topological proof for the spherical space form groups in general.

5.2. *M. Kreck and S. Stolz*

? **804. Problem.** Suppose that  $G$  and  $G'$  are compact simple Lie groups. Is it true that homeomorphic homogeneous spaces  $G/H$  and  $G'/H'$  are diffeomorphic?

(If  $G$  and  $G'$  are not simple there are counterexamples (KRECK and STOLZ [1988]) giving a negative answer to the corresponding general problem posed by W. C. and W. Y. Hsiang in 1966.)

5.3. *M. Kreck, A. Libgober, and J. Wood*

? **805. Problem.** Is the diffeomorphism type of a complete intersection  $X_n$  in  $\mathbb{C}P_{n+r}$  determined by its dimension, total degree, intersection pairing or Arf invariant (undefined or  $\pm 1$ ), and Stiefel-Whitney and Pontrjagin classes?

5.4. *A. Libgober and J. Wood*

? **806. Problem.** Is a compact Kähler manifold which is homotopy equivalent to  $\mathbb{C}P_n$  necessarily analytically equivalent to  $\mathbb{C}P_n$ ?

[Yes  $n = 2$  (Yau),  $n = 4, 6$  (Libgober-Wood).]

## 6. Transformation Groups

### 6.1. W. Browder

Under certain circumstances, I have shown that equivariant homotopy equivalence implies isovariant homotopy equivalence, in particular with a strong “gap” hypothesis that

- (a)  $\dim M^H < \frac{1}{2} \dim M^K$ , and
- (b)  $(\dim M^H, \dim M^K) \neq (1, 4)$

for every pair of isotropy groups  $K \subseteq H$ , where  $G$  is finite, acting  $PL$ . Weakening condition (a), tom Dieck and Löffler have shown that a linking invariant between fixed point sets can occur which is an obstruction to equivalence.

**Problem (i)** Describe the first non-trivial obstructions for this problem, and can they be expressed as linking phenomena? 807. ?

**Problem (ii)** Remove the  $(1, 4)$ -condition (b). 808. ?

**Problem (iii)** If  $G = \prod^k \mathbb{Z}_p$  acts freely on  $S^{n_1} \times \dots \times S^{n_\ell}$ , is  $k \leq \ell$ ? (Classical) 809. ?

If  $n_1 = \dots = n_\ell$ , this is known by work of CARLSSON [1982] when  $G$  acts trivially on homology, and ADEM and BROWDER [1988] when  $n_i \neq 1, 3, 7$  or  $p \neq 2$ .

### 6.2. M. Morimoto

Let  $G$  be a compact Lie group (possibly a finite group). In the following we treat only *smooth* actions. If a  $G$ -manifold has exactly one  $G$ -fixed point, then the action is called a one fixed point action. E. STEIN [1977], T. PETRIE [1982], E. LAITINEN and P. TRACZYK [1986] and M. MORIMOTO [19∞a, 19∞b] studied one fixed point actions on spheres. We know that  $S^6$  and  $S^7$  and some higher dimensional spheres have one fixed point actions. Recently M. FURETA [19∞] showed that  $S^4$  does not have one fixed point orientation preserving actions of finite groups, by observation of a moduli space of self-dual connections of some principal  $SO(3)$ -bundle over  $S^4$ , and his idea originates from Donaldson’s work. Assuming Furuta’s result, M. Morimoto showed that  $S^4$  does not have one fixed point actions (MORIMOTO [19∞b]).

**Problem A.** Does there exist a one fixed point action on  $S^3$ ? 810. ?

We note that some 3-dimensional homology spheres have one fixed point actions of  $A_5$ , the alternating group on five letters.

? 811. **Problem B.** *Does there exist a one point fixed point action on  $D^4$ ?*

? 812. **Problem C.** *Does there exist a one fixed point action on  $S^5$ ?*

? 813. **Problem D.** *Does there exist a one fixed point action on  $S^8$ ?*

From the results of R. OLIVER [1979] and W.-Y. HSIANG and E. STRAUME [1986], it holds that  $S^8$  does not have one fixed point actions of compact connected Lie groups.

In the above,  $S^n$  means the standard  $n$ -dimensional spheres and  $D^4$  means the standard 4-dimensional disk.

**Note:** Recently, S. Demichelis has shown that a finite group acting locally linearly and preserving orientation on a closed  $\mathbb{Z}$ -homology 4-sphere has fixed point set a sphere.

### 6.3. R. Schultz: Problems on low-dimensional group actions

? 814. A) *Let  $M^4$  be a closed topological 4-manifold with a topological circle action. Is the Kirby-Siebenmann invariant of  $M$  trivial?*

**Comments:** Work of Kwasik and Schultz shows that topological  $S^1$ -manifolds satisfy many of the same global restrictions as in the smooth category. Also, the answer is yes for free circle actions. Both of these suggest the answer is yes in general. Finally, the answer to this question will be yes in many cases if the same is true for the next question.

? 815. B) *Is every topological circle action on a 4-manifold concordant to a smooth action?*

This is related to Problem 6.9 in the list of problems in the proceedings of the Boulder Conference on Group Actions (A.M.S. Contemporary Mathematics Vol. 36, p. 544).

? 816. C) *Let  $M^4$  be the closed manifold homotopy equivalent but not homeomorphic to  $CP^2$ . Does  $M^4$  admit a nontrivial involution?*

**Comments:** Work of Kwasik and Vogel shows that there are no locally linear involutions, but for  $k$  odd there is a rich assortment of  $\mathbb{Z}_4$  actions (references are given below).

? 817. D) *Classify all 4-dimensional  $h$ -cobordisms up to homeomorphism or diffeomorphism. In particular, if  $\pi$  is the fundamental group of a closed 3-manifold  $M^3$ , which elements of the Whitehead torsions of  $h$ -cobordisms with one end equal to  $M$ ?*

**Comments:** The results of Freedman have led to new results in this direction when  $\pi$  is small, including the existence of exotic  $s$ -cobordisms (see CAPPELL and SHANESON [1985], KWASIK [1986b] and forthcoming work of Kwasik and Schultz). However, our overall understanding is far from complete. For  $\pi$  finite the realization question is connected to the existence of nonlinear free finite group actions on homotopy 3-spheres (KWASIK [1986a]). Here is a related question.

E) Which elements of the projective class group can be realized as the finite-ness obstructions for tame ends of topological 4-manifolds? **818. ?**

#### 6.4. Tammo tom Dieck

A classical theorem of Jordan about finite subgroups  $G$  of  $O(n)$  states: There exists an integer  $j(n)$ , independent of  $G \subseteq O(n)$ , such that  $G$  has an Abelian normal subgroup  $A$  with  $|G/A| < j(n)$ . The following conjecture would be a homotopical generalization.

**Conjecture A:** Given a natural number  $n$ . Let  $X$  be an  $n$ -dimensional homotopy representation of the finite group  $G$  with effective action. Then there exists an integer  $J(n)$  such that  $G$  has an Abelian normal subgroup  $A$  with  $|G/A| < J(n)$ . **819. ?**

(TOM DIECK and PETRIE [1982]).

The equivariant finiteness obstruction yields a homomorphism from the Picard group of the Burnside ring to projective class groups

$$s: \text{Pic } A(G) \rightarrow \prod_{(H) \subseteq G} \tilde{K}_0(\mathbb{Z}NH/H).$$

The definition of  $s$  uses the geometric definition of  $\text{Pic } A(G)$  as a subgroup of the homotopy representation group.

**Problem B:** Give an algebraic definition of  $s$ . **820. ?**

This should be a generalized Swan homomorphism (= boundary in a Mayer-Vietoris sequence) and would require a  $K_1$ -definition of  $\text{Pic } A(G)$  (TOM DIECK [1985]).

**Problem C:** Give a classification of 3-dimensional homotopy representations. **821. ?**

## 7. K. Pawalowski

Let  $G$  be a compact Lie group whose identity connected component  $G_0$  is abelian (i.e.,  $G_0$  is either a trivial group or a torus  $T^k$  with  $k \geq 1$ ) and assume that the quotient group  $G/G_0$  has a normal, possibly trivial, 2-Sylow subgroup. If  $G$  acts smoothly on  $M = D^n$  or  $\mathbb{R}^n$ , there is the following restriction on the set  $F$  of points in  $M$  left fixed by  $G$ . Namely  $F$  is a stably complex manifold in the sense that  $F$  has a smooth embedding into some Euclidean space such that the normal bundle of the embedding admits a complex structure; cf. EDMONDS and LEE [1975]. In particular,  $F$  is orientable and all connected components of  $F$  are either even or odd dimensional.

Assume further that either (i)  $G/G_0$  is of prime power order or (ii)  $G/G_0$  has a cyclic subgroup not of prime power order. In case (i), it follows from Smith Theory that  $F$  is  $\mathbb{Z}$ -acyclic when  $G = G_0 = T^k$  with  $k \geq 1$ , and  $F$  is  $\mathbb{Z}_p$ -acyclic when  $|G/G_0| = p^a$  with  $p$  prime,  $a \geq 1$ . In case (ii), if  $M = D^n$ , Oliver's work implies that  $\chi(F) \equiv 1 \pmod{n_G}$ , where  $n_G$  is the Oliver integer of  $G$ . It turns out that for  $G$  as above, these restrictions on  $F$  are both necessary and sufficient for a compact smooth manifold  $F$  (resp., a smooth manifold  $F$  without boundary) to occur as the fixed point set of a smooth action of  $G$  on a disk (resp., Euclidean space). This raises the question which smooth manifolds can occur as the fixed point sets of smooth actions of  $G$  on disks (resp., Euclidean spaces) for other compact Lie groups  $G$ . In particular, the following related problems are still unsolved.

- ? 822. **Problem A.** *Let  $G$  be a compact Lie group such that  $G/G_0$  is not of prime power order but each element of  $G/G_0$  has prime power order. Is there a smooth action of  $G$  on a disk (resp., Euclidean spaces) such that the fixed point set is not a stably parallelizable manifold?*
- ? 823. **Problem B.** *Is there a compact Lie group  $G$  which can act smoothly on a disk (resp. Euclidean space) with fixed point set  $F$  consisting both of even and odd dimensional connected components? Can some of them be nonorientable manifolds? In particular, can  $F$  be the disjoint union of a point, a circle, and the closed (resp., open Möbius band)?*
- ? 824. **Problem C.** *Is there a compact Lie group  $G$  such that each component smooth manifold (resp., each smooth manifold without boundary) can occur as the fixed point set of a smooth action of  $G$  on a disk (resp., Euclidean space)?*

### Comments.

**Ad. 1.** If such a  $G$  acts smoothly on  $D^n$  or  $\mathbb{R}^n$ , then at any two fixed points, the representations of  $G$  are equivalent. In particular, each fixed point set connected component has the same dimension; cf. PAWALOWSKI [1984].

**Ad. 2.** According to the above discussion, if such a finite group  $G$  exists,  $G$  has a cyclic subgroup not of prime power order and  $G$  is an even order group whose 2-Sylow subgroup is not normal.

**Ad. 3.** Again, according to the above discussion, if such a finite group  $G$  exists,  $G$  is as in Ad. 2 and  $n_G = 1$  in the case of smooth actions of  $G$  on disks.

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## CHAPTER 29

### Problems in Knot Theory

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## 0. Introduction

This paper is an introduction to knot theory through a discussion of research problems. Each section (there are eleven) deals with a specific problem, or with an area in which problems exist. No attempt has been made to be either complete or particularly balanced in the composition of these problems. They reflect my combinatorial bias, and my conviction that many problems in graph theory (such as the four color problem) are really problems in the theory of knots.

In this sense the theory of knots goes beyond topology into the combinatorial structures that underpin topology. In the same sense, knot theory is also deeply related to contexts in theoretical physics, and we have touched on some of these connections, particularly in relation to the Jones polynomial and its generalizations.

Knot theory had its inception in a combinatorial exercise to list all possibilities for vortex atoms in the aether. It has always lived in the multiple worlds of combinatorics, topology and physics. This is every bit as true as it was a century ago. And the plot thickens!

I shall let the problems speak for themselves. Earlier problems introduce information and terminology that occurs (with appropriate reference) in the later problems. In retrospect, a few fascinating classes of problems have not been touched here, so I shall mention them in this introduction. They are the problems of the understanding of frictional properties of knots (give a good mathematical model for it), understanding knotted orbits in dynamical systems, understanding physical configurations of knots and links under various conditions (tensions, fields,  $\dots$ ), and the applications of knot theory and differential geometry to chemistry and molecular biology.

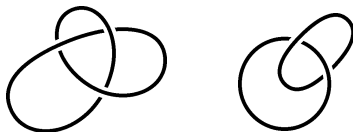
It gives me great pleasure to thank the following people for very helpful conversations: G. Spencer-Brown, John Conway, Louis Crane, Jim Flagg, Ivan Handler, Vaughan Jones, Carol Mariani, Ken Millett, Kunio Murasugi, Mario Rasetti, and Dan Sandin.

## 1. Reidemeister Moves, Special Moves, Concordance

For our purposes, a *knot* is a differentiable embedding of a circle into three dimensional space, and a *link* is an embedding of a collection of circles. Two links are said to be *ambient isotopic*, if there is a continuously parametrized (over the interval  $[0, 1]$ ) family of such embeddings so that the first link corresponds to the parameter value 0, and the second link corresponds to the parameter value 1.

Any link may be projected to a plane in three-space to form a link diagram. The link diagram is a locally 4-valent planar graph with extra structure at the vertices of this graph indicating which segment of the diagram crosses over

the other in the three dimensional embedding. The usual convention for this information is to indicate the undercrossing line by drawing it with a small break at the crossing. The over-crossing line goes continuously through the crossing and is seen to cleave the under-crossing line:

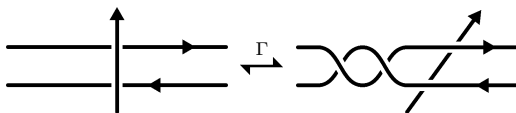


These diagrams can be used to formulate a purely combinatorial theory of links that is equivalent to the theory of link embeddings up to ambient isotopy in three dimensional space. The combinatorial theory is based on the Reidemeister moves (REIDEMEISTER [1948]). (See Figure 1.) These moves (along with the topological moves on the 4-valent planar graph underlying the link diagram) generate ambient isotopy for knots and links in three dimensional space. Two diagrams are related via a sequence of Reidemeister moves if and only if the link embeddings that they represent in three-space are ambient isotopic. (See BURDE and ZIESCHANG [1986] for a modern proof of this fact.)



Figure 1: Reidemeister Moves

One can add extra moves to the Reidemeister moves, thereby getting larger equivalence classes and (in principle) invariants of ambient isotopy. For example, consider the following *switch move* (called the Gamma move in KAUFFMAN [1983]) on oriented diagrams:

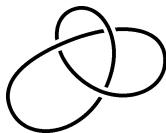


(An orientation of a diagram consists in assigning a direction of travel to each link component. This is indicated by arrows on the diagram.)

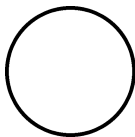
In KAUFFMAN [1983] it is shown that the equivalence relation generated by the Reidemeister moves plus the switch (call it *pass equivalence*) has exactly

two equivalence classes for knots. (There is a similar statement for links but I shall not go into it here.)

The trefoil knot



represents one class, and the trivial knot



represents the other class.

Switch equivalence is interesting because *every ribbon knot is pass equivalent to the unknot*.

A *ribbon knot* is a knot that bounds a disk immersed in three space with only ribbon singularities. A *ribbon singularity* consists in a transverse intersection of two non-singular arcs from the disk: One arc is interior to the disk; one arc has its endpoint on the boundary of the disk. Examples of ribbon knots are shown in Figure 2.

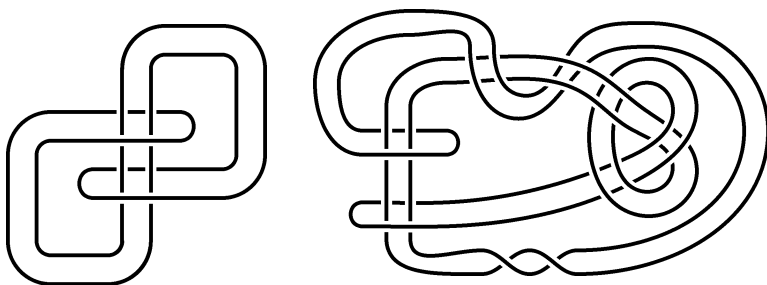


Figure 2: Ribbon Knots

In the diagram of a ribbon knot a sequence of switches can be used to remove all the ribbon singularities. Thus a ribbon knot is pass equivalent to an unknot. Since the trefoil knot is not pass equivalent to the unknot, we conclude that the trefoil is not ribbon.

The interest in the problem of ribbon knots lies in the fact that *every ribbon knot is slice*. That is, every ribbon knot bounds a smoothly embedded disk in the upper four space  $H^4$  (if the knots and links are in the Euclidean space  $\mathbb{R}^3$ , then  $H^4 = \mathbb{R}^+ \times \mathbb{R}^3$  where  $\mathbb{R}^+ = \{r : r \geq 0, r \text{ a real number}\}$ .) A knot that bounds a smoothly embedded disk in upper four space is called a *slice knot*.

(FOX and MILNOR [1966]). We would really like to characterize slice knots. In fact, it remains an open question:

? 825. **Problem 1.** *Is every slice knot a ribbon knot?*

The problem of detecting slice knots is very deep, with distinct differences between the case in dimension three and the generalizations to higher dimensional knots. The most significant work on the slice problem in dimension three, definitively discriminating it from higher dimensions, is due to CASSON and GORDON [1978]. While their invariants detect significant examples of non-slice knots that are algebraically slice (from the viewpoint of the Seifert pairing), the method is in general very difficult to apply. Thus: One would like to have new and computable invariants to detect slice knots. (See Problem 2 below.)

The appropriate equivalence relation on knots and links for this matter of slice knots is the notion of *concordance*. Two knots are said to be *concordant* if there is a smooth embedding of  $S^1 \times I$  into  $S^3 \times I$  with one knot at one end of the embedding and the other knot at the other end. A slice knot is concordant to the unknotted circle. (The embedding of the slice disk can rise only a finite height into four-space by compactness. Locate a point of maximal height and exercise a small disk. This produces the concordance.)

Concordance is generated by the Reidemeister moves, in conjunction with the passage through saddle point singularities, and the passage through minima and maxima. A minimum connotes the birth of an unknotted circle, and a maximum connotes the death of an unknotted circle.

Of course, the entire history of the concordance is constrained to trace out an annulus ( $S^1 \times I$ ) embedded in the four-space. It is this constraint that makes the subject of knot and link concordance so difficult to analyze. It is easy to construct slice knots, but very hard to recognize them!

Later, we shall raise this question of slice knots and behaviour under concordance with respect to various invariants such as the Jones polynomial (JONES [1985]). The question is:

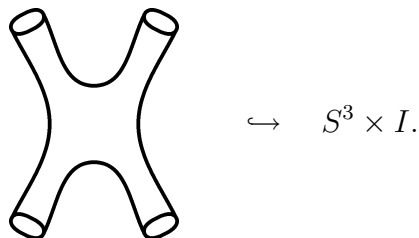
? 826. **Problem 2.** *Are there any new and simple invariants of concordance?*

It is possible that we are overlooking the obvious in this realm.

## 2. Knotted Strings?

String Theory is usually formulated in dimensions that forbid the consideration of knots. We can, however, imagine string-like particles tracing out world sheets in four dimensional spacetime. A typical string vertex will then be an

embedding of a sphere with four holes in  $S^3 \times I$  so that two holes are in  $S^3 \times 0$  and two holes are in  $S^3 \times 1$ :



Just as with knot concordance (Section 1), the embedding can be quite complex, and this complexity will be indexed by the appearance of singularities in the hyperspace cross sections  $S^3 \times t$  for  $t$  between zero and one. The singularities are births, deaths and saddle points. It is interesting to note that in this framework, a knot and its mirror image can interact to produce two unknots! See Figure 3.

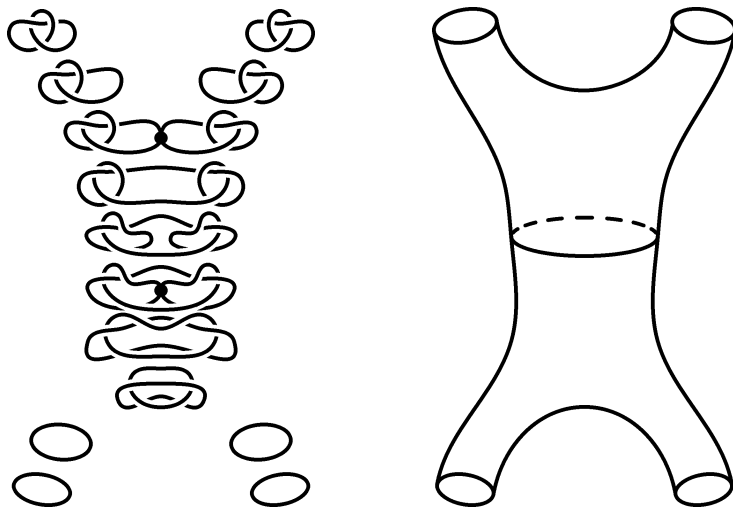


Figure 3: Interaction.

Thus, this embedded string theory contains a myriad of “particle states” corresponding to knotting and the patterns of knot concordance (Section 1).

While the physical interpretation of knotted strings is ambiguous, the mathematics of interacting knots and links is a well-defined and unexplored territory.

**Problem 3.** *Investigate knotted strings and four-space interactions.*

**827. ?**

### 3. Detecting Knottedness

? **828. Problem 4.** *Does the original Jones polynomial (JONES [1985]) detect knottedness?*

There are many polynomial invariants of knots and links that generalize the original Jones polynomial (AKUTSU and WADATI [1988], FREYD, YETTER, HOSTE, LICKORISH, MILLETT and OCNEAUNU [1985], HO [1985], HOSTE [1986], JONES [1987, 1989], KAUFFMAN [1989a, 1990a, 1990a], LICKORISH and MILLETT [1987], LICKORISH [1988], RESHETIKHIN [1987, 1989], TURAEV [1987], WITTEN [1989]), and the same problem can be addressed to them. Nevertheless, the problem is most charming when phrased to the original Jones polynomial. It is, at base, a combinatorial question about the structure of the bracket state model (KAUFFMAN [1987b]) that calculates this polynomial.

Recall the bracket,  $[K]$ . It is, at the outset a well-defined three variable polynomial for unoriented link diagrams—defined by the equations

$$\begin{aligned} 1. \quad [\text{X}] &= A[\text{Y}] + B[\text{Z}] \\ 2. \quad [\text{O}] &= d[\text{K}], \quad [\text{O}] = d \end{aligned}$$

In these equations the small diagrams stand for parts of otherwise identical larger diagrams. The second equation is to be interpreted as saying that an isolated loop (Jordan curve) contributes a factor of  $d$  to the polynomial. Since we assume that  $A$ ,  $B$  and  $d$  commute, it follows easily that  $\langle K \rangle$  is well-defined on unoriented diagrams. Call this polynomial the *three-variable bracket*. It is not an ambient isotopy invariant as it stands, but a specialization of the variables yields the Jones polynomial.

To be precise, one easily finds the following formula:

$$[\text{X}] = AB[\text{Z}] + (ABd + A^2 + B^2)[\text{Y}].$$

Hence, if we choose  $B = A^{-1}$  and  $d = -A^2 - A^{-2}$ , and define the *topological bracket*,  $\langle K \rangle$ , by the formula

$$\langle K \rangle(A) = [K](A, A^{-1}, -A^2 - A^{-2}) / [\text{O}]$$

then

$$\langle \text{X} \rangle = \langle \text{Z} \rangle$$

achieving invariance under the second Reidemeister move. It is then easy(!) to see that this *topological bracket* is invariant under the third Reidemeister move as well. Finally, we get the formulas

$$\begin{aligned} \langle \text{Y} \rangle &= (-A^3) \langle \text{W} \rangle \\ \langle \text{Z} \rangle &= (-A^{-3}) \langle \text{W} \rangle \end{aligned}$$



(Thus  $\langle K \rangle$  is not invariant under the first Reidemeister move. It is invariant under II and III. This is called invariance under *regular isotopy*.)

**3.1. THEOREM** (KAUFFMAN [1987b]). *The original Jones polynomial  $V_K(t)$  is a normalized version of the special bracket. In particular,*

$$V_K(t) = f_K(t^{-\frac{1}{4}})$$

where  $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle$ ,  $K$  is oriented,  $w(K)$  is the sum of the crossing signs of  $K$ , and  $\langle K \rangle$  is the topological bracket evaluated on  $K$  by forgetting  $K$ 's orientation.  $\square$

We can restate the question at the beginning:

**Problem 4.1.** *Does there exist a knot  $K$  ( $K$  is assumed to be knotted) such that  $\langle K \rangle$  is a power of  $A$ ?* **829. ?**

Such a knot would have extraordinary cancellations in the bracket calculation.

One way to begin to look into this problem is to consider the structure of a state summation for the bracket. That is, we can give a specific formula for the bracket as a combinatorial summation over certain configurations of the link diagram. I call these configurations “states” of the diagram—in analogy to the states of a physical system in physical mechanics. In a sense, each model for a link invariant has its own special set of states. The states for the bracket are particularly simple: let  $U$  be the four-valent plane graph underlying a given link diagram  $K$ . A (bracket) state of  $U$  is a collection of Jordan curves in the plane that is obtained by splicing each crossing of  $U$  in one of the two possible ways—as shown in Figure 4.

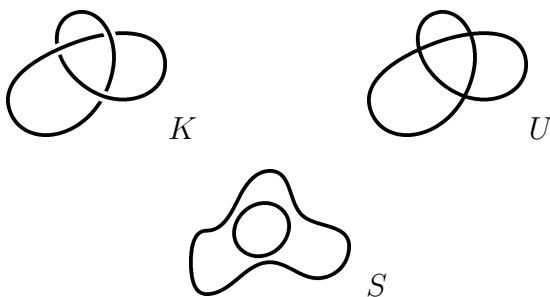


Figure 4:  $S$  is a *state* of  $U$

For a given link diagram  $K$ , each state  $S$  of  $K$  has *vertex weights*  $A$  or  $B$  at each crossing of  $K$ . These weights depend upon the relationship of the local state configuration and the crossing in the link diagram. If  $C$  is a crossing

and  $Q$  is a local state configuration, then I let  $[C|Q]$  denote the vertex weight contributed by  $C$  and  $Q$ . The rules are as shown below:

$$\begin{aligned} [\text{X} | \text{X}] &= A \\ [\text{X} | \text{C}] &= B \end{aligned}$$

If  $K$  is a link diagram and  $S$  is a state of  $K$ , then  $[K|S]$  denotes the product of the vertex weights from  $K$  and  $S$  over all the crossings of  $K$ .

$$\left. \begin{array}{c} K \\ S \end{array} \right\} \Rightarrow \begin{aligned} [K|S] &= A^3 \\ \|S\| &= 2 \end{aligned}$$

We then have the specific formula

$$[K] = \sum [K|S] d^{\|S\|}$$

where the summation extends over all states of the diagram, and  $\|S\|$  denotes the number of Jordan curves in the state  $S$ .

In the case of the special bracket, this summation becomes

$$\langle K \rangle = \sum A^{i(S)-j(S)} (-A^2 - A^{-2})^{\|S\|-1}.$$

Here the summation extends over all the states of the diagram, and  $i(S)$  and  $j(S)$  denote the number of sites in the states that receive vertex weights of  $A$  and  $A^{-1}$  respectively. From this formula, we see that the whole difficulty in understanding cancellation phenomena in the bracket is concentrated in the presence of the signs  $(-1)^{\|S\|}$  in the state summation. In the cases of alternating links (KAUFFMAN [1987b], MURASUGI [1987]) and adequate links (LICKORISH [1988]) it is possible to see directly that there is no non-trivial cancellation (i.e., the polynomial  $\langle K \rangle$  detects knottedness for alternating and adequate knots and links). In general, it is quite possible that there is a topologically knotted diagram  $K$  with enough cancellation to make  $\langle K \rangle$  into a power of  $A$ .

? 830. **Problem 6.** *Where is this culprit  $K$ ?*

(The culprit would answer Problem 4.1.)

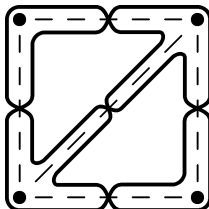
#### 4. Knots and Four Colors

A simple, classical construction relates arbitrary planar graphs and (projected) link diagrams. This is the construction of the *medial graph* associated to any planar graph embedding. See Figure 5.

The medial graph is obtained as follows: In each region of the plane graph  $G$  (A *plane graph* is a graph that is embedded in the plane), draw a Jordan curve that describes the boundary of the region. For ease of construction, the curve should be drawn near the boundary. Now each edge of the graph will appear as shown below (with the dotted line representing the original edge, and the solid lines representing the Jordan curves).



Once for each edge in  $G$ , replace the parallel Jordan curve segments with a crossing as shown below and in Figure 5.



The resulting locally four-valent plane graph is the medial graph,  $M(G)$ .



Figure 5: The graph  $G$  and its medial graph  $M(G)$

The upshot of this medial graph construction is that the class of locally four-valent plane graphs is sufficiently general to capture all the properties of the entire class of plane graphs. Since knots and links project to locally four-valent plane graphs, this means that *in principle, all combinatorial problems about plane graphs are problems about link diagrams*. A problem about link diagrams may or may not be a problem about the topology of links, but it is interesting and possibly very significant to see the relationship between combinatorial problems and their topological counterparts.

A first example of this correspondence is the chromatic polynomial,  $C_G(q)$ . This is the number of vertex colorings by  $q$  colors of the graph  $G$  such that vertices that share an edge receive distinct colors (i.e.,  $C_G(q)$  is the number of proper vertex colorings of  $G$  with  $q$  colors.). It is easy to see that  $C_G$  satisfies the following formulas

$$\begin{aligned} C_{\text{---}\diagup\text{---}} &= C_{\text{---}\diagdown\text{---}} - C_{\text{---}\times\text{---}} \\ C_{\bullet G} &= qC_G \end{aligned}$$

Here  $\bullet G$  denotes the disjoint union of  $G$  with an isolated point, and the small diagrams indicate (in order from left to right) an edge in the graph  $G$ , the deletion of this edge, the contraction of this edge to a point. Thus the first formula states that

$$C_G = C_{G'} - C_{G''}$$

where  $G$ ,  $G'$  and  $G''$  stand for the original graph, the graph with a specific edge deleted, and the graph with this edge contracted to a point, respectively.

On translating this formula to the medial graph we find

$$C_{\text{---}\times\text{---}} = C_{\text{---}\supset\subset\text{---}} - C_{\text{---}\asymp\text{---}}$$

Since one must keep track of the direction of splitting in terms of the original graph it is best to work with a shaded medial, thus:

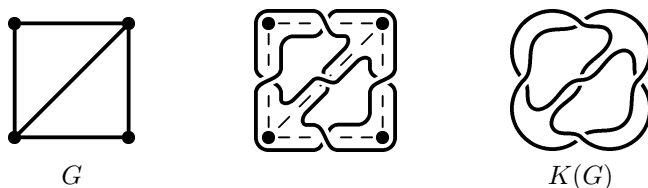
$$\begin{aligned} C_{\triangle} &= C_{\angle} - C_{\text{---}\text{---}} \\ C_{\text{---}\bigcirc\text{---}} &= C_{\text{---}\bigcirc\text{---}} - C_{\text{---}\bigcirc\text{---}} \end{aligned}$$

This scheme is quite convenient for working with colorings of graphs. In particular, it suggests that the chromatic polynomial is very similar to the bracket polynomial. In fact, *we can use the crossings of a knot diagram to encode the chromatic polynomial as a bracket calculation.* (See KAUFFMAN [1989d].) The result is as follows: Associate to each plane graph  $G$  an alternating link diagram  $K(G)$  by taking the medial  $M(G)$ , and arranging a link diagram over  $M(G)$  with crossings chosen to be of “A-type” for each edge of  $G$  (See Figure 6 for this convention.) Define a special bracket via

$$\begin{aligned} \{\text{---}\times\text{---}\} &= \{\text{---}\supset\subset\text{---}\} - q^{-1/2}\{\text{---}\asymp\text{---}\} \\ \{\text{---}\bigcirc\text{---}\} &= q^{1/2}\{\text{---}\bigcirc\text{---}\} \end{aligned}$$

Then  $C_G(q) = q^{N/2}\{K(G)\}$  where  $N$  denotes the number of vertices of the original graph  $G$ .

This formula shows that the chromatic polynomial for a plane graph can be put into exactly the same framework as the Jones polynomial for a given link. Now the classical combinatorial problem about the chromatic polynomial for

Figure 6:  $G$  and its alternating link diagram  $K(G)$ 

plane graphs is to show that it does not vanish for  $q = 4$  when  $G$  has no loop and no isthmus (See KEMPE [1879], WHITNEY [1988]). We see from this reformulation that this difficulty is very similar to the difficulty in showing that the Jones polynomial detects knottedness.

These remarks solve neither the knot detection problem nor the coloring problem, but it is significant to find that these problems share the rung in the inferno.

## 5. The Potts Model

The chromatic polynomial of Section 4, is a special case of the *dichromatic polynomial*, a polynomial  $W_G(q, v)$  in two variables,  $q$  and  $v$ , associated with an arbitrary graph  $G$  via the formulas

$$\begin{aligned} W_{\succleftarrow} &= W_{\succ} \prec - v W_{\times} \\ W_{\bullet G} &= q W_G \end{aligned}$$

That is,  $W$  is a generalization of the chromatic polynomial. It specializes to the chromatic polynomial when  $v = 1$ .

Just as we expressed the chromatic polynomial as a bracket calculation, we can also express the dichromatic polynomial in a similar way. Generalize the special bracket of Section 4 via the rules:

$$\begin{aligned} \{\infty\} &= \{\supset\subset\} + vq^{-1/2}\{\asymp\} \\ \{\bigcirc\mathbf{K}\} &= q^{1/2}\{\mathbf{K}\} \end{aligned}$$

Then one has the formula

$$W_G(q, v) = q^{N/2}\{K(G)\},$$

where  $N$  denotes the number of vertices of the graph  $G$  ( $G$  is a plane graph for this discussion.) and  $K(G)$  is the alternating link diagram associated with the plane graph  $G$  via the medial construction (See Section 4).

Now it is well-known (BAXTER [1982]) that the dichromatic polynomial of a graph  $G$  can be interpreted as the partition function of a statistical mechanics

model based on  $G$ . This model, known as the *Potts model* depends upon  $q$  local states at the vertices of the graph, and the variable  $v$  is related to the temperature in the model via the equation  $z = \exp(\frac{1}{kT}) - 1$  (anti-ferromagnetic case) where  $T$  denotes temperature, and  $k$  is a constant (Boltzman's constant). The partition function is a summation over the physical states of the model of probability weighting for these states. The weights depend upon energy, temperature and Boltzman's constant. In this very simple model, a state  $\sigma$  is an assignment of values (colors, spins, ...) to each vertex of the graph. The energy,  $E(\sigma)$ , of the state  $\sigma$  is then defined to be the number of coincidences of spins for pairs of vertices that are connected by an edge in the graph. The partition function is the summation

$$Z_G = \sum_{\sigma} \exp \left( \left( \frac{1}{kT} \right) E(\sigma) \right)$$

where the sum extends over all states of the given graph.

The basic result is that  $Z_G = W_G(q, \exp(\frac{1}{kT}) - 1)$ . Hence

$$Z_G = q^{\frac{N}{2}} \{K(G)\} \{q, \exp(\frac{1}{kT}) - 1\}.$$

Note that this formula says that the Potts partition function at zero temperature is the chromatic polynomial.

For  $G$  a rectangular lattice in the plane it is conjectured (BAXTER [1982]) that the Potts model has a phase transition (in the limit of large lattices) for the temperature value that symmetrizes the model with respect to graph and dual graph in the plane. In terms of the special bracket link diagram representation of the model, this means that we demand that  $q^{\frac{1}{2}} = \exp(\frac{1}{kT}) - 1$  since this creates the symmetry

$$\{\infty\} = \{\supset\subset\} + \{\asymp\}$$

corresponding in link diagrams to the desired duality.

Many problems about the Potts model find their corresponding formulations in terms of this special bracket for linked diagrams. In particular, it is at once obvious from the special bracket expansion for the Potts model that the Potts model can be expanded over the Temperley-Lieb algebra—with this algebra represented diagrammatically via braid monoid elements of the form

$$\bigcup \big| \cdots \big|, \big| \bigcup \big| \cdots \big|, \dots, \big| \cdots \big| \bigcap.$$

There are a number of important questions about the relationship of the Temperley-Lieb algebra and other structures of the model near criticality. For example, one would hope that this approach sheds light on the relationship with the Virasoro algebra in the continuum limit of the Potts model at criticality. In general we can ask:

**Problem 7.** *Does the link diagrammatic approach lend insight into properties of the Potts model?* **831. ?**

We can also ask whether the concepts of statistical mechanics can be used in the topological context. For example,

**Problem 8.** *What does the phenomena of phase transition mean in the context of calculating link polynomials for large links?* **832. ?**

**Problem 9.** *Is there a way to extract topological information from the dynamical behaviour of a quasi-physical system associated with the knot?* **833. ?**

Finally, to return directly to the knot theory, one might wonder:

**Problem 10.** *Is the Potts partition function viewed as a function on alternating link diagrams a topological invariant of these diagrams?* **834. ?**

Instead of being nonsense, this turns out to be a deep question! It requires interpretation. The apparently correct conjecture is this:

**Problem 11.** *Let  $K$  be a reduced alternating diagram, then we conjecture that  $[K](A, B, d)$  is an ambient isotopy invariant of  $K$ .* **835. ?**

(A diagram is reduced if there are no simplifying type I moves and it is not a connected sum of two non-trivial diagrams.) This conjecture has its roots in the classical conjectures of Tait, Kirkwood and Little who suggested that two reduced alternating projections of the same link are related by a sequence of higher-order moves called *flypes*. A flype takes a tangle with two crossed input strands and two output strands, and turns the tangle by one half twist (180 degrees). This moves takes alternating projections to alternating projections. *It is easy to see that the full three-variable bracket polynomial is invariant under flying.* Thus the Flying Conjecture of Tait, Kirkwood and Little implies the topological invariance of the full bracket for the reduced alternating projections. This, in turn, implies the invariance of the Potts partition function for an associated reduced alternating link.

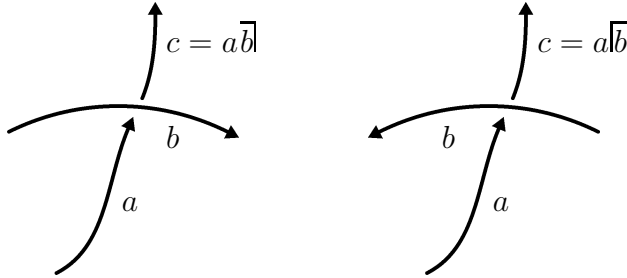
It appears that the Potts partition function contains real topological information. Perhaps eventually it will be seen that the Tait Flying Conjecture follows from subtle properties of statistical mechanics.

## 6. States, Crystals and the Fundamental Group

The fundamental group of the complement of the link can be described as a special sort of state of the link diagram. In order to illustrate this point and to ask questions related to it, I shall describe a structure that simultaneously

generalizes the fundamental group, the Alexander module, and the Quandle (JOYCE [1982]). We shall call this algebraic structure related to an oriented link diagram the *crystal of  $K$* ,  $C(K)$ . (See KAUFFMAN [1987a, 1987b]).

The crystal is obtained by assigning an algebra element to each arc in the diagram, and writing a relation at each crossing in the form shown below:



In this formalism the mark,  $\overline{\phantom{x}}$  (or  $\overline{\phantom{x}}$ , (there are left and right versions of the mark) is a formal operator that is handled like a root sign ( $\sqrt{\phantom{x}}$ ) in ordinary algebra. That is, the mark has the role of operator and parenthesis. It acts on the expression written within it, and it creates a parenthetical boundary for the result of the operation. The concatenation  $a\overline{b}$  is regarded as a non-commutative product of  $a$  and  $\overline{b}$ . The crystal is a formal algebra that is given to be associative and (possibly) non-commutative.

Products in the crystal are built via the following rules:

- (1) If  $a$  and  $b$  are in  $C$ , then  $a\overline{b}$ ,  $a\overline{b}$ ,  $\overline{a}b$  and  $\overline{a}b$  are in  $C$ .
- (2) The labels for the arcs on the link diagram are in  $C$ .
- (3) All elements of  $C$  are built via these three rules.

The crystal axioms are:

2.  $x\overline{a\overline{a}} = x$  for all  $x$  and  $a$  in  $C$ .
3.  $x\overline{b\overline{a\overline{b}}} = x\overline{ab}$  for all  $x$ ,  $a$  and  $b$  in  $C$ . (and the variants motivated below.)

The axioms are labelled 2. and 3. to correspond to the Reidemeister moves 2. and 3. The diagrams in Figure 7 show this correspondence with the moves. Here we have used a modified version of the type III move (a detour) that is valid in the presence of the type two move.

By our assumption about the Crystal Axioms, the crystal acts on itself via

$$C \times C \longrightarrow C : (a, b) \longrightarrow a\overline{b} \text{ (or } \overline{a}b\text{)}.$$

Given the associativity of the concatenation operation in the crystal, we see that Axiom 2 asserts that the *operator subset*

$$C^* = \{x \in C : x = \overline{a} \text{ or } x = \overline{a} \text{ for some } a \in C\}$$



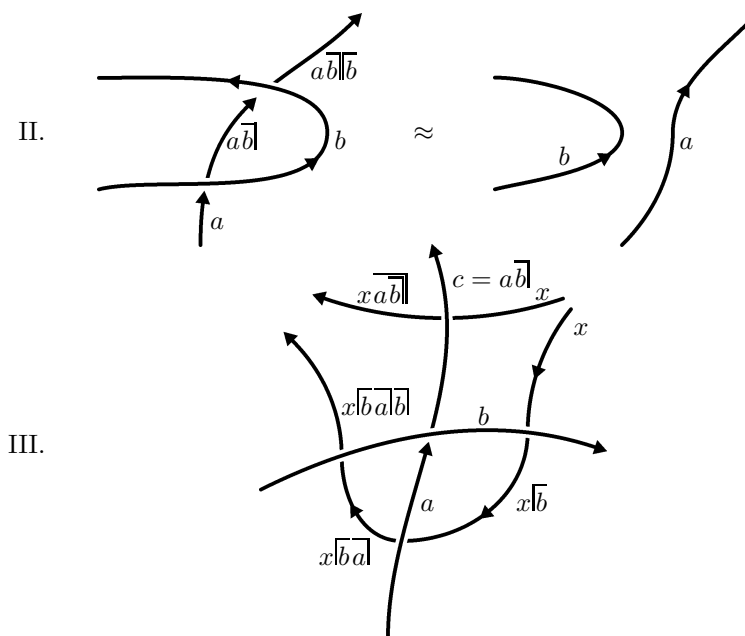


Figure 7: Crystal Axioms and Reidemeister Moves

is a group (of automorphisms of  $C$ ) under the crystal multiplication. *This group is the fundamental group of the link complement.* (Compare the formalism with that of the Wirtinger presentation (CROWELL and FOX [1963]) of the fundamental group.) If we wish to emphasize this group structure then we can write the axioms as:

2.  $\overline{a}a = 1$ ,  $a\overline{a} = 1$ ,
3.  $\overline{a\bar{b}} = \overline{b\bar{a}}\bar{b}$ ,  $\overline{a\bar{b}} = \overline{b\bar{a}}\bar{b}$ ,  $\overline{a\bar{b}} = \overline{b\bar{a}}\bar{b}$ ,  $\overline{a\bar{b}} = \overline{b\bar{a}}\bar{b}$ .

These are the operator identities, but the important point to see is that we associate one group element to each arc of the diagram, and that there is one relation for each crossing in the form shown below:

$$\left. \begin{array}{c} \text{Diagram of a crossing with arcs } a, b, c = a\bar{b} \end{array} \right\} \Rightarrow \begin{array}{l} \alpha = \overline{a} \\ \beta = \overline{b} \\ \gamma = \overline{c} \\ \gamma = \overline{a\bar{b}} \\ \quad = \overline{b\bar{a}}\bar{b} \\ \gamma = \beta^{-1}\alpha\beta \end{array}$$

This is the familiar Wirtinger relation for the fundamental group.

The quandle (JOYCE [1982], WINKER [1984]) is generated by *lassos*, each consisting in an arc emanating from a basepoint in the complement of the link, plus a disk whose boundary encircles the link—the interior of the disk is punctured once transversely by the link itself. One lasso acts on another to form a new lasso  $a * b$  (for lassos  $a$  and  $b$ ) by changing the arc of  $a$  by first travelling down  $b$ , around its disk, back to basepoint, then down the original arc of  $a$ . Since one can travel around the disk in two ways, this yields two possible operations  $a * b$  and  $a \bar{*} b$ . These correspond formally to our abstract operations  $a\overline{b}$  and  $a\bar{\overline{b}}$ . See Figure 8.

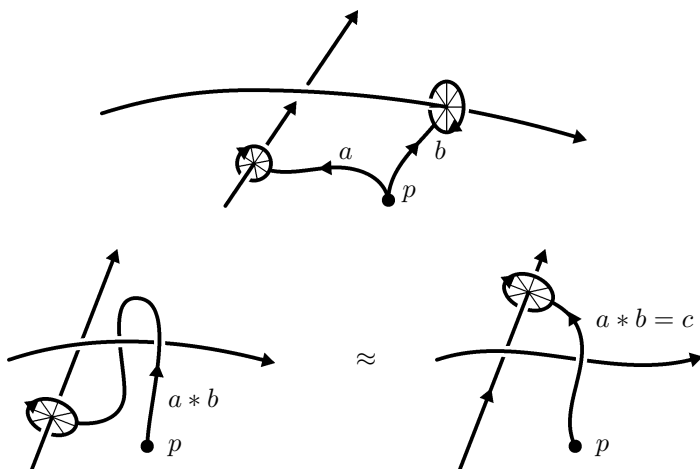
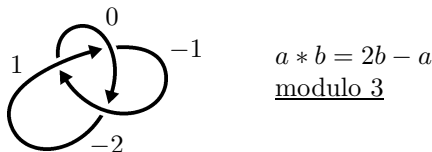


Figure 8: Lassos

The crystal contains more than the fundamental group, and in fact it classifies knots up to mirror images. (See JOYCE [1982].) I have defined the crystal so that it is a regular isotopy invariant (invariant under the second and third Reidemeister moves. It is nevertheless the case that the group  $C^*(K)$  is invariant under all three Reidemeister moves. The quandle is a quotient of the crystal. We translate to the quandle by writing  $a * b = a\overline{b}$ . This has the effect of replacing a non-commutative algebra with operators by a non-associative algebra.

Simple representations of the crystal show its nature. For example, label the arcs of the link diagram with integers and define  $a\overline{b} = a\bar{\overline{b}} = 2b - a$ . (This operation does not depend upon the orientation of the diagram.) Then each link diagram will have a least modulus (not equal to 1) in which the crossing

equations can be solved. For example, the modulus of the trefoil is three:



This shows that the number three is an invariant of the trefoil knot, and it shows that we can label the arcs of a trefoil with three colors  $(0, 1, 2)$  so that each crossing sees either three distinct colors or it sees only one color. The invariance of the crystal tells us that any diagram obtained from the trefoil by Reidemeister moves can be colored in the same way (i.e., according to the same rules). In general, there is a coloring scheme corresponding to each modulus, and any knot can be colored with (sufficiently many) labels. Note that for a diagram isotopic to a given diagram there may appear different colors, since not all the colors will necessarily be used on a given diagram (even for a fixed modulus).

In any case, this modular approach to link invariants shows us a picture of a link invariant arising as a property of a special sort of “state” of the diagram (The state is a coloring of the arcs according to the crystal rules.). That property is the modulus. It is the least integer that annihilates all the state labels. The states themselves are arranged so that if the diagram is changed by a Reidemeister move, then there is a well-defined transition from the given state to a state of the new diagram.

The same picture holds for the classical Alexander polynomial. Here the crystal represents the Alexander module by the equations

$$\begin{aligned} a\overline{b} &= ta + (1-t)b \\ a\overline{b} &= t^{-1}a + (1-t^{-1})b \end{aligned}$$

Note that when  $t = -1$  we have the formalism of the modular crystal described above. The labels  $a, b, \dots$  on the arcs of the link diagram are generators of a module (hence additively commutative) over the ring  $\mathbb{Z}[t, t^{-1}]$ . Each crossing in the diagram gives a relation that must hold in the module. The classical Alexander polynomial is defined (up to units in  $\mathbb{Z}[t, t^{-1}]$ ) as the generator of the annihilator ideal of the Alexander module. Once again, we have generalized states of the diagrams (labellings from the Alexander module) and a topological invariant arising from the properties of these states.

**Problem 12.** *A fundamental problem is to find new ways to extract significant topological information from the crystal. We would like to find a useful generalization of the crystal that would completely classify links—including the information about mirror images.* **836. ?**

I have taken the time to describe this crystalline approach to the classical invariants because it is fascinating to ask how the classical methods are related to the new methods that produce the Jones polynomial and its generalizations. At the present time there seems to be no direct relationship between the Jones polynomial and the fundamental group of the knot complement, or with a structure analogous to the Alexander module. This means that although the newer knot polynomials are very powerful, they do not have access to many classical techniques. A direct relation with the fundamental group or with the structure of the crystal would be a real breakthrough.

There is a theme in this quest that is best stated in the metaphors of mathematical physics. A given physical system has physical states. As time goes on, and as the system is changed (possibly as the system is topologically deformed) these states undergo transitions. The patterns of the state transitions reflect fundamental properties of the physics of the system. The usual method of statistical mechanics is to consider not the transitions of the states, but rather the gross average of probability weighting over all the possible states. This average is called the partition function of the system. The two points of view—transition properties and gross averages—are complementary ways of dealing with the physics of the system. They are related. For example, one hopes to extract information about phase transition from the partition function. Phase transition is a significant property of state changes in the system.

In the knot theory we have the same schism as in the physics—between the contexts of state transition and state averaging. The tension between them will produce new mathematics and new relations with the physics.

## 7. Vacuum-Vacuum Expectation and Quantum Group

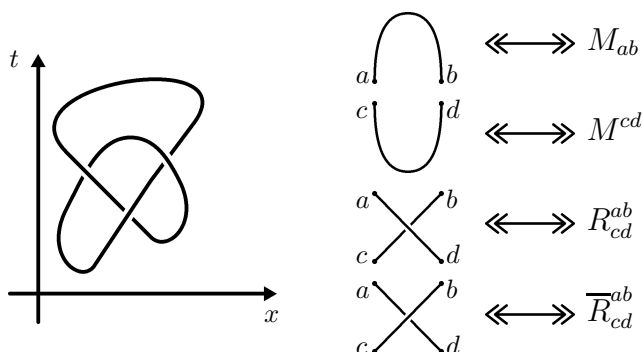
An intermediate position related to the philosophy at the end of Section 6 is the fact that a statistical mechanics model in  $d + 1$  dimensions of space can be construed as a quantum statistical mechanics model in  $d$  dimensions of space and 1 dimension of time. (This is called  $d + 1$ -dimensional space-time.)

In the case of the knot invariants, this philosophy leads to the invariant viewed as a vacuum-vacuum expectation for a process occurring in  $1 + 1$ -dimensional space-time, with the link diagram in the Minkowski plane. For knots in three-space, the process occurs in a  $2 + 1$ -dimensional space-time. Here the picture is quite intuitive. One visualizes a plane moving up through three dimensional space. This is the motion through time for the flatlanders living in the plane. The flatlanders observe a complex pattern of particle creation, interaction and annihilation corresponding to the intersection of the moving plane with a link embedded in the three dimensional space.

In order to calculate the vacuum-vacuum expectation of this process, the flatlanders must know probability amplitudes for different aspects of the

process—or they must have some global method of computing the amplitude. In the case of the Jones polynomial and its generalizations the global method is provided by Witten’s topological quantum field theory (See WITTEN [1989]).

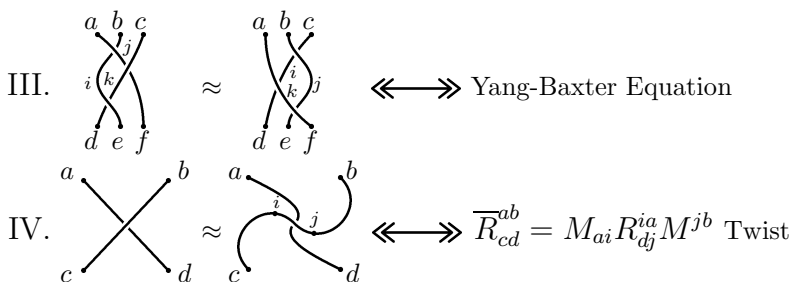
For now we shall rest content with a simpler calculation more suited to linelanders than to flatlanders. The simplest version of a quantum model of this kind is obtained from the planar knot and link diagrams. There we can call attention to creations, annihilations and interactions in the form of cups, caps and crossings. Note that the crossings go over and under the plane of the diagram. This model has just a bit more than one dimension of space in its space-time.



As illustrated above, I have associated each cup, cap, or crossing with a matrix whose indices denote the “spins” of the particles created or interacting, and whose value denotes a generalized quantum amplitude taking values in an (unspecified) commutative ring.

Following the principles of quantum mechanics, the amplitude is the sum (over all possible configurations of spins) of the products of the amplitudes for each configuration. In order for this amplitude to be an invariant of regular isotopy, we need matrix properties that correspond to topological moves. Thus we require

$$\begin{aligned}
 \text{I.} \quad & \begin{array}{c} b \\ \curvearrowright \\ a \end{array} \approx \begin{array}{c} b \\ / \\ a \end{array} \quad \Longleftrightarrow \quad M_{ai} M^{ib} = \delta_a^b \text{ (sum on } i) \\
 \text{II.} \quad & \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ i \quad j \\ \diagup \quad \diagdown \\ c \quad d \end{array} \approx \begin{array}{c} a \\ | \\ c \end{array} \begin{array}{c} b \\ | \\ d \end{array} \quad \Longleftrightarrow \quad R_{ij}^{ab} \bar{R}_{cd}^{ij} = \delta_c^a \delta_d^b
 \end{aligned}$$



(See KAUFFMAN [1990b, 1990a].)

Many link polynomials fall directly in this framework. For example, the bracket model for the Jones polynomial (See Section 3) is modelled via

$$(M_{ab}) = (M^{ab}) = M = \begin{bmatrix} 0 & \sqrt{-1} A \\ -\sqrt{-1} A^{-1} & 0 \end{bmatrix}$$

and

$$R_{cd}^{ab} = A M^{ab} M_{cd} + A^{-1} \delta_c^a \delta_d^b.$$

Now, the remarkable thing about this approach is that it is directly related to the non-commutative Hopf algebra constructions (called quantum groups) of Drinfeld and others (See DRINFELD [1986], MANIN [1988], and RESHETIKHIN [1987]). In particular, the so-called Double Construction of Drinfeld exactly parallels these extended Reidemeister moves (extended by the conditions related to creation and annihilation). (For example, the twist move corresponds to the existence of an antipode in the Hopf algebra via the use of the Drinfeld universal solution to the Yang-Baxter Equation.)

This context of link invariants as vacuum-vacuum amplitudes is a good context in which to ask the question:

? **837. Problem 13.** *Do these vacuum-vacuum amplitude invariants completely classify knots and links?*

The abstract tensor formalism of cup, cap and interaction satisfying only the properties we have listed does give a faithful translation of the regular isotopy classes of knots and links into a category of formal tensor products. In order to calculate an invariant these tensor symbols must be replaced by actual matrices.

? **838. Problem 14.** *I conjecture that for a given pair of links that are distinct, there exists a representation of the abstract tensor formalism that distinguishes them. (In fact I conjecture that there is a representation of the Drinfeld double construction, i.e., a quantum group, that distinguishes them.)*

The abstract tensor structures are related to the duality structure of conformal field theories, and to invariants of three manifolds obtained in a number

of related ways (CRANE [1989], RESHETIKHIN [1989], WITTEN [1989]). There is not space in this problem list to go into the details of all these constructions. However, the basic idea behind the constructions of the three-manifold invariants in the Reshetikhin-Turaev approach is to add extra conditions to the link polynomials so that they become invariants of framed links and so that they are further invariant under the Kirby moves (See KIRBY [1978]). This insures that the resulting polynomials are invariants of the three manifold obtained by surgery on the framed link. The fundamental group of the three manifold is obtained as a quotient of the fundamental group of the given link complement.

**Problem 15.** *We now face the important question of the sensitivity of these new invariants of three-manifolds to the fundamental group of the three-manifold.* **839. ?**

If the new three manifold invariants can be non-trivial on simply connected compact three-manifolds, then there will exist a counterexample to the classical Poincaré Conjecture. The structure of these new invariants will provide a long sought after clue to the solution of this venerable conundrum. (The *Poincaré Conjecture* asserts that a compact simply connected three manifold is homeomorphic to the standard three dimensional sphere.)

## 8. Spin-Networks and Abstract Tensors

Another relationship between quantum networks and three dimensional spaces occurs in the Penrose theory of spin networks (PENROSE [1971]). Here the formalism of spin angular momentum in quantum mechanics is made into a purely diagrammatic system. Each spin network is assigned a combinatorially computed norm. (The Penrose norm has the form of a vacuum-vacuum expectation for the whole network, but here the network is not embedded in a space-time. This bears an analogical relation with the amplitudes for knots and links that depend only upon the topology of the embedding into space-time and not upon any given choice for an arrow of time.) These norms, in turn can be used to compute probabilities of interaction between networks, or between parts of a given network.

Probabilities for interaction lead to a definition of angle between networks. The angle is regarded as well-defined if two repeated measurements yield the same result. The upshot of the Penrose work is the *Spin-Geometry Theorem* that states that *well-defined angles between subnetworks of a (large) network obey the dependency relation of angles in a three dimensional space*. In other words, the properties of three dimensional space begin to emerge from the abstract relations in the spin networks.

One would hope to recover distances and even space-time in this fashion. The Penrose theory obtains only angles in a fundamental way.

? 840. **Problem 16.** *I conjecture (KAUFFMAN [1990b]) that a generalization of the spin networks to networks involving embedded knotted graphs will be able to realize the goal of a space-time spin geometry theorem.*

Here it must be understood that the embedding space of the knotted graphs is not the final space or three manifold that we aim to find. In fact, it may be possible to use a given embedding of the graph for calculating spin network norms, but that these norms will be essentially independent of the embedding (just so the Penrose spin nets are calculated through a planar immersion of the net, but they depend only on the abstract net and cyclic orders attached to the vertices).

In this vision, there will be constructions for new three dimensional manifolds, and these manifolds will carry the structure of significant topological invariants in the networks of which they are composed.

In order to bring this discussion down to earth, let me give one example of how the spin networks are already generalized by the vacuum-vacuum, amplitude models for the Jones polynomial. If we take the bracket (Section 3) at the value  $A = -1$ , then the bracket relation becomes

$$\langle \text{X} \rangle + \langle \text{Y} \rangle + \langle \text{Z} \rangle = 0.$$

This relation is identical to the generating relation for the *Penrose binor calculus*—a translation of  $SL(2, C)$  invariant tensors into diagrammatic language. The binor calculus is the underpinning of the spin networks. As  $A$  is deformed away from  $-1$  (or from  $1$ ) the symmetry of these networks becomes the quantum group  $SL(2)q$ . ( $A = \sqrt{\varepsilon}$ )

Thus the link diagrams as abstract tensor diagrams already show themselves as a generalization of the spin networks.

## 9. Colors Again

To come fully down to earth from Section 8, here is a spin network calculation that computes the number of edge colorings of a trivalent plane graph:

Associate to each vertex in the graph the tensor  $\sqrt{-1}\epsilon_{abc}$  where  $\epsilon_{abc}$  denotes the alternating symbol—that is,  $a, b$  and  $c$  run over three indices  $\{0, 1, 2\}$ ; the epsilon is zero if any two indices are the same, and it is the sign of the permutation  $abc$  when the three indices are distinct. Call an assignment of indices to all the edges of the graph an *em edge coloring* if each vertex receives three distinct indices. For each edge coloring  $\sigma$  of  $G$ , let  $\|\sigma\|$  denote the product of the values  $\sqrt{-1}\epsilon_{abc}$  from each vertex. Thus  $\|\sigma\|$  is the product of the vertex weights assigned by this tensor to the given edge coloring. Define the *norm*,  $\|G\|$  of a graph  $G$  to be the sum of these products of vertex weights, summing over all edge colorings of the graph.

Then one has the following result:

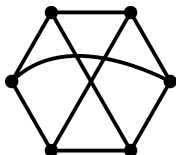


**9.1. THEOREM** (Penrose [1971]). *If  $G$  is a trivalent plane graph, then the norm of  $G$ ,  $\|G\|$ , is equal to the number of edge colorings of  $G$ . In fact the norm of each coloring is  $+1$  if  $G$  is planar. In general, the norm for immersed graphs (with edge-crossing singularities) obeys the following equation*

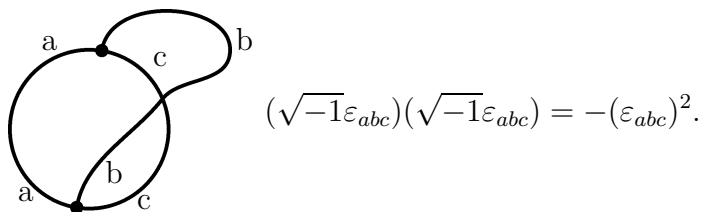
$$\|\text{X}\| = \|\text{Y}\| - \|\text{Z}\|$$

with the value of a collection of (possibly overlapping) closed loops being three (3) raised to the number of loops.  $\square$

In this theorem, the norm can be evaluated for non-planar graphs by choosing a singular embedding of the graph in the plane, and then computing the norm as before. (Crossing lines may be colored differently or the same. We are actually coloring the abstract graph.) The immersion of the graph in the plane gives a specific cyclic order to the edges of each vertex, and this determines the norm computation. Of course, any graph with no colorings receives a norm zero, but non-planar graphs that have colorings can also receive norm zero. For example, the graph below has zero norm:



An embedding with singularities may not enumerate all the colorings with positive signs. The simplest example is



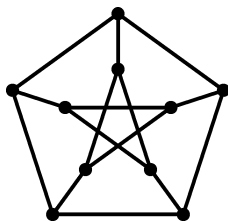
Note how the recursion formula works:

$$\|\text{O}\| = \|\text{D}\| - \|\text{X}\| = 3^2 - 3 = 6.$$

**Remark:** The problem of edge colorings for trivalent graphs is well-known to be equivalent to the four color problem for arbitrary plane graphs. Thus the spin network evaluation is another instance of the four color problem living in relation to a context composed of combinatorics, knot theory and mathematical physics. The relation with the knot theory could be deepened

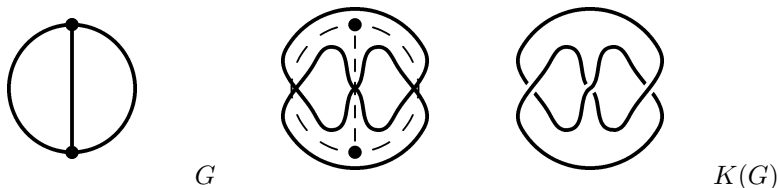
if we could add crossing tensors to the Penrose formula so that it computed the coloring number for arbitrary (not necessarily planar) trivalent graphs. This is a nice challenge for the knot diagrammatic approach.

The simplest known *snark* (a snark is a non-edge colorable trivalent graph) is the Petersen graph—shown below:



The Petersen graph is to combinatorics as the Möbius strip is to topology—a ubiquitous phenomenon that insists on turning up when least expected. Tutte has conjectured that the Petersen graph must appear in any snark.

This chromatic spin-network calculation can be reformulated in terms of link diagrams if we restrict ourselves to plane graphs. Then the medial construction comes directly into play: Take the medial construction for the trivalent graph.



Associate link diagrammatic crossings to the crossings in the medial construction to form an alternating link (as we have done in Section 4). Define a state expansion on link diagrams via

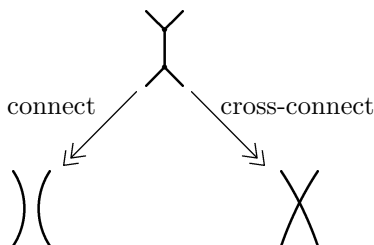
$$\|\text{X}\| = \|\text{Y}\| - \|\text{Z}\|$$

where the value of a collection of loops (with singular crossings) is three to the number of loops. This norm computes the same coloring number as the Penrose number. (Exercise!)

## Heuristics

One of the advantages of the coloring problem in relation to our concerns about state models and topology is that, while the coloring problem is very difficult, there are very strong heuristic arguments in favor of the conjecture that four colors suffice to color a plane map, (and equivalently that trivalent plane maps without loop or isthmus can be edge colored with three colors distinct at each edge.)

At an edge in a trivalent map I shall define two operations to produce smaller maps. These operations are denoted *connect* and *cross-connect* as illustrated below:

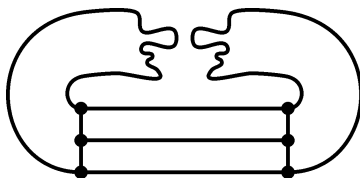


Call a trivalent map that has no edge coloring and that is minimal with respect to this property *critical*. It is obvious that if  $G$  is critical and we form  $H$  by connecting or cross-connecting an edge of  $G$ , then *the two local edges produced by the operation must receive the same color in any coloring of  $H$* . (If they are different, then it is trivial to produce a coloring of  $G$  by using a third color on the edge deleted by the operation.) Call a pair of edges *twins* if they must receive the same color in any coloring of a graph  $H$ . Say that  $G$  *forces twins at an edge  $e$  of  $G$*  if the edges resulting from both connect and cross-connect at  $e$  are twins.

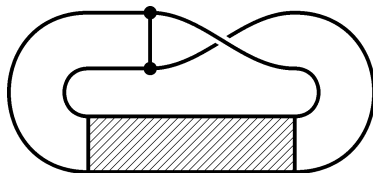
Thus

**9.2. THEOREM.** *A critical trivalent map  $G$  forces twins at every edge of  $G$ .  $\square$*

In order to design a critical map it is necessary to create maps with twins. A simple example of a pair of twins is shown below.

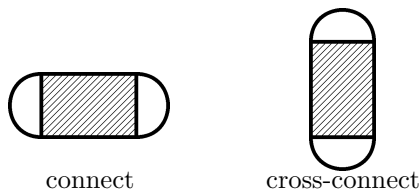


To give a feel for the design problem, suppose that the diagram below represents a trivalent critical map. (I have drawn it in a non-planar fashion for convenience. We are discussing the matter of design of critical maps in the abstract)

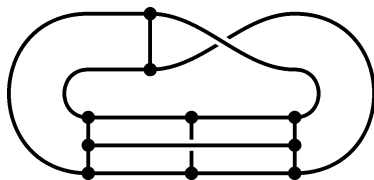


If this map is critical then the edge that is shown must force twins. The

pattern of the forced pairs simplifies as shown below.



The simplest example that I can devise to create forcing from both pairs is as shown below



This graph is isomorphic to the Petersen graph.

Thus we have seen how one is lead inevitably to the Petersen graph in an attempt to design critical trivalent maps. The design side is a strong arena for investigating the coloring problem.

A similar arena exists in knot theory via the many examples that one can construct and compute, however I do not yet see the problem of designing knots that are undetectable in any similar light. The graph theory may yield clues. Time will tell.

? 841. **Problem 17.** *Can Knot Theory solve the Four Color problem and what does the truth of the four color theorem imply for three-dimensional topology?*

## 10. Formations

A diagrammatic approach to coloring trivalent maps clarifies some of the issues of Problem 17, and allows us to raise a central issue about map coloring. This diagrammatic technique goes as follows. Regard the three colors as red (————), blue (-----) and purple (-----). That is, regard one color (purple) as a superposition of the other two colors, and diagram red by a solid line, blue by a dotted line, and purple by a combination dotted and solid line.

With this convention, any edge three coloring of a trivalent graph has the appearance of two collections of Jordan curves in the plane. One collection consists in red curves. The other collection has only blue curves. The red curves are disjoint from one another, and the blue curves are disjoint from one another. Red and blue curves share segments corresponding to the edges that are labelled purple. Thus red curves and blue curves can interact by either sharing a segment without crossing one another (a *bounce*), or by sharing in the form of a crossing (a *cross*).

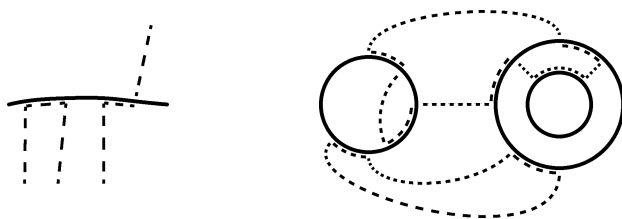


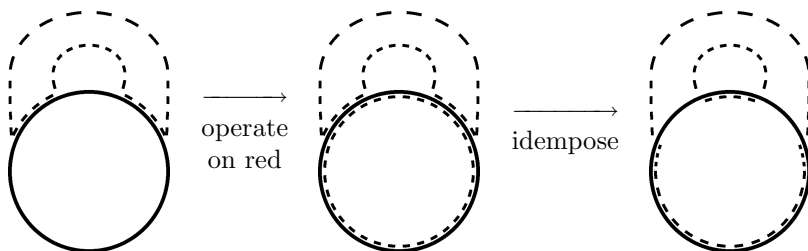
Figure 9: Bouncing and Crossing

These two forms of interaction are illustrated in Figure 9. I call a coloring shown in this form of interacting Jordan curves a *formation*. The terminology formation and the idea for this diagrammatic approach to the coloring problem is due to G. SPENCER-BROWN [1979].

The existence of edge colorings for trivalent plane maps is equivalent to the existence of formations for these maps. This point of view reveals structure. For example, we see at once that the product of imaginary values (for a given coloring) in the spin network calculation of the norm,  $\|G\|$ , for planar  $G$  is always equal to one. For each bounce contributes 1, while each crossing contributes  $-1$ , and the number of crossings of a collection of Jordan curves in the plane is even.

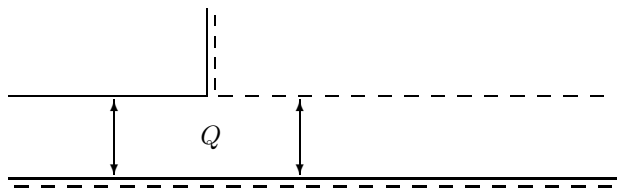
A deeper result has to do with parity. In an edge three coloring the parity of the total number of alternating color cycles (called here *cinguli*) remains unchanged under the operation of switching a pair of colors along a cingulus (a *simple operation*). One can use the language of formations to prove this result (see SPENCER BROWN [1979], KAUFFMAN [1986], and compare with TUTTE [1948]).

In the language of formations a simple operation is accomplished by drawing a curve of one color along a curve of the opposite color (red and blue are opposite, as are red/blue alternating and purple). (Note that in a formation the red cinguli index cycles of alternating red and purple, while the blue cinguli index cycles of alternating blue and purple.) After the curves are superimposed, common colors are cancelled. This cancellation is called *idemposition*. For example



While parity is preserved under simple operations, the parity necessarily changes under a *Spencer-Brown switching operation* (G. SPENCER-BROWN

[1979]) at a five-region. Spencer-Brown's operation is performed to replace one extension problem by another. We are given a configuration as shown below:



This configuration, I shall call a  $Q$ -region. It has two missing edges denoted by arrows. If these edges could be filled in to make a larger formation, the result would be a coloring of a larger map.

The problem corresponds to having a map that is all colored except for one five-sided region.

- ? 842. **Problem 18.** *One wants to rearrange the colors on the given map so that the coloring can be extended over the five-sided region.*

If one can always solve this problem then the four-color theorem follows from it.

In the category of non-planar edge three-colorings it is possible for a  $Q$ -region problem to have no solution involving only simple operations.

The switching operation replaces the  $Q$ -region by another  $Q$ -region, and changes the parity in the process. In the language of formations, the switch is performed by drawing a red curve that replaces one missing edge, idemposes one edge, and travels along a blue cingulus in the original formation to complete its journey. See the example below.

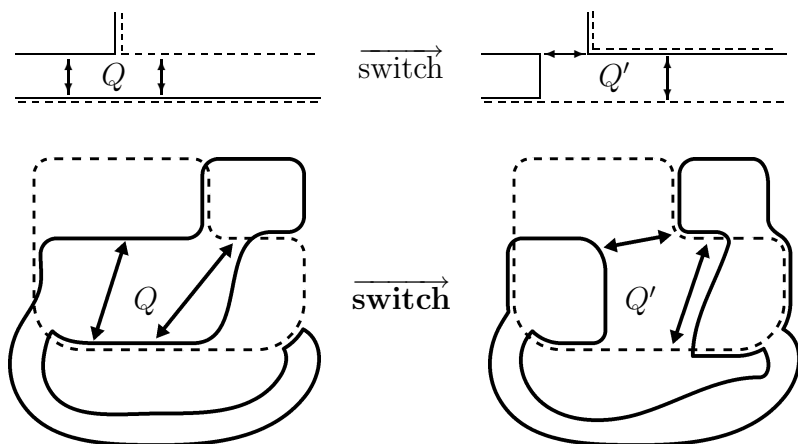
- ? 843. **Problem 19. Switching Conjecture:** *I conjecture that a  $Q$ -region in a planar formation that is unsolvable by simple operations before the Spencer-Brown switch becomes solvable by simple operations after the switch.*

Of course this conjecture would solve the four color problem, and one might think that it is too good to be true. I encourage the reader to try it out on formations of weight three (that is with exactly one red, one blue, and one red/blue alternating cingulus).  $Q$ -regions at weight three are always unsolvable by simple operations. Switch a weight three  $Q$ -region problem and you will find higher weight (since parity changes and weight can't go down).

The switching conjecture aside, it is now possible to indicate a proof of the four color theorem that is due to G. Spencer-Brown.

### Spencer-Brown's Proof:

It suffices to consider a formation with a  $Q$ -region. If the weight is larger than



three then there is an extra cingulus (red, blue or red/blue alternator) other than the three cinguli involved at the  $Q$ -region. If this extra cingulus can be used in a sequence of simple operations to solve the  $Q$ -region then we are done. If this cingulus can not be used in any such sequence, then the extra cingulus is *ineffective*, and from the point of view of the  $Q$ -region it is invisible to the problem. Hence the formation with an ineffective cingulus is structurally smaller, and hence is solved by induction. If there is no extra cingulus in the formation, then the weight is equal to three. Apply the switching operation. Now the weight is greater than three. Hence there is an extra cingulus, and the first part of the argument applies. **Q.E.D.**

**Problem 20.** *Understand this proof!*

844. ?

The crux of the matter in bringing this proof to earth lies in understanding the nature of an effective cingulus. The proof is an extraordinary guide to understanding the map color problem. In G. SPENCER-BROWN [1979] the argument is extended to show that a formation with an extra cingulus can always be solved by complex operations.

Of course one would like to know what is the relationship among quantum physical, statistical mechanical and topological structures and these deep combinatorial matters of the coloring problem. Full understanding of the four color theorem awaits the unfolding of these relationships.

## 11. Mirror-Mirror

The last problem on this set is a conjecture about alternating knots that are *achiral*. A knot is achiral if it is ambient isotopic to its mirror image.

We usually take the mirror image as obtained from the original diagram by switching all the crossings. The mirror is the plane on which the diagram is drawn.

Let  $G(K)$  denote the graph of the diagram  $K$ . That is,  $G(K)$  is obtained from a checkerboard shading of the diagram  $K$  (unbounded region is shaded white). Each black region determines a vertex for  $G(K)$ .  $G(K)$  has an edge for each crossing of  $K$  that is shared by shaded regions.

Let  $M(K)$  denote the cycle matroid of  $G(K)$  (See WELSH [1988] for the definition of the matroid.). Let  $M^*(K)$  denote the dual matroid of  $M(K)$ .

**? 845. Problem 21. Conjecture.**  *$K$  is alternating, reduced and achiral, if and only if  $M(K)$  is isomorphic to  $M^*(K)$  where  $M(K)$  is the cycle matroid of  $G(K)$  and  $M^*(K)$  is its dual.*

This conjecture has its roots in the observation that for all the achiral reduced alternating knots of less than thirteen crossings, the graphs  $G(K)$  and  $G^*(K)$  (the planar dual) are isomorphic. One might conjecture that this is always the case, but Murasugi has pointed out that it is not so (due to flying—compare with Section 5). The matroid formulation of the conjecture avoids this difficulty.

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## CHAPTER 30

### Open Problems in Infinite Dimensional Topology

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## 1. Introduction

This chapter is intended as an update and revision of the problem set written by Ross Geoghegan entitled *Open Problems in Infinite-Dimensional Topology*, which appeared in GEOGHEGAN [1979]. That problem set was the result of a satellite meeting of infinite-dimensional topologists held at the 1979 Spring Topology Conference in Athens, Ohio, for the purpose of updating an earlier list published as an appendix to T. A. Chapman's volume, *Lectures on Hilbert Cube Manifolds*, in the C.B.M.S. series of the American Mathematical Society (CHAPMAN [1976]). That volume, the monograph *Selected Topics in Infinite-Dimensional Topology*, by Cz. BESSAGA and A. PEŁCZYŃSKI [1975], and the new text by J. VAN MILL [1989b] are the basic introductory sources for the subject. However, the interested reader might well want to consult ANDERSON [1972] and several other basic references for overlapping fields, such as BORSUK [1967, 1975], DYDAK and SEGAL [1978], MARDEŠIĆ and SEGAL [1982], HUREWICZ and WALLMAN [1948], NAGATA [1965], DAVERMAN [1986], CHAPMAN [1983b], ANDERSON and MUNKHOLM [1988], ANOSOV [1969], BALLMANN ET AL [1985], GROMOV [1979, 1987], BROWN [1982, 1989], BEAUZAMY [1988].

In view of the amount of progress in the field, it was generally recognized five years ago that a revision of GEOGHEGAN [1979] was desirable; however, the revolution in the field wrought by the work of A. N. Dranišnikov during the last two years has made it imperative, since he has solved the problem that was perhaps the main focus of GEOGHEGAN [1979]. His negative solution in DRANIŠNIKOV [1988a] of P. S. ALEXANDROFF's [1936] question *Does integral cohomological dimension equal Lebesgue covering dimension on metric compacta?* solved at one stroke the most fundamental and resistant question in the dimension theory of separable metric spaces and one of the most difficult questions of the theory of finite- and infinite- dimensional manifolds, as well as one of the most difficult problems of shape theory. Specifically, Dranišnikov proved the following:

**1.1. THEOREM.** *There exists a compact metric space  $X$  of integral cohomological dimension  $c - \dim_{\mathbb{Z}}(X) = 3$  and  $\dim(X) = \infty$ .*

**1.2. COROLLARY.** (EDWARDS [1978], WALSH [1981]). *There exists a cell-like map  $f : X \rightarrow Y$  of a 3-dimensional compactum onto an infinite dimensional compact metric space.*

Dranišnikov, Dydak, and Walsh are planning a book on cohomological dimension theory which should be indispensable to an understanding of this subject.

Infinite-dimensional topology aspires to encompass all topological aspects of all infinite-dimensional spaces and subsets of interest in mathematics and the applications of the ideas, techniques, and philosophy of the field in finite-

dimensional settings. Of course, the possibilities, even obviously promising ones, are far too numerous for the relatively small number of mathematicians currently active in the field to cover. As a result, the topics that have been developed reflect in part the interests and backgrounds of the people in the field, in part the problems and developments of related disciplines, and in part sheer chance. Thus, interactions with stable homotopy theory and  $K$ -theory have included spectacular results (CHAPMAN [1974], TAYLOR [1975], FERRY [1977b], TORUŃCZYK [1977], FARRELL and HSIANG [1981], and DRANIŠNIKOV [1988a]), as well as applications of  $K$ -theoretic invariants to the geometric topology of  $Q$ -manifolds analogous to the finite-dimensional theory. On the other hand, the algebraic topology of function spaces and automorphism groups has been essentially left to others, and, after the early 1970's, a burgeoning investigation of topological questions connected with Differential Topology, Differential Geometry, and Global Analysis has almost ceased after early sweeping early successes.

Nevertheless, as emphasized by Geoghegan, there is a recognizable unity of technique and of central problems to be met, and the field has a distinct internal coherence despite the tendencies of individual researchers to concentrate on problems associated with very different disciplines, e.g., Dimension Theory, Geometry of Banach Spaces and allied Functional Analysis, Geometric Topology of  $n$ -manifolds, Shape Theory, and Point-Set Topology.

The Problem List is traditionally limited to metric topology. I have made no effort to expand its scope. It is also usually broken into various sections, to which format I have adhered. I have also followed the custom of not attributing problems to the people who submitted them. However, I want to thank those who have done so as well as those who have given detailed discussions of the current status of problems in GEOGHEGAN [1979]. Without their help, this problem list and update would not have been possible. These people include C. Z. Bessaga, K. Brown, S. Ferry, R. Daverman, J. Dijkstra, A. Dranišnikov, J. Dydak, T. Dobrowolski, F. T. Farrell, A. Fathi, S. Ferry, R. Geoghegan, J. Henderson, M. R. Holmes, C. B. Hughes, J. Keesling, N. Hingston, G. R. Livesay, E. Michael, J. van Mill, J. Mogilski, Nguyen To Nhu, D. Repovš, J. Rogers, L. Rubin, K. Sakai, E. Ščepin, H. Toruńczyk, Vo Thanh Liem, J. Walsh, R. Wong, and D. Wright. The usual caveat should be repeated here: with probability one, I have made errors of substance in interpreting what is known and what is open. Also, some of the problems may be mis-formulated, or trivial, or already solved. If an important topic is omitted, as several are, it is because it was not represented in the submitted problems and I did not feel competent to write on the topic myself. As a result, for example, shape theory developments are primarily left to other sections of this book, and the exciting area termed "Geometric Methods in Group Theory" together with shape questions at the ends of  $Q$ -manifold  $K(\pi, 1)$ 's is almost totally neglected, with regret. It will be treated in a forthcoming

book by Geoghegan, however, which should be consulted by all readers of this article.

The symbol  $\approx$  will denote “is homeomorphic with”, and  $\simeq$  means “homotopy equivalent with”.

## 2. CE: Cell-Like Images of ANR's and $Q$ -Manifolds

### 2.1. General Discussion

The acronym CE comes from Cellular Equivalent. A set is cell-like (CE) provided that it is compact, metric, and deforms to a point in each of its neighborhoods in some, hence every, ANR in which it is embedded. (These are the sets with the shape of a point.) A proper map is a CE map if it is surjective and each point inverse is cell-like. The following theorem summarizes various theorems about CE maps by Moore, Lacher, Armentrout, Siebenmann, Chapman, and Kozłowski (cf. LACHER [1977] for an excellent discussion and bibliography on the topic).

**2.1.1. THEOREM.** *For a proper map  $f: X \rightarrow Y$  between locally compact metric spaces,*

- (1) *if  $X$  and  $Y$  are ANR's, the following are equivalent:*
  - (a)  *$f$  is CE.*
  - (b)  *$f$  is an hereditary homotopy equivalence.*
  - (c)  *$f$  is an hereditary shape equivalence.*
  - (d)  *$f$  is a fine homotopy equivalence.*
- (2) *if  $Y$  is finite-dimensional, the following are equivalent:*
  - (a)  *$f$  is CE.*
  - (b)  *$f$  is an hereditary shape equivalence.*
- (3) *if  $X$  and  $Y$  are  $Q$ -manifolds or  $n$ -manifolds, the following are equivalent: (unless  $n = 3$  and counter-examples to the Poincaré Conjecture are present),*
  - (a)  *$f$  is CE.*
  - (b)  *$f$  is a uniform limit of homeomorphisms  $h: X \rightarrow Y$ .*

$f$  is an *hereditary homotopy equivalence* provided that  $f: f^{-1}(U) \rightarrow U$  is a proper homotopy equivalence for each open set  $U \subset Y$ ; it is an *hereditary shape equivalence* if  $f: f^{-1}(A) \rightarrow A$  is a shape equivalence for each closed subset  $A \subset Y$ ; it is a *fine homotopy equivalence* if it is an  $\alpha$ -equivalence for all open covers  $\alpha$  of  $Y$ ; it is an  $\alpha$ -*equivalence* if it has an  $\alpha$ -homotopy inverse, i.e., a map  $g: Y \rightarrow X$  and homotopies  $F: g \circ f \simeq id_X$  and  $G: f \circ g \simeq id_Y$  such that  $G$  is limited by  $\alpha$  and  $F$  is limited by  $f^{-1}(\alpha) = \{f^{-1}(U) \mid U \in \alpha\}$ . ( $G$  is *limited* by  $\alpha$  provided that each homotopy track  $G(x \times I)$  lies in some member of  $\alpha$ .)

CE maps lie at the heart of manifold topology and ANR theory. Note that by their very nature, the existence of a CE map  $f: X \rightarrow Y$  onto an ANR



or manifold gives little information about the local properties of the domain (although, being acyclic,  $f$  is a Čech cohomology isomorphism by the Vietoris-Begle Theorem). The natural questions, then, concern what may be inferred about the target from information about the domain. These questions are extremely important in applications.

This is seen in its purest form in the theory of  $Q$ -manifolds, and there, in the proof of H. Toruńczyk's topological characterization of Hilbert cube manifolds TORUŃCZYK [1980] (cf. EDWARDS [1978]):

**2.1.2. THEOREM.** *A locally compact ANR is a  $Q$ -manifold if and only if each pair  $g, h: Q \rightarrow X$  of maps of  $Q$  into  $X$  may be approximated by maps with disjoint images.*

The proof proceeds by first finding a CE resolution  $f: M^Q \rightarrow X$  of  $X$  by a  $Q$ -manifold, and then by approximating  $f$  with a homeomorphism. (There is an analogous program for characterizing  $n$ -manifolds, beginning with a compact ANR homology manifold, employing Quinn's Resolution Theorem (QUINN [1983, 1987]) to obtain a CE resolution by an  $n$ -manifold, and Edwards' Theorem (DAVERMAN [1986]) or (Quinn's depression of it into 4-manifolds 1982b) to approximate the CE map by a homeomorphism if the ANR has Cannon's Disjoint Discs Property ( $DD^2P$ ), which is the same as the approximation hypothesis in Toruńczyk's Theorem except  $Q$  is replaced by  $D^2$ . At the moment, it is unknown whether the integral obstruction to CE resolution encountered by Quinn is ever realized (Good Problem!), but the theory is extraordinarily successful in its present form owing to the fact that the obstruction vanishes if any point of  $X$  has an Euclidean neighborhood.)

Surprisingly, in the absence of an hypothesis such as finite-dimensionality or ANR on both domain and range, CE maps may fail to be even shape equivalences. The following collects work of TAYLOR [1975], KEESLING [1975], EDWARDS [1978], DRANIŠNIKOV [1988a, 1988b], VAN MILL [1981], KOZŁOWSKI ET AL [1981], and WALSH [1981]:

**2.1.3. THEOREM.**

- (1) *There exists a CE map from an infinite dimensional metric compactum onto the Hilbert cube that is not a shape equivalence.*
- (2) *There exists a CE map from the Hilbert cube to a compactum every point inverse of which is a Hilbert cube that is not a shape equivalence.*
- (3) *For each  $n \geq 6$ , there exist CE maps from compact  $n$ -dimensional manifolds to infinite-dimensional compacta.*

The main questions in this section remain as in GEOGHEGAN [1979]: Let  $f: X \rightarrow Y$  be CE, where  $X$  and  $Y$  are locally compact.

? 846. **Question 1** *If  $X$  is an ANR, under what conditions is  $Y$  an ANR?*

**Question 2** *If  $X$  is a  $Q$ -manifold, under what circumstances is  $Y$  also a  $Q$ -manifold?* **847. ?**

A most common occurrence of CE maps is in decompositions or as quotient maps. In this context, one usually knows the domain and that the map is CE, but one may not know that the map is a shape equivalence (TAYLOR [1975]), much less that it preserves the ANR property (KEESLING [1975], KOZŁOWSKI [19∞]). In fact, a CE map  $f: X \rightarrow Y$  between locally compact metric spaces with  $X$  finite dimensional will preserve the ANR property if and only if it is an hereditary shape equivalence which is equivalent to the requirement that it raises the dimension of no saturated closed subset of  $X$  (as hereditary shape equivalences cannot raise dimension) (KOZŁOWSKI [19∞] (cf. ANCEL [1985]), ADDIS and GRESHAM [1978], GRESHAM [1980]). There are good discussions in GEOGHEGAN [1979], WALSH [1981], DRANIŠNIKOV and ŠČEPIN [1986], and MITCHELL and REPOVŠ [19∞].

## 2.2. Progress on Problems of Section CE

The 1979 Problem List gave 11 problems connected with Questions 1 and 2. Here, they will be denoted by (79CE1), etc. Eight of these have been explicitly solved in the ensuing decade. In particular, Question 2 above is now well understood due to the efforts primarily of Daverman and Walsh. The thrust of this development is that by Toruńczyk's Characterization of  $Q$ -manifolds, Question 2 reduces first to Question 1 and then to consideration of the general position property that any two maps of  $Q$  into  $Y$  should be approximable by maps with disjoint images. This is easily seen to be equivalent to the *disjoint  $n$ -discs property for all  $n$*  ( $DD^n P$ ), which is the same general position property for each  $n$ -cell. Daverman and Walsh provided a reduction of that property, and hence of Toruńczyk's Criterion, to  $DD^2 P$  plus for each  $n > 2$  a disjoint Čech homology carriers property which is vastly more easy to apply, which they did.

**CE 1** (79CE1) *If  $f: X \rightarrow Y$  is surjective with  $X$  an ANR and each  $f^{-1}(y)$  is an AR, must  $Y$  be an AR?*

No, VAN MILL [1981]. This problem is originally from BORSUK [1967]. Adding work of KOZŁOWSKI ET AL [1981] and DRANIŠNIKOV [1988a] provides counter-examples with all  $f^{-1}(y) \approx D^7$ .

**CE 2** (79CE2) *If  $f: X \rightarrow Y$  is approximately right invertible, and  $X$  is a compact ANR, must  $Y$  be an ANR? an FANR?* **848. ?**

Open. Toruńczyk (unpublished) has proved this under the additional hypothesis that  $Y$  be  $LC^1$ .  $Y$  must be movable. ( $Y$  is approximately right invertible

if for all  $\epsilon > 0$  there is a map  $g: Y \rightarrow X$  such that  $f \circ g$  is  $\epsilon$ -close to the identity. FANR is the Shape version of ANR, and is equivalent to shape domination by a finite CW-complex.)

- ? 849. **CE 3** (79CE3) *If  $f: X \rightarrow Y$  is a refinable map and  $X$  is a compact ANR, must  $Y$  be an ANR? an FANR?*

Open. Ford and Rogers introduced refinable maps in FORD and ROGERS [1978] and showed that under these hypotheses,  $f$  is approximately right invertible. Ford and Kozłowski showed that  $Y$  is an ANR if  $X$  is finite-dimensional and  $Y$  is  $LC^1$  (FORD and KOZŁOWSKI [1980]). (*Refinable* maps are those that may be uniformly approximated by maps with point inverses all of diameter less than  $\epsilon$  for each  $\epsilon > 0$ .) With  $X$  a compact ANR,  $f$  must be approximately right invertible. The problem is reduced by Toruńczyk to determining whether  $Y$  is  $LC^1$  (cf. remark to CE2.)

**CE 4** (79CE4) *Let  $f: D^n \rightarrow Y$  be a CE map. Must  $Y$  be an AR?*

No. This is open for  $n = 4$  and 5 (CE12). Kozłowski showed CE4 is in general equivalent to the non-existence of dimension-raising CE mappings of metric compacta in KOZŁOWSKI [19 $\infty$ ] (cf. VAN MILL [1986], NOWAK [1985] for explicit alternative analyses); Edwards and Walsh showed that the existence of a CE dimension-raising map defined on a compactum of dimension  $n$  is equivalent to the existence of a counter example to Alexandroff's Problem of integral cohomology dimension  $n$  (EDWARDS [1978], WALSH [1981]); Dranišnikov gave a counter-example to Alexandroff's Problem of integral cohomological dimension 3 in DRANIŠNIKOV [1988a] and a counter example to CE4 for  $n = 6$  in DRANIŠNIKOV [19 $\infty$ a] and has recently eliminated the case  $n = 6$  in DRANIŠNIKOV [19 $\infty$ b]. It is known that no CE map of a space of dimension 1 raises dimension. Dimension 2 is open and already a good problem (cf. DAVERMAN [19 $\infty$ ] for current progress.) However, no cell-like map defined on a 2-dimensional AR raises dimension SCHORI [1980]. For manifolds, R. L. Moore's great theorem MOORE [1932] rules out dimension 2 and KOZŁOWSKI and WALSH [1983] rules out dimension 3.

- ? 850. **CE 5** (79CE5) *If in (79CE4) the non-degenerate point inverses are arcs, must  $Y$  be an AR?*

Open. Of interest because of its connection with decomposition space questions.

**CE 6** (79CE6) *If  $X = D^n$  or  $\mathbb{R}^n$  and  $f: X \rightarrow Y$  is CE, must  $Y$  be contractible?*

No. VAN MILL [1986] and NOWAK [1985] showed this would imply the non-existence of dimension-raising CE maps. The precise range of the phenomenon is unknown.

**CE 7** (79CE7) *Produce a direct proof that if  $f: M^{2n+1} \rightarrow Y$  is a CE map,  $M$  is a manifold without boundary, and  $Y$  is finite-dimensional (equivalently, is an ANR) and has the DDP, then  $Y$  has the  $DD^n P$ .*

Done by BRYANT [1986].

**CE 8** (79CE8) *If  $f: Q \rightarrow Y$  is a CE map onto an AR, must  $Y \approx Q$  if*

- (1) *the collection of non-degenerate point inverses is null (at most finitely many of diameter  $> \epsilon$  for each  $\epsilon > 0$ ) or*
- (2) *the closure in  $Y$  of the non-degeneracy set (the set of points with non-degenerate point inverses) is zero-dimensional?*

No. DAVERMAN and WALSH [1981], or more concretely by applying the inflation technique of DAVERMAN [1981] to DAVERMAN and WALSH [1983a].

**CE 9** (79CE9) *Let  $f: Q \rightarrow Y$  be a CE map with  $Y$  an AR. Suppose that  $Y \times F \approx Q$  for some finite-dimensional compactum  $F$ . Must  $Y \times I^n \approx Q$  for some  $n$ ? What about  $n = 2$  or  $n = 1$ ?* **851. ?**

Yes. DAVERMAN and WALSH [1981] showed that  $Y \times I^2 \approx Q$ . The case  $n = 1$  is open.

**CE 10** (79CE10) *If  $f: Q \rightarrow Y$  is a CE map with a countable set of non-degenerate point inverses, each of which is cellular, is  $Y \approx Q$ ? What about  $Y \times I$ ?*

No. The second counter-example to (CE8) is a counter to this; in this case,  $Y \times I \approx Q$ . (Here, *cellular* means  $\bigcap_{i=1}^{\infty} K_i$ , where the  $K_i$ 's are a nested sequence of co-dimension zero Hilbert cubes with bi-collared boundaries.)

**CE 11** (79CE11) *If  $f: Q \rightarrow Q$  is a CE map with zero-dimensional non-degeneracy set, is each point inverse cellular?*

No. Daverman has done this.

### 2.3. More Problems on Cell-Like Mappings

Here are more currently interesting open questions in (CE).

- ? **852. CE 12** *Let  $f: I^n \rightarrow Y$  be a CE map. Can  $\dim Y$  be greater than  $n$  when  $n = 5$ ? What about  $n = 4$ ?*
- ? **853. CE 13** *Let  $f: I^n \rightarrow Y$  be CE map. If all point inverses are cells of dimension  $\leq 6$  must  $\dim Y$  be  $\leq n$ ? What if all point inverses are cells of dimension  $\leq 5$ ?  $\leq 1$ ?*
- ? **854. CE 14** *If  $f: Q \rightarrow Y$  is a CE map onto an ANR with the  $DD^2P$  and if each point inverse is finite dimensional, must  $Y \approx Q$ ?*
- ? **855. CE 15** *Give an explicit construction of a CE dimension-raising map.*

Other problems involving CE maps appear in other sections.

## 3. D: Dimension

### 3.1. Introduction

This is another section that has seen great advances in the past decade. Dimension Theory is a mature and complex subject, and this List only discusses some aspects of it that seem closely tied to the infinite-dimensional manifolds either through interest or common technique. The connection has been primarily through the CE mapping problem and related questions. At present, the interaction is more intense than ever before. Some references in dimension theory, some with open questions and extensive bibliography are HUREWICZ and WALLMAN [1948], NAGATA [1965], ENGELKING [1978], PASYNKOV ET AL [1979], ENGELKING [1980], ENGELKING and POL [1983], WALSH [1981], RUBIN [1986], DRANIŠNIKOV and ŠČEPIN [1986].

The three most outstanding results are:

- The Edwards-Walsh proof (EDWARDS [1978], WALSH [1981]) that there exist cell-like dimension-raising mappings of manifolds if and only if there exist infinite-dimensional metric compacta of finite integral cohomological dimension.
- DRANIŠNIKOV's [1988a] proof that there exists an infinite-dimensional compact metric space  $X$  with integral cohomological dimension three.

- POL's [1981] construction of an infinite-dimensional compact metric space that is neither countable dimensional nor strongly infinite dimensional.

A space  $X$  is *strongly infinite dimensional* provided that it contains an infinite *essential family*  $\{(A_i, B_i)\}_{i=1}^\infty$ , i.e., a sequence of disjoint pairs of closed subsets with the property that if  $\{S_i\}_{i=1}^\infty$  is a collection of closed *separators* of the  $(A_i, B_i)$ 's ( $S_i$  separates  $A_i$  and  $B_i$ ), then  $\bigcap_{i=1}^\infty S_i \neq \emptyset$ . It is *weakly infinite dimensional* if not strongly infinite dimensional. There are two variants of weak infinite dimensionality:  $X$  is *weakly infinite dimensional in the sense of Alexandroff* (AWID) provided that for each sequence  $\{(A_i, B_i)\}_{i=1}^\infty$  as above, there is a sequence  $\{S_i\}_{i=1}^\infty$  of separators with  $\bigcap_{i=1}^\infty S_i = \emptyset$ ; it is *weakly infinite dimensional in the sense of Smirnov* (SWID) provided that the separators may always be chosen so that  $\bigcap_{i=1}^n S_i = \emptyset$  for some  $n$  depending on  $\{(A_i, B_i)\}_{i=1}^\infty$ . If the  $n$  is independent of  $\{(A_i, B_i)\}_{i=1}^\infty$ , then  $X$  has dimension  $\leq n$ . A space is said to be *of countable dimension* provided that it is a countable union of finite-dimensional subsets. Note that there is an obvious formulation of these essential families definitions in terms of mappings into  $I^n$  and  $Q$ .

A space  $X$  is *of cohomological dimension  $\leq n$  with respect to the abelian group  $G$*  ( $\dim_G(X) \leq n$ ) provided that for each closed subset  $A \subset X$  the inclusion homomorphism

$$i^*: H^n(X; G) \rightarrow H^n(A; G)$$

is surjective. There is an obvious extension for any cohomology theory.

It is well known that all the above definitions agree for finite dimensional compact metric spaces with Lebesgue's covering dimension and inductive dimension (HUREWICZ and WALLMAN [1948], WALSH [1981]) provided that one uses  $G = \mathbb{Z}$ . Since  $H^n(A; G)$  is naturally equivalent to the set of homotopy classes  $[A, K(G, n)]$  of maps into an Eilenberg-MacLane space of type  $(G, n)$ , this may be given a unified treatment (cf. WALSH [1981]).

### 3.2. Progress on Problems of Section D

**D 1 (79D)** If  $f: X \rightarrow Y$  is a CE map with  $X$  compact and finite dimensional, must  $Y$  be finite-dimensional?

No. By combined efforts of DRANIŠNIKOV [1988a], EDWARDS [1978], and WALSH [1981]. Yes, if  $X$  is a 3-manifold, KOZŁOWSKI and WALSH [1983].

**D 2 (79D1)** (Alexandroff) Is there an infinite-dimensional compact metric space with finite integral cohomological dimension?

Yes. DRANIŠNIKOV [1988a].

- ? **856. D 3** (79D2) *Do there exist positive integers  $n$  and  $p_i$  and maps  $f_i: S^{n+p_i} \rightarrow S^n$  such that in the following sequence every finite composition is essential?*

$$\dots \rightarrow S^{n+p_1+p_2} \xrightarrow{\sum^{p_1} f_2} S^{n+p_1} \xrightarrow{f_1} S^n$$

Open. The primary interest in this problem was a claim by R. Edwards that such a system implies the existence of a dimension-raising cell-like map. It is at present unknown whether there exists any infinite sequence of dimension lowering maps of spheres each finite composition of which is essential.

- ? **857. D 4** (79D3) *Classify “Taylor examples”. In other words, what kinds of compacta can occur as CE images of  $Q$ ?*

Open. TAYLOR [1975] used ADAMS’ [1966] inverse system similar to the one above to produce a map from the inverse limit to  $Q$ . There is relevant work in DAVERMAN and WALSH [1983b]. DEVINATZ ET AL [1988] probably implies a classification of all examples of the sort constructed by Taylor.

- ? **858. D 5** (79D4) *Does every infinite-dimensional compact ANR contain  $n$ -dimensional closed sets for each  $n$ ?*

Open.

- ? **859. D 6** (79D5) *Let  $X$  be a compact AR such that for every finite-dimensional compact subset  $A \subset X$  and every open set  $U \subset X$   $H_*(U, U - A) = 0$ . If  $X$  has the  $DD^2P$ , then must  $X$  have the  $DD^nP$  for all  $n$ ?*

Open.

**D 7** (79D6) *Does there exist an infinite-dimensional compactum which is neither of countable dimension nor strong infinite dimension?*

Yes. This question of P. Alexandroff is answered by the example in POL [1981].

### 3.3. More Problems on Dimension

- ? **860. D 8** *Characterize those metric compacta that are the CE images of finite dimensional ANR’s (or manifolds).*
- ? **861. D 9** *Give an explicit construction of a Dranišnikov compactum (i.e., infinite-dimensional but of finite integral cohomological dimension).*

**D 10** (P.S. Alexandroff) *Must a product of two  $A$ -weakly infinite dimensional compacta be  $A$ -weakly infinite-dimensional?* **862. ?**

**D 11** *Must every AWID compactum have Property C?* **863. ?**

Property C lies between countable dimensionality and AWID. A space  $X$  has *Property C* provided that for each sequence  $\{\mathcal{U}\}_{i=1}^{\infty}$  of open covers there is an open cover  $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ , where each  $\mathcal{V}_i$  is a collection of pairwise disjoint open sets refining  $\mathcal{U}_i$ . It is known (ROHM [190a, 190b]) that products of  $\sigma$ -compact C spaces are C-spaces and that the product of an AWID compactum with a C-space is AWID.

**D 12** *Is there a compact metric space of Pol type (i.e., neither strongly infinite dimensional nor countably infinite dimensional) containing no strongly infinite dimensional subspace? What about strongly countable dimensional subspaces?* **864. ?**

Pol's example has both; it also has Property C.

**D 13** *Is there an AWID compactum  $X$  with  $\dim_{\mathbb{Z}}(X) < \infty$ ?* **865. ?**

**D 14** *Is there an infinite dimensional compactum  $X$  with* **866. ?**

(1)  $\dim_{\mathbb{Z}}(X) = 2$ ?

(2)  $\dim_{\mathbb{Z}}(X \times X) = 3$ ?

**D 15** *If  $\dim_{\mathbb{Z}}(X \times X) = 3$ , can a cell-like mapping  $f: X \rightarrow Y$  raise dimension?*

(Added in proof: Daverman [190] has just announced a negative answer to D15.)

**D 16** *Is there a (generalized) homology theory  $h_*$  with  $h_*(CP^{\infty}) = 0$ ?* **867. ?**

If so, then Dranišnikov can show that the answer to part (2) of (D14) above is "Yes".

**D 17** *Is  $\dim(X) = \dim_{\mathcal{S}}(X)$  where  $\mathcal{S}$  denotes stable cohomotopy?* **868. ?**

**D 18** *Is there an infinite-dimensional compact ANR  $X$  with  $\dim_G(X) < \infty$  for some group  $G$ ?* **869. ?**



Edwards, Kozłowski, and Walsh know that  $G \neq \mathbb{Z}$ .

- ? 870. **D 19** *Let  $X$  be separable metric with  $\dim_{\mathbb{Z}}(X) < \infty$ . Is there a metric compactification of  $X$  of finite integral cohomological dimension? of the same integral cohomological dimension?*
- ? 871. **D 20** *Is there for each  $n$  a universal metric compactum  $X$  for the class of metric compacta of integral cohomological dimension  $n$ ?*

## 4. SC: Shapes of Compacta

### 4.1. Introduction

Shape Theory and Infinite-Dimensional Topology have been interacting deeply since the inception of each. Quite possibly the most fruitful interaction was also one of the earliest: Chapman's Complement Theorem, which states that two metric compacta are of the same shape if and only if, when embedded in the Hilbert cube as  $Z$ -sets (i.e.,  $\pi_*(U, U - X) = 0$  for each open set  $U \subset Q$ ), their complements are homeomorphic and goes on to provide a category isomorphism with a weak proper homotopy category of the complements (CHAPMAN [1972]). This suggested the question: If we use proper homotopy, what is the relation to Shape Theory? This spawned what is now known as Strong Shape. Quite a few of the deeper questions in Shape Theory are particularly difficult or false for the infinite-dimensional spaces, e.g., Whitehead theorems, pointed versus unpointed shape, and CE maps. Note that Dranišnikov's example together with the analyses of KOZŁOWSKI [19 $\infty$ ] and EDWARDS [1978], WALSH [1981] imply that CE maps of finite-dimensional compacta need not be Shape equivalences if the images are not finite-dimensional. A point of philosophy that bears repetition is that many phenomena of homotopy theory that involve infinite sequences of spaces of ever increasing dimension may be realized as geometrical properties of the infinite dimensional compacta or at the ends of locally compact manifolds.

For maps, Shape equivalence and Strong Shape equivalence have quite concrete characterizations: an inclusion of metric compacta  $i: A \rightarrow X$  is a Shape equivalence provided that

1. each map  $f: A \rightarrow K$  extends to a map  $F: X \rightarrow K$ , whenever  $K$  is a CW-complex, and
2. all such extensions are homotopic.

A map  $f: X \rightarrow Y$  of metric compacta is a *shape equivalence* provided that the inclusion of  $X$  into the mapping cylinder  $M_f$  of  $f$  is a shape equivalence;  $f$  is

a *strong shape* equivalence if in addition, the inclusion of  $X$  into the double mapping cylinder  $M_f \cup_X M_f$  is a shape equivalence.

The problems in this section are limited to those that are directly concerned with constructions and phenomena arising in the  $Q$ -manifold setting, or directly related to it. For shape theory *per se* and its applications to the theory of continua, the reader should consult other sources.

#### 4.2. Progress on Problems of Section SC

**SC 1 (79SC1)** *Can cell-like maps raise dimension?*

Yes. DRANIŠNIKOV [1988a].

**SC 2 (79SC2)** *If  $f: X \rightarrow Y$  is a CE map with non-degeneracy set of countable dimension, must  $f$  be a shape equivalence?*

No. DAVERMAN and WALSH [1983a].

**SC 3 (79SC3)** *Let  $M \xrightarrow{f} X \xleftarrow{g} N$  be a diagram of CE maps, where both  $M$  and  $N$  are compact  $Q$ -manifolds and  $X$  is a metric compactum. Is there a homeomorphism  $h: M \rightarrow N$  such that  $g \circ h$  is arbitrarily close to  $f$ ?* **872. ?**

Open. Ferry has provided an uncontrolled homeomorphism (but not a controlled one)  $M \approx N$ , which allows the assignment of a unique simple homotopy type to  $X$ , that carried by  $M$ . He has also shown that the natural idea of declaring CE maps of compacta to be simple homotopy equivalences produces only homotopy theory, even if one restricts attention to finite-dimensional spaces dominated by compact polyhedra FERRY [1980b], cf. FERRY [1981a].

**SC 4 (79SC4)** *If  $X$  and  $Y$  are shape equivalent  $UV^1$  compacta, is there a finite diagram* **873. ?**

$$X = X_0 \longleftrightarrow X_1 \longleftrightarrow \dots \longleftrightarrow X_n = Y$$

where each  $\longleftrightarrow$  is an hereditary shape equivalence either from  $X_i$  to  $X_{i+1}$  or from  $X_{i+1}$  to  $X_i$ ?

Open. An *hereditary shape equivalence* is a map  $f: X \rightarrow Y$  such that for each closed subset  $A$  of  $Y$ ,  $f|_{f^{-1}(A)}: f^{-1}(A) \rightarrow A$  is a shape equivalence. This problem is solved affirmatively if  $X$  and  $Y$  are 1-dimensional (DAVERMAN and VENEMA [1987a]). This sort of chain of equivalences defines the concept “hereditary shape equivalent”. In this way, we get “CE equivalent”, etc. KOZŁOWSKI [19∞] showed that for finite dimensional spaces, hereditary shape equivalence coincides with CE equivalence, cf. DYDAK and SEGAL [1978]. FERRY [1980a] has shown that the  $UV^1$  condition is necessary,

even for 1-dimensional continua. A compact metric space  $X$  is  $UV^k$  provided it embeds in  $Q$  so that for each neighborhood  $U$  of  $X$  in  $Q$ , there is a smaller one  $V$  such that the inclusion  $V \hookrightarrow U$  is zero on  $\pi_i$  for all  $i \leq k$ . There is much relevant work in this general area. See TAYLOR [1975], EDWARDS and HASTINGS [1976b], FERRY [1980b, 1980c], HASTINGS [1983], DAVERMAN and VENEMA [1987a, 1987b], FERRY [1987], MROZIK [1980a, 1980b] KRASINKIEWICZ [1977, 1978], FERRY [1980b]. In particular,

- TAYLOR [1975] shows that CE equivalent infinite dimensional continua need not be shape equivalent.
- EDWARDS and HASTINGS [1976b] show shape equivalent compacta are strong-shape equivalent.
- FERRY [1980b] shows that for compacta homotopy equivalence implies CE equivalence.
- FERRY [1980c] shows that a compactum  $X$  is shape equivalent with an  $LC^n$  continuum if and only if  $\text{pro-}\pi_k(X)$  is stable for each  $k < n$  and is Mittag-Leffler for  $k = n$ . It also shows that a compactum  $X$  with  $\text{pro-}\pi_1(X)$  trivial is shape equivalent with a compactum  $Y$  for which the shape and strong shape are indistinguishable by finite dimensional compacta.
- HASTINGS [1983] shows that suspensions of strong shape equivalences are CE equivalences, so that suspensions of shape equivalent compacta are CE equivalent.
- DAVERMAN and VENEMA [1987a] show that CE equivalence agrees with shape equivalence for locally connected, one dimensional compacta and generalizes the example of FERRY [1980a] to give for each  $n \geq 1$ ,  $n$ -dimensional  $LC^{n-2}$  continua shape equivalent with  $S^n$  but not CE equivalent with  $S^n$ .
- CHIGOGIDZE [1989] shows that shape equivalent  $LC^n$  compacta are  $UV^n$ -equivalent.
- DAVERMAN and VENEMA [1987b] show that locally connected continua are homotopy equivalent if and only if they are CE equivalent in the category of locally connected continua.
- FERRY [1987] shows that  $UV^k$ -equivalent  $k$ -dimensional compacta are shape equivalent and that, conversely, if  $X$  is a continuum with  $\text{pro-}\pi_1(X)$  profinite, then continua shape equivalent with  $X$  are  $UV^k$  equivalent with it for all  $k$ .

- MROZIK [19∞a, 19∞b] shows that if  $X$  is a continuum with  $\text{pro-}\pi_1(X)$  not profinite, then there are continua shape equivalent with  $X$  that are not CE equivalent with  $X$ .
- KRASINKIEWICZ [1977] shows that continuous images of pointed-1-movable continua are pointed-1-movable, and KRASINKIEWICZ [1978] shows that pointed-1-movable continua are shape equivalent with locally connected continua.
- FERRY [19∞b] shows that for continua, profiniteness of  $\text{pro-}\pi_1$  is equivalent to every continuum in the shape class being the continuous image of a CE set, which is equivalent to every continuum in the shape class being the continuous image of a  $UV^1$  continuum.

**SC 5 (79SC5)** *Characterize ANR divisors. Is the property of being an ANR divisor invariant under shape domination?* **874. ?**

Open.  $P$  is an *ANR divisor* if there is an embedding of  $P$  in some ANR  $X$  such that  $X/P$  is an ANR. Dydak has shown finite shape dimensional  $P$  are ANR divisors if and only if they are nearly 1-movable and have stable pro-homology (cf. DYDAK [1978, 1979]).

**SC 6 (79SC6)** *When is the one-point compactification of a locally compact ANR an ANR?* **875. ?**

Open. Note that if  $X$  is compact and embedded in  $Y$ , then  $Y/X$  is the one-point compactification of  $Y - X$ . If  $Y^*$  is the one-point compactification of  $Y$ , then Dydak's characterization of finite dimensional ANR divisors extends to (SC6):  $Y^*$  is an ANR if and only if the end of  $Y$  is nearly 1-movable and has stable pro-homology. The interesting case is the infinite dimensional one. Dydak has an example of an ANR divisor of infinite shape dimension and thus not an FANR (i.e., not shape dominated by a finite complex). This problem is quite important. For example, the classification of compact Lie group actions on  $Q$  that are free off a single fixed point needs this as an ingredient. (See Section GA in this Problem List and the discussion in GEOGHEGAN [1979].)

**SC 7 (79SC7)** *Is there a theory like that of CHAPMAN and SIEBENMANN [1976] for completing a non-compact  $Q$ -manifold into a compact one by adding a shape  $Z$ -set?* **876. ?**

A closed subset  $A \subset X$  of a locally compact ANR is a  *$Z$ -set* if for every open cover  $\mathcal{U}$  of  $X$  there is a homotopy  $F: X \times I \rightarrow X$  that is limited by  $\mathcal{U}$  and stationary off the star  $st(A)$  of  $A$  in  $\mathcal{U}$  such that  $f_0 = id_X$  and for each  $t > 0$ ,  $f_t(X) \subset X - A$ . If we require only  $f_t(X - A)$  to miss  $A$  for  $t > 0$ , and only that  $f$  be supported on  $st(A, \mathcal{U})$ , then  $A$  is a *shape  $Z$ -set*.

- ? 877. **SC 8** (79SC8) *Are there versions of Chapman's Complement Theorem for shape  $Z$ -sets?*

Open. However, for  $n$ -manifolds, the complement theorem of LIEM and VEN-EMA [19 $\infty$ ] captures the essence of this question and goes beyond it.

- ? 878. **SC 9** (79SC9) *Let  $X$  be an FANR and a  $Z$ -set in  $Q$ , and let  $h$  be a homeomorphism of  $X$  that is homotopic to the identity in each of its neighborhoods in  $Q$ . Is there a nested sequence  $M_i \supseteq M_{i+1}$  of compact  $Q$ -manifold neighborhoods of  $X$  with  $\bigcap_{i=1}^{\infty} M_i = X$  and an extension of  $h$  to a homeomorphism  $H$  of  $Q$  such that for all  $i$ ,  $H(M_i) = M_i$ ?*

Open. It implies all compact FANR's are pointed FANR's, a result obtained by HASTINGS and HELLER [1982].

- ? 879. **SC 10** (79SC10) *If  $X$  and  $Y$  are  $Z$ -sets in  $Q$  and if  $f: Q - X \rightarrow Q - Y$  is a proper map that is a weak proper homotopy equivalence, is it a proper homotopy equivalence? In other words, is every shape equivalence of metric compacta a strong shape equivalence?*

Open. Two proper maps  $f, g: X \rightarrow Y$  are *weakly properly homotopic* provided that there is for each compact set  $C \subseteq Y$  a compact set  $K \supseteq f^{-1}(C) \cup g^{-1}(C)$  and a homotopy  $F: X \times I \rightarrow Y$  from  $f$  to  $g$  such that  $F((X - K) \times I) \cap C = \emptyset$ . D. Edwards and Hastings have proved that every weak proper homotopy equivalence is weakly properly homotopic to a proper one (EDWARDS and HASTINGS [1976b]). This shows that shape equivalent compacta are strong-shape equivalent.

- ? 880. **SC 11** (79SC11) *Let  $X$  and  $Y$  be connected  $Z$ -sets in  $Q$ . Let base rays be chosen for the ends of  $Q - X$  and  $Q - Y$ . Let  $f: Q - X \rightarrow Q - Y$  be a proper, base ray preserving map that is invertible in "base ray preserving weak proper homotopy" theory. Is  $f$  a proper homotopy equivalence?*

Open. This would be true if  $Q$  were finite dimensional. DYDAK and GEOGHEGAN [1986a, 1986b] have made significant progress on this topic. For shape theoretic reformulations of (79SC10) and (79SC11), see DYDAK and SEGAL [1978], page 141. The following is a pleasant one:

- ? 881. **SC 12** (79SC12) *Let  $i: X \rightarrow Y$  be an embedding of one compactum in another. Suppose that  $i$  is a shape equivalence. Must it follow that whenever  $f, g: Y \rightarrow P$  are maps into an ANR with  $f|_X = g|_X$ , then  $f \simeq g$  rel.  $X$ ?*

- ? 882. **SC 13** (79SC13) *Can one choose a representative in the shape class of each*

$UV^1$  compactum so that on this class, strong shape equals homotopy theory?

Open. But see the discussion after (SC4).

**SC 14** (79SC14) *Let  $(X, x)$  be a pointed, connected compactum with stable  $\text{pro-}\pi_i(X, x)$  for all  $i$ . Is  $X$  shape equivalent with a locally contractible compactum?* **883.** ?

Yes, for finite-dimensional  $X$  (EDWARDS and GEOGHEGAN [1975], FERRY [1980c]).

**SC 15** (79SC15) *Let  $X$  be compact, locally connected and dominated by a finite complex. Must  $X$  be homotopy equivalent with one? What if  $X$  is locally 1-connected?* **884.** ?

Open. The connectivity hypothesis is necessary by FERRY [1980b].

#### 4.3. More Problems on Shapes of Compacta

**SC 16** *If  $X$  and  $Y$  are shape equivalent  $LC^k$  compacta, are they  $UV^k$  equivalent?* **885.** ?

This is true for compacta such that  $\text{pro-}\pi_1$  is profinite (FERRY [1987]).

**SC 17** *Let  $\Omega$  be the set of all countable ordinals. Does there exist a function  $\beta: \Omega \rightarrow \Omega$  such that if  $f: X \rightarrow Y$  is an hereditary shape equivalence between two countable dimensional compacta, then  $\text{ind}(Y) \leq \beta(\text{ind}(X))$ ?* **886.** ?

**SC 18** *If  $X$  and  $Y$  are shape equivalent continua with  $\text{pro-}\pi_1$  profinite, are  $X$  and  $Y$  CE equivalent? Are they  $UV^\omega$ -equivalent?* **887.** ?

**SC 19** *Let  $X$  be a finitely dominated compactum with Euler characteristic  $\chi(X) = 0$ . Is the Nielsen number of the identity map of  $X$  always zero?* **888.** ?

In GEOGHEGAN [1981], it is shown that this is equivalent with a conjecture of H. Bass concerning the integrality of the Hattori trace. He shows that it is true provided that  $\tilde{K}_0(Z[\pi_1(X)])$  is torsion, which obtains, for example, when  $\pi_1(X)$  is finite. Geoghegan's argument makes use of Ferry's solution (FERRY [1981a, 1980b]) of a problem of J.H.C. Whitehead which shows that for all finitely presented groups  $G$ , all elements of  $\tilde{K}_0(Z[G])$  may be realized as the obstructions to homotopy-finiteness of finitely dominated metric continua  $X$  with  $\pi_1(X) = G$ . See also EDWARDS and GEOGHEGAN [1975].

- ? 889. **SC 20** Let  $f: X \rightarrow Y$  be a map between compact metric spaces with Euler characteristics defined. Suppose that  $\chi(f^{-1}(y)) = 0$  for all  $y \in Y$ . Under what circumstances must  $\chi(X) = \chi(Y)$ ?

One's Euler characteristic depends on one's homology or cohomology theory.

## 5. ANR: Questions About Absolute Neighborhood Retracts

### 5.1. Introduction

The term ANR will be used indiscriminately to denote “absolute neighborhood retract for metric spaces” and “absolute neighborhood extensor for metric spaces”. The characterization of infinite dimensional ANR's in a practical way is one of the most resistant of problems. This is now the hard part of identifying infinite-dimensional manifolds. The majority of the problems in this entire problem set are concerned with ANR's and the ANR property and are distributed under the other headings, so this section is short, but not by neglect of the topic. Repeated here are three useful sufficient conditions collected in GEOGHEGAN [1979] that do not appear in the textbooks:

- A locally contractible metric space that is of countable dimension or has Property C is an ANR (GEOGHEGAN and HAVER [1976]), ADDIS and GRESHAM [1978], GRESHAM [1980], ANCEL [1985]).
- $X$  is an ANR iff for some space  $E$ ,  $X \times E$  has a basis of open sets such that the intersection of any finite subcollection is either empty or path connected and with trivial homotopy groups (TORUŃCZYK [1978]).
- $Y$  is an ANR if there are an ANR  $X$  and a map  $f: X \rightarrow Y$  onto a dense subset of  $Y$  such that for any open cover  $\mathcal{V}$  of  $Y$  there is a homotopy  $h_t$  from  $f$  to  $f \circ g$  that is limited by  $\mathcal{V}$ . KOZŁOWSKI [1980], CORAM ET AL [1985].

On the other hand, two excellent counter examples to two reasonable conjectures are:

- It is not sufficient to postulate a basis of contractible open sets, even for compacta BORSUK [1967, Chapter 5, Section 11] (Cf. DAVERMAN and WALSH [1983b]).
- It is not sufficient to postulate that maps defined on compact sets may be extended continuously (VAN MILL [1986]).

## 5.2. Progress on Problems of Section ANR

**ANR 1** (79ANR1) *If a metric space has a basis of contractible open neighborhoods must it be an ANR?*

No. Examples by BORSUK [1967] (Chapter V, Section 11) and by DAVERMAN and WALSH [1983b].

**ANR 2** (79ANR1a) *If a topological group has a basis of contractible open neighborhoods, is it an ANR?* **890. ?**

Open.

**ANR 3** (79ANR2) *Is a metric space every open subset of which is homotopically dominated by a CW complex necessarily an ANR?* **891. ?**

Open.

**ANR 4** (79ANR3) *Is  $X - X_0$  hazy in  $X$  when* **892. ?**

- *$X$  is a separable linear metric space and  $X_0$  is the linear hull of a countable dense subset, or*
- *$X$  is the component of the identity in the homeomorphism group  $\mathcal{H}(M)$  of a closed PL manifold  $M$  of dimension  $\geq 5$  and  $X_0$  consists of all PL homeomorphisms in  $X$ ?*

Open. A subset  $A$  of  $X$  is termed *hazy* (Kozłowski) if the inclusion  $U - A \hookrightarrow U$  is a homotopy equivalence for each open set  $U$  of  $X$ . The point is that Kozłowski has shown that if  $X - X_0$  is hazy and  $X_0$  is an ANR, then  $X$  is an ANR, and the two  $X_0$ 's are known to be ANR's, HAVER [1973], KEESLING and WILSON [1975]. In the second part, it is known that the relevant inclusions are weak homotopy equivalences (GEOGHEGAN and HAVER [1976]).

**ANR 5** (79ANR4) *Let  $X$  be a non trivial homogeneous contractible compactum. Is  $X$  an AR? Is  $X$  a Hilbert cube?* **893. ?**

Open.

**ANR 6** (79ANR5) *Let  $X$  be a separable contractible homogeneous complete non-locally compact metric space. Is  $X$  an AR? Is  $X \approx s$ ?* **894. ?**

Open. It is not known whether  $X$  must be an ANR, but it need not be  $s$  by ANDERSON ET AL [1982].

**ANR 7** (79ANR6) *When are homogeneous spaces ANR's?* **895. ?**



### 5.3. More Problems on ANR's

? 896. **ANR 8** Is  $Q$  the only homogeneous non-degenerate compact AR?

? 897. **ANR 9** Is  $Q$  the only homogeneous continuum homeomorphic with its own cone?

This is true for compact AR's.

? 898. **ANR 10** Let  $f: A \rightarrow X$  be a CE map from an ANR  $A$  onto  $X$ . If  $X$  is not strongly infinite-dimensional, must  $X$  be an ANR?

? 899. **ANR 11** Are the Banach-Mazur compacta  $Q(n)$  AR's? Are they Hilbert cubes?

$Q(n)$  is the set of isometry classes of  $n$ -dimensional Banach spaces topologized by the metric

$$d(E, F) = \ln \inf \{ \|T\| \cdot \|T^{-1}\| \mid T: E \rightarrow F \text{ is an isomorphism} \}$$

It is known that  $Q(n)$  is compact, metric, and contractible. For  $n = 2$ , it is known to be locally contractible.  $Q(n) \approx C(\mathbb{R}^n)/\sim$ , where  $C(\mathbb{R}^n)$  is the hyperspace of all compact convex bodies in  $\mathbb{R}^n$  with the Hausdorff metric and  $\sim$  is the equivalence relation induced by the natural action of  $GL(n)$ .

? 900. **ANR 12** Let  $X$  be an ANR with Toruńczyk's strong discrete approximation property (SDAP). Is there a completion  $X'$  of  $X$  with  $X' - X$  locally homotopically negligible in  $X'$  and such that  $X'$  enjoys the strong discrete approximation property? (b) What if  $X$  is merely  $LC^{n-1}$  with the  $n$ -SDAP, and we ask for  $X' - X$  to be locally  $n$ -homotopy negligible and  $X'$  to have the  $n$ -SDAP,  $n = 1, 2, \dots, \infty$ ?

$X$  enjoys the strong discrete approximation property (SDAP) provided that given a map  $f: Q \times Z \rightarrow X$  and an open cover  $\mathcal{U}$  of  $X$ , there is an embedding  $g: Q \times Z \rightarrow X$   $\mathcal{U}$ -close to  $f$  such that the collection  $\{g(Q \times n)\}$  is discrete.  $n$ -SDAP uses  $n$ -cells instead of  $Q$ .  $A$  is locally homotopically negligible in  $X$  provided that for each open set  $U$  of  $X$ , the inclusion  $U - A \hookrightarrow X - A$  is a weak homotopy equivalence; it is locally  $n$ -homotopy negligible in  $X$  if  $U - A \hookrightarrow X - A$  is an isomorphism on  $\pi_j$ ,  $0 \leq j \leq n$ .

? 901. **ANR 13** Is every  $\sigma$ -compact space with the compact extension property an ANR?

$X$  has the *compact extension property* provided that each map into  $X$  defined on a compact metric space  $A$  extends to any separable metric space containing  $A$ . This is proposed in light of VAN MILL's [1986] example which is a space with the compact extension property that is not an ANR.

Many more problems and questions concerning ANR's occur throughout this chapter. See especially the sections on homeomorphism spaces and linear spaces.

## 6. QM: Topology of $Q$ -manifolds

### 6.1. Introduction

For background, the two generally available sources are CHAPMAN [1976] and the new book by VAN MILL [1989b]. The essential advance covered in van Mill's book but not in Chapman's is also the most important theorem in the subject: Toruńczyk's Characterization Theorem (TORUŃCZYK [1980]) which states that a locally compact ANR  $X$  is a  $Q$ -manifold if and only if for each  $n$  and each  $\epsilon > 0$  each pair of maps  $f, g: I^n \rightarrow X$  may be  $\epsilon$ -approximated by maps with disjoint images. (It is interesting that Toruńczyk and Cannon divined the same concept at the same time, and apparently totally independently.) This result, and its refinement by DAVERMAN and WALSH [1981] to a question of maps of  $D^2$  and disjoint carriers for Čech homology  $n$ -cycles, provides an extremely effective tool for identifying  $Q$ -manifolds. (The difficulty now rests primarily in verification of the ANR property, which is still a major difficulty for strongly infinite-dimensional spaces.) The emphasis has thus shifted from questions of identification to questions of structure and maps.

### 6.2. Progress on Problems of Section QM

**QM 1 (79QM1)** Let  $M$  be a compact  $Q$ -manifold, and let  $f: M \rightarrow M$  be a map such that  $f^2$  is homotopic to the identity. When is  $f$  homotopic to an involution? **902. ?**

Open.

**QM 2 (79QM2)** Let  $\pi$  be a group such that there exist compact  $K(\pi, 1)$   $Q$ -manifolds  $M$  and  $N$ . Must they be homeomorphic? **903. ?**

Open, but see work by FARRELL and HSIANG [1981, 1983], FARRELL and JONES [1986a, 1986b, 1988a, 1988b, 1990, 1989].

**QM 3 (79QM3)** Let  $\alpha$  be an open cover of  $N$  a compact  $Q$ -manifold. What conditions imply that any  $\alpha$ -equivalence  $f: M \rightarrow N$  is homotopic to a homeomorphism? **904. ?**

Here,  $f$  is an  $\alpha$ -equivalence provided it is a homotopy equivalence and there are a homotopy inverse  $g$  of  $f$  and homotopies  $F: gf \simeq id_N$  and  $G: fg \simeq id_M$  that are limited by the open covers  $\alpha$  and  $f^{-1}(\alpha)$ , respectively. This is still a very good problem. There has been much recent work on the general topic of Controlled topology, Cf. CONNELL and HOLLINGSWORTH [1969], QUINN [1979, 1982a, 1982b, 1983, 1987], CHAPMAN [1983b, 1984], DAVERMAN [1986], ANDERSON and MUNKHOLM [1988]. FERRY [1977b] and TORUŃCZYK [1977] proved that the homeomorphism group of a compact  $Q$ -manifold is an ANR, from which is drawn in FERRY [1977b] the fact that sufficiently fine covers  $\alpha$  will do. (This is one version of Ferry's  $\alpha$ -approximation theorem. Cf. CHAPMAN and FERRY [1979], FERRY [1979].) In general, it is important to know the analog of this for the more general controlled set-up where  $N$  is equipped with a control map  $p: N \rightarrow X$ , for various kinds of spaces  $X$ . The general problem is obstructed, but there are many cases in which it is not. (See also CHAPMAN [1980, 1981a, 1982, 1983a, 1983c], ANDERSON and HSIANG [1976, 1977, 1980], CHAPMAN and FERRY [1977, 1978, 1990, 1979, 1983], HUGHES [1983, 1985, 1987, 1988], HUGHES ET AL [1990a], FERRY and PEDERSEN [1990a, 1978], PEDERSEN [1984], PEDERSEN and WEIBEL [1985], QUINN [1988], FARRELL and JONES [1988a, 1986b, 1988a, 1988b, 1990, 1989], STEINBERGER and WEST [1985, 1986, 1987a, 1987b, 1990a], TORUŃCZYK and WEST [1989]).

- ? 905. **QM 4** (79QM4) *Let  $M$  be a compact  $Q$ -manifold and  $\mathcal{U}$  a finite open cover of  $M$  by contractible open sets such that the intersections of subcollections of  $\mathcal{U}$  are either contractible or void. Is  $M$  homeomorphic with  $N(\mathcal{U}) \times Q$ ?*

Open. Here,  $N(\mathcal{U})$  is the nerve. The hypotheses guarantee that the barycentric maps  $M \rightarrow N(\mathcal{U})$  are homotopy equivalences. The  $\alpha$ -approximation theorem says "Yes" for sufficiently fine  $\mathcal{U}$ . Does CAUTY [1986] bear on this?

- ? 906. **QM 5** (79QM5) *Is there a  $Q$ -manifold version of Quinn's End theorems?*

A version of this was done in CHAPMAN [1982]. There is continuing work on controlled topology of  $Q$ -manifolds within which there is room for a re-working of this result. (See, for example, a sequence of papers by CHAPMAN [1980] and HUGHES [1983, 1985, 1987], HUGHES ET AL [1990a].) There is more to be said on this topic. The Farrell-Jones controlled  $h$ -cobordism theorem with foliated control space with hyperbolic leaves (FARRELL and JONES [1988a, 1988b]) should also be investigated in this context. The issue is structure on the control space involving objects where there is no obstruction and control is transverse to them. On the general topic, Quinn has suggested that his Finite Structure Spectrum in QUINN [1979, 1982a, 1982b] be reworked from a  $Q$ -manifold point of view. Perhaps TORUŃCZYK and WEST [1989] will be useful here.

**QM 6** (79QM6) *Let  $B$  be a compact polyhedron, and let  $\epsilon > 0$  be given. Is there a  $\delta > 0$  such that if  $M, N$  are compact  $Q$ -manifolds and  $f: M \rightarrow N$  and  $p: N \rightarrow B$  are maps such that (1)  $p$  is  $UV^1$  and (2)  $f$  is a  $p^{-1}(\delta)$ -equivalence, then  $f$  is  $p^{-1}(\epsilon)$ -homotopic to a homeomorphism?*

Yes. CHAPMAN [1983a] has done this in a more general form over finite dimensional compact metric  $B$  with fiber conditions more general than the  $UV^1$  condition here. Of interest is the case that  $B$  is not of finite dimension. The problem is reformulated with  $B$  an arbitrary metric compactum is (QM26).

**QM 7** (79QM7) *If  $Y$  is a locally compact polyhedron, when can one add a compactum  $A$  to  $Y$  so that  $Y \cup A$  is a compact ANR and  $A$  is a  $Z$ -set in  $Y \cup A$ ?*

If and only if the CHAPMAN and SIEBENMANN [1976] obstruction vanishes. This is equivalent to the stabilized question. See the discussion following.

**QM 8** (79QM8) *If  $Y$  is a locally compact ANR such that the  $Q$ -manifold  $Y \times Q$  can be compactified by adding a compact  $Z$ -set  $A$  (in which case the compactification is a  $Q$ -manifold), is it possible to compactify  $Y$  by adding a compact  $Z$ -set?*

Yes. The closure in  $Y \times Q \cup A$  of  $Y \times \{0\}$  is the sought ANR compactification of  $Y$ . The discussion in CHAPMAN and SIEBENMANN [1976] concerning finite domination and infinite mapping cylinders now is relevant, and provides necessary and sufficient conditions.

**QM 9** (79QM9) *Let  $p: M \rightarrow B$  be a locally trivial bundle, where  $B$  is a 907. ? locally compact polyhedron and the fibers are  $Q$ -manifolds. Does there exist a locally compact polyhedron  $P$ , a  $PL$  map  $q: P \rightarrow B$ , and a fiber preserving homeomorphism  $h: M \rightarrow P \times Q$ ?*

Open. The compact-fiber case was established by in CHAPMAN and FERRY [19 $\infty$ ] (cf. BURGHELEA [1983].) The general case does not appear to be in the literature. There is now no reason to restrict the question to polyhedral bases.

**QM 10** (79QM10) *If  $E \rightarrow S^1$  is a locally trivial bundle with fiber  $F$  a non-compact  $Q$ -manifold admitting a compactification, when does there exist a locally trivial  $Q$ -manifold bundle each fiber of which is a compactification of  $F$ ?*

Here, compactification is taken in the sense of CHAPMAN and SIEBENMANN [1976]. This was done over any compact polyhedral base in METCALF [1985].

**QM 11** (79QM11) *Is every Hurewicz fibration over a compact ANR base with compact  $Q$ -manifold fibers a locally trivial bundle?*

No. In TORUŃCZYK and WEST [1989] is given an example of a fibration of a Hilbert cube by convex Hilbert cubes over a Hilbert cube that is not a bundle. Also given is a fibred version of Toruńczyk's criterion that detects bundles among the fibrations with compact ANR fibers (In CHAPMAN and FERRY [1977] it is shown that if the base is locally finite dimensional, then the fibration is always a bundle.)

? **908. QM 12** (79QM12) *Let  $p: M \rightarrow B$  be a Hurewicz fibration where  $M$  is a compact  $Q$ -manifold. Is  $B$  an ANR?*

Open. Note that the answer is “No” if “Hurewicz fibration” is weakened to “cell-like map with discs as point inverses”, as a consequence of DRANIŠNIKOV [1988a].

? **909. QM 13** (79QM13) *Let  $p: M \rightarrow S^2$  be an approximate fibration, where  $M$  is a compact  $Q$ -manifold. Must the fiber have the shape of a finite complex?*

Open. This is not true for  $p: M \rightarrow S^1$ , FERRY [1977a].

**QM 14** (79QM14-16) *Let  $p: M \rightarrow B$  be an approximate fibration with  $M$  a compact  $Q$ -manifold and  $B$  a compact polyhedron. What is the obstruction to:*

- (1) *approximating  $p$  by an  $h$ -block bundle map,*
- (2) *deforming  $p$  to an  $h$ -block bundle map,*
- (3) *approximating  $p$  by locally trivial bundle maps,*
- (4) *deforming  $p$  to a locally trivial bundle map?*

An  $h$ -block bundle map  $p': M \rightarrow B$  is such that there is a space  $F$  and for each simplex  $\sigma$  of  $B$ , a homotopy equivalence  $h_\sigma: p'^{-1}(\sigma) \rightarrow \sigma \times F$  such that for each face  $\tau$  of  $\sigma$ ,  $h_\sigma \mid p'^{-1}(\tau): p'^{-1}(\tau) \rightarrow \tau \times F$  is a homotopy equivalence. In CHAPMAN and FERRY [1983], examples are given over  $S^2$  of approximate fibrations that are homotopic to bundle projections, yet cannot be approximated by them. In HUGHES [1983], Hughes shows that approximate fibrations may be approximated by bundle maps if and only if they are homotopic to bundle maps through approximate fibrations. In HUGHES ET AL [1988a], Hughes, Taylor and Williams give a classification of manifold approximate fibrations over a finite-dimensional manifold and show that the problem of approximation by bundle maps may be viewed as a lifting problem, hence is obstructed by certain cohomology classes in the base. In more recent work, they have identified the obstructions to approximation by block

bundle projections. (This has been done for  $M$  finite-dimensional, too. See also QUINN [1979, 1982a, 1982b].)

**QM 15** (79QM17) *Let  $f: M \rightarrow B$  be a map, with  $M$  a compact  $Q$ -manifold and  $B$  a compact polyhedron. What is the obstruction to homotoping  $f$  to an approximate fibration?* **910. ?**

Open. (Is this contained in HUGHES ET AL [19∞a] and the other work of Hughes and Hughes, Taylor and Williams?)

**QM 16** (79QM18) *Does there exist a  $Q$ -manifold pair of codimension greater than 2 having two non-isotopic tubular neighborhoods?*

Yes, NOWELL [19∞].

**QM 17** (79QM19) *Does there exist a  $Q$ -manifold pair having open tubular neighborhoods but no closed subtubes?*

Yes, stringing together CHAPMAN [1978], NOWELL [19∞], and BROWDER [1966].

**QM 18** (79QM20) *Let  $X$  be a connected pointlike compactum in  $Q$  that is the closure of its interior and such that  $Q - X$  is pointlike. Is there a compactification  $Y$  of  $Q - X$  such that  $Y - (Q - X)$  is a  $Q$ -manifold and each homeomorphism  $h: (Q, X) \rightarrow (Q, X)$  extends to one on  $Y$ ?* **911. ?**

Open.

### 6.3. More Problems on $Q$ -manifolds

This section contains additional problems. There are others in the section on group actions, ANR's, and natural phenomena.

**QM 19** *Suppose that  $A$  is a closed finite-dimensional subset of  $Q$  that can be instantly isotoped off itself. Must  $A$  be a  $Z$ -set?* **912. ?**

$A$  can be *instantly isotoped off itself* provided that there is an isotopy

$$H: Q \times I \rightarrow Q \text{ with } h_0 = id_Q \text{ and for } t > 0, h_t(A) \cap A = \emptyset.$$

It is known that in  $\mathbb{R}^n$ , a Cantor set is tame if and only if it can be isotoped off itself instantly (WRIGHT [1976]).

**QM 20** *Is there a Cantor set  $C$  in  $Q$  and  $\epsilon > 0$  such that for each homeomorphism  $h: Q \rightarrow Q$  that is  $\epsilon$ -close to the identity,  $h(C) \cap C \neq \emptyset$ ?* **913. ?**

There is a wild Cantor set in  $Q$  with contractible complement. Daverman (unpublished) has constructed one by infinite inflation of a crumpled cube that is ideally flat modulo a Cantor set and has contractible interior.

- ? 914. **QM 21** *Is every homeomorphism of  $s$  the composition of two conjugates of homeomorphisms that extend to  $Q$ ?*

$s$  is the countably infinite product of lines. At least two are in general necessary (VAN MILL [1989a]).

- ? 915. **QM 22** *Is there a Čech-cohomology version of the fibred general position theory of TORUŃCZYK and WEST [1989] along the lines of DAVERMAN and WALSH [1981] that detects the  $Q$ -manifold bundles among the ANR-fibrations with compact fibers?*

- ? 916. **QM 23** *Is there a tameness condition at infinity that will extend the “general position fibrations are bundles” theorem of TORUŃCZYK and WEST [1989] to more bases?*

- ? 917. **QM 24** *Let  $p: B \times Q \rightarrow B$  be projection. Let  $A \subset B \times Q$  be closed. Under what conditions is  $p|_B: B \times Q - A \rightarrow B$  an Hurewicz fibration? An ANR-fibration in the sense of TORUŃCZYK and WEST [1989]? A bundle?*

FERRY [19∞a] has shown that if  $p|_A: A \rightarrow B$  is a “cohomology fibration”,  $B$  is a finite-dimensional polyhedron, and  $A$  is finite dimensional, then  $A$  may be embedded fiber-wise in  $B \times S^n$  so that its complement is a fibration, i.e., “S-duals of cohomology fibrations are Hurewicz fibrations”.

- ? 918. **QM 25** *Extend (79QM6) to arbitrary metric bases.*

- ? 919. **QM 26** *Can (79QM10) be extended to a fibred end theorem for  $Q$ -manifold fibrations over complete metric ANR’s? over arbitrary separable metric spaces?*

Note here the implications of TORUŃCZYK and WEST [1989].

- ? 920. **QM 27** *Let  $M$  be a compact manifold or polyhedron, and let  $N$  a compact  $Q$ -manifold. Classify the embeddings  $f: M \rightarrow N$  such that the pair  $(N, M)$  is homotopically stratified in the sense of QUINN [1988]. (i) Start with  $M = S^1, N = Q$  as in TORUŃCZYK and WEST [1978], (ii) How does the case of  $M$  a  $Q$ -manifold differ from the finite-dimensional case?*

**QM 28** *Classify  $Q$ -manifold approximate fibrations in the sense of HUGHES ET AL [19∞a] over compact  $Q$ -manifolds  $B$ .* **921.** ?

In the finite-dimensional case, the classification theorem of HUGHES ET AL [19∞a] uses essentially the topological tangent bundle of  $B$ , which has no obvious substitute in this case.

**QM 29** *Extend the program of Hughes, Taylor, and Williams over bases  $B$  that are compact polyhedra, not compact manifolds.* **922.** ?

**QM 30** *Let  $p: M \rightarrow B$  be an approximate fibration of a compact  $Q$ -manifold over an aspherical manifold. If  $p$  is homotopic to a bundle projection, may it be approximated by them?* **923.** ?

Hughes, Taylor, and Williams prove this if  $B$  is a closed Riemannian manifold of nowhere positive sectional curvature (HUGHES ET AL [19∞b]).

**QM 31** *Let  $M$  be a compact  $Q$ -manifold, and let  $Q_s$  be a Lipschitz-homogeneous Hilbert cube. Are all Lipschitz  $Q_s$ -structures on  $M$  equivalent (via bi-Lipschitz homeomorphisms)?* **924.** ?

In connection with the above problem, Väisälä and Hohti (HOHTI [1985]) have established a theory of Lipschitz-homogeneous metrics on the countably infinite product of intervals, and J. Luukainen has discussed a certain type of uniqueness in LUUKAINEN [1985]. See also LUUKAINEN [1977].

**QM 32** *Give a characterization of the Lipschitz-homogeneous convex compacta in normed linear spaces.* **925.** ?

**QM 33** *For a standard Lipschitz-homogeneous  $Q_s$ , is there a class of Lipschitz  $Z$ -sets (i.e., such that every bi-Lipschitz homeomorphism between them extends to one of  $Q_s$ ) large enough to contain homeomorphs of all finite polyhedra?* **926.** ?

**QM 34** *Give a  $Q$ -manifold version of Quinn's Finite Structure Spectrum in Ends II.* **927.** ?

(QUINN [1979, 1982a, 1982b].)

**QM 35** *Using this, is it then possible to unify QUINN [1979, 1982a, 1982b],* **928.** ?



ANDERSON and MUNKHOLM [1988], CHAPMAN [1983b, 1982], HUGHES ET AL [1980a], FARRELL and JONES [1988a, 1988b]?

The fact that several constructions preserve  $Q$ -manifolds but not  $n$ -manifolds should result in a cleaner treatment.

**? 929. QM 36** (Ganea) *Let  $M$  be a compact  $Q$ -manifold. Let  $C_z(M)$  denote the smallest integer  $k$  such that  $M$  may be covered with  $k$  open sets each homeomorphic with  $Q \times [0, 1)$ . Is it always true that  $C_z(M \times S^n) = C_z(M) + 1$ , where  $S^n$  is the  $n$ -sphere?*

MONTEJANO [1987] has shown that  $C_z(M) = \text{cat}(M) + 1$ , where  $\text{cat}(M)$  denotes the Lusternik-Schnirelmann category of  $M$ . WONG [1988] has proved that  $C_z(M \times [0, 1)) = \text{cat}(M)$ . SINGHOF [1979] has proved  $\text{cat}(L \times S^1) = \text{cat}(L) + 1$  in the case that  $L$  is a closed manifold. This is a reformulation of a problem of GANEA [1971].

## 7. GA: Group Actions

### 7.1. Introduction

In this section, the terminology “ $G$ –” will mean either “equivariant” or “in the category of spaces with actions of  $G$  and equivariant maps”.

The single most fundamental transformation group problem involving (possibly) infinite dimensional spaces is perhaps P. Smith’s generalization of Hilbert’s Fifth Problem, generally known as “The Hilbert-Smith Conjecture”.. It asks whether a locally compact topological group acting effectively on an  $n$ -manifold must be a Lie group. The connection with infinite dimensions is that the problem is reduced in MONTGOMERY and ZIPPIN [1955] to whether there is an effective action of the  $p$ -adic integers

$$A_p = \varprojlim (\mathbb{Z}_p \leftarrow \mathbb{Z}_{p^2} \leftarrow \dots)$$

on an  $n$ -manifold  $M^n$ , and that if there is, then the homological dimension of the orbit space  $M^n/A_p = n + 2$ , whence

$$\text{ind}(M^n/A_p) = n + 2 \text{ or } \infty$$

(YANG [1960], BREDON ET AL [1961]). (As of this moment, it is not clear whether the Hilbert-Smith Conjecture is open: there is a new paper recently circulated by L. McAuley claiming to contain a proof, but it has not as of this writing been verified; anyone interested in this question should consult that manuscript, if not McAuley.)

There has been relatively little progress as of this writing on the group actions problems posed in the 1979 Problem List GEOGHEGAN [1979]. Its exposition is sufficiently good that it needs no updating, except to note that the papers of Vo Thanh Liem discussed have now appeared VO THANH LIEM [1979, 1981, 1983]. The progress that has been made is due to him. The interested reader should consult GEOGHEGAN [1979] and these papers. The primary focus there was on actions on  $Q$  that are free off a single fixed point, called “based-free” actions. This focus derived from the idea that these should be the simplest after the free actions on  $Q$ -manifolds and ought to be relatively easy to classify. After twenty years of inconclusive work using elementary methods, it appears that this is not the case, but that the based-free actions form instead another location in the theory of  $Q$ -manifolds and infinite dimensional topology where serious input from stable homotopy theory is needed before the most basic questions such as the Anderson Conjecture are settled. (This was the case in Taylor’s example as well as in Dranišnikov’s.)

The topic of group actions on  $Q$ -manifolds has been developed in another direction, which has proven to be much more manageable as well as fruitful for applications to the finite-dimensional context. M. Steinberger and J. West have investigated a class of actions which are universal for equivariant topology in the same way that  $Q$  is universal for separable metric spaces. For a finite group  $G$ , let  $Q_G = \prod_{i>0} D_i$ , where  $D_i$  is the unit disc of the regular real representation  $V = R[G] = \{\sum_{g \in G} r_g g \mid r_g \in R\}$ , with  $h \cdot \sum r_g g = \sum r_g \cdot hg$ .  $Q_G$  is a Hilbert cube that contains every irreducible real representation of  $G$  infinitely many times. It is an absolute retract for  $G$ -spaces and every separable metric  $G$ -space embeds in it equivariantly. It is also equivariantly homogeneous, and it is reasonable to consider manifolds that are locally homeomorphic with  $Q_G$ , i.e.,  $Q_G$ -manifolds. This hypothesizes that the actions are locally linear and thus cannot exhibit the local difficulties that have made the based-free case so difficult.

Steinberger and West have established for  $Q_G$ -manifolds all the basic theorems of inequivariant  $Q$ -manifold theory that do not rely upon the vanishing of algebraic  $K$ -theoretic invariants (STEIBERGER and WEST [1986, 198b, 1985]). The point of this discussion is to motivate an extension of this theory to Lie groups. For compact Lie groups,  $G$ , there is a good model for a universal linear action on a Hilbert cube,  $Q_G$ . In particular, if  $G$  is compact and Lie, then let  $Q_G = \prod_{i>0, \rho} D_{\rho, i}$  be the product of the unit balls of all the irreducible orthogonal representations of  $G$ , each representation disc being represented infinitely often. Let  $G$  act on  $Q_G$  by simultaneous action on all the factors. Some questions about this action appear below.

A major development in the interface between topology and group theory is the area beginning to be called “Geometric Methods in Group Theory”. The idea is to understand the structure of groups by use of spaces upon which they act. An example is group cohomology—introduced as the cohomology of

an Eilenberg-MacLane space  $K(G, 1) = E/G$  where  $E$  is a contractible space on which  $G$  acts freely (see BROWN [1982]). If  $G$  has torsion, then it is known that  $\dim(E) = \infty$ . Another is Tits' "buildings" (BROWN [1989]). See also SERRE [1980], GROMOV [1987]. Many questions about precisely the topological and combinatorial properties of the simplest spaces  $E$  for a given class of groups are of interest to a wide audience. This is an attractive area currently undergoing explosive growth that should interact strongly with infinite-dimensional topology. (GA11) is an example of questions in this area, as is (GA29). So is (HS23).

## 7.2. Progress on Problems of Section GA

? 930. **GA 1** (79GA1) (Anderson) *Let  $h: Q \rightarrow Q$  be a based-free involution. Must  $h$  be topologically conjugate with the linear action " $-1$ "?*

(Here,  $Q = \prod_{i>0} [-1, 1]_i$ .)

Open. This holds for  $h$  if and only if the fixed point has a basis of contractible invariant neighborhoods (WONG [1974]) or, equivalently, the space of free orbits is movable at infinity (BERSTEIN and WEST [1978]), or the full orbit space is dominated by a CW-complex, which is equivalent to its being an AR (WEST and WONG [1979]). This is ensured if the action factors as a product of finite-dimensional actions (BERSTEIN and WEST [1978].)

? 931. **GA 2** (79GA2) *Same as above with the action of any finite period.*

Open. Same comments as above.

? 932. **GA 3** (79GA3) *Does there exist a sequence*

$$(E_1, S^\infty) \xleftarrow{\tilde{f}_1} \dots \leftarrow (E_{i-1}, S^\infty) \xleftarrow{\tilde{f}_{i-1}} (E_i, S^\infty) \leftarrow \dots$$

*of principal  $\mathbb{Z}_2$  bundles of CW complexes and bundle maps, each the identity on  $S^\infty$ , such that if  $f_{i-1}: (B_i, RP^\infty) \rightarrow (B_{i-1}, RP^\infty)$ , is the map of base pairs covered by  $\tilde{f}_i$  then (i) each  $(B_i, RP^\infty)$  is relatively finite, (ii) each  $(B_i, RP^\infty)$  is relatively 1-connected, (iii) each  $\tilde{f}_i$  is null homotopic, and (iv) each finite composition of  $f_i$ 's is essential (as maps of pairs)?*

Open. This is equivalent to (GA1) above (BERSTEIN and WEST [1978]).

? 933. **GA 4** (79GA4) *Let the compact Lie group  $G$  act semifreely on  $Q$  in two ways such that their fixed point sets are identical. If the orbit spaces are ANR's, are the actions conjugate?*

Open.

**GA 5** (79GA5) *If  $\alpha$  is a standard action of a finite cyclic group  $G$  on  $Q$ , is  $\mathcal{H}_G(Q)$  a Hilbert manifold? Conversely, if  $\mathcal{H}_G(Q)$  is a Hilbert manifold and  $\alpha$  is a based-free action on  $Q$ , is  $\alpha$  standard?* **934. ?**

Open.  $\mathcal{H}_G(Q)$  denotes the equivariant homeomorphisms of  $Q$ . VO THANH LIEM [1981] has shown that for the standard based-free action of a finite or toral group on  $Q$ ,  $\mathcal{H}_G(Q)$  is locally contractible.

**GA 6** (79GA7) *Under what conditions can a non-free action  $\alpha$  of a compact group on a  $Q$ -manifold  $M$  be factored as a diagonal action* **935. ?**

$$\beta \times \gamma: G \times (N \times Q) \rightarrow N \times Q$$

where  $N$  is a finite-dimensional manifold, polyhedron, or ANR? (Is there any difference between these questions?)

Open. VO THANH LIEM [1979] has shown that for free actions of finite groups, this is always possible.

**GA 7** (79GA8) *Let  $\alpha$  be a semi-free action of a finite group  $G$  on  $Q$  with fixed point set  $F$  a Hilbert cube  $Z$ -set. When is  $\alpha$  equivalent with the product  $\sigma \times id_F$ , where  $\sigma$  is the standard action of  $G$  on  $Q$ ? What if  $F = I^n$ ?* **936. ?**

Open. VO THANH LIEM [1981] has shown that if  $\alpha$  is conjugate to a fiber-preserving action over  $I^n$  of  $Q \times I^n$  and if for each  $t \in I^n$ ,  $\alpha|_{Q \times t}$  is standard, then the answer is “Yes” in the second case. In general, he has proved that for a finite group  $G$  acting on a  $Q$ -manifold  $M$ , if the fixed point sets  $M^H$  of the subgroups of  $G$  are locally flat Hilbert cube submanifolds of  $M$ , then the orbit space  $M/G$  is a  $Q$ -manifold (VO THANH LIEM [1981]).

### 7.3. More Problems on Groups of Transformations

**GA 8** *Is there an effective action by a  $p$ -adic group on an  $n$ -manifold?* **937. ?**

See discussion above.

**GA 9** *Does there exist a smooth minimal diffeomorphism on separable Hilbert space or on each connected separable Hilbert manifold? What about a minimal smooth flow?* **938. ?**

Fathi has shown that there exist minimal *homeomorphisms* on each connected separable Hilbert manifold (FATHI [1984]). A homeomorphism or flow is *minimal* provided that every orbit is dense. C. Read has recently constructed a bounded linear operator on  $l_1$  such that the orbit of every point but 0 is dense. (cf. BEAUZAMY [1988], pp. 75ff, 345, 358.)

- ? 939. **GA 10** *Does every compact connected  $Q$ -manifold with zero Euler characteristic admit a minimal homeomorphism? Does it admit a minimal flow?*

The Euler characteristic zero hypothesis is necessary, as otherwise fixed point theory shows there is a fixed point. GLASNER and WEISS [1979] have shown that  $S^1 \times Q$  admits a minimal homeomorphism.

- ? 940. **GA 11** *Let  $G$  be the group of  $PL$  homeomorphisms of the unit interval with singularities in  $\mathbf{Z}[1/6]$  and slopes in the multiplicative group generated by  $2/3$ . Is  $G$  finitely generated?*

Discussion: BROWN and GEOGHEGAN [1984] discovered the first interesting example of an infinite dimensional discrete group  $G$  with the finiteness property  $F_\infty$ . This means that there is a  $K(G, 1)$  complex with only finitely many cells in each dimension. (It is the same group that Dydak showed was a universal detector of “non-splittable” homotopy idempotents on  $\pi_1$  (DYDAK [1977]), and the demonstration of the infinite dimensionality of the  $K(\pi, 1)$  of which by HASTINGS and HELLER [1982] showed that “non-splittable” homotopy idempotents occur only on infinite dimensional spaces, so that compact FANR’s are pointed FANR’s.) The intuition of Brown and Geoghegan was that this group should be the first example in a theory of “infinite dimensional arithmetic groups”. This intuition was reinforced when Brown later constructed three infinite families of infinite dimensional  $\mathcal{F}_\infty$  groups, with the original  $G$  as the first member of one of these families (BROWN [1987]). All of the groups in these families can be viewed as “arithmetically defined”  $PL$  homeomorphism groups of an interval, a circle, or a Cantor set, and the results obtained about them were obtained by using a suitable notion of “triangulation” to construct contractible complexes on which the groups act.

The long-term goal is to develop a general theory of infinite dimensional arithmetic groups analogous to the Borel-Serre theory for ordinary arithmetic groups. The first step, however, is to understand some slight variants of Brown’s three families, where one knows practically nothing. There are many questions one could ask about the groups, but the above is a concrete one which illustrates how little is currently known. The problem in trying to answer such a simple-looking question is that one does not know how to construct a useful complex on which a  $PL$  homeomorphism group like  $G$  acts, except in the special case where the group of allowable slopes is generated by integers. For an analysis of the finite-piecewise-linear homeomorphisms of the line, see BRIN and SQUIER [1985] (cf. BROWN and GEOGHEGAN [1984], BROWN [1987], BIERI and STREBEL [1990], GHYS and SERGIESCU [1987], GREENBERG [1987], STEIN [1990]).

If  $G$  is compact and Lie, then let  $Q_G = \prod_{i>0,\rho} D_{\rho,i}$  be the product of the unit balls of all the irreducible real representations of  $G$ , each representation

disc being represented infinitely often. (By the Peter-Weyl theorem, the irreducible representations of compact Lie groups are all finite dimensional and may be assumed to be into the orthogonal groups,  $O(n)$ , cf. MONTGOMERY and ZIPPIN [1955], and are countable.) Let  $G$  act on  $Q_G$  by simultaneous action on all the factors.

**GA 12** *Does the basic  $Q$ -manifold theory as expounded in CHAPMAN [1976] extend to  $Q_G$  manifolds? In particular, is there a  $Q_G$ -manifold version of the material of Chapters I-IV of CHAPMAN [1976] (Approximation of mappings by equivariant embedding, Equivariant  $Z$ -set Unknotting, Equivariant Stability)?* **941. ?**

In the case that  $G$  is finite, Steinberger and West have established for  $Q_G$ -manifolds all the basic theorems of inequivariant  $Q$ -manifold theory that do not rely upon the vanishing of algebraic  $K$ -theoretic invariants (STEINBERGER and WEST [1986, 1986b, 1985]). In particular, every theorem in CHAPMAN [1976] prior to Corollary (29.5) is true in this setting. Additionally, if  $X$  is a locally compact metric  $G$ -ANR, then  $X \times Q_G$  is a  $Q_G$ -manifold, using  $Q_G$  yields a Toruńczyk-style Characterization of  $Q_G$ -manifolds, and the group of equivariant homeomorphisms of a compact  $Q_G$ -manifold is a  $G$ -ANR.

However, Equivariant Handle Straightening, Equivariant Triangulation, and Topological Invariance of Equivariant Whitehead Torsion (defined using stable matrix algebra or equivariant simplicial moves) are false. Although compact  $Q_G$ -manifolds are equivariantly dominated by finite  $G$ -CW complexes (here  $G$  must act by cell-permutation), they may fail to have the equivariant homotopy types of finite  $G$ -CW-complexes QUINN [1979, 1982a, 1982b], STEINBERGER and WEST [1986, 1985]. Moreover, although not equivariantly triangulable, all compact  $Q_G$ -manifolds are equivariantly homeomorphic with manifolds of the form  $M^n \times Q_G$ , where  $M^n$  carries a locally linearizable action of  $G$ .

They have been able to apply this theory to aid the analysis of the locally linearizable  $G$  actions on  $n$ -manifolds (STEINBERGER and WEST [1986, 1985, 1986b, 1987a, 1987b, 1988a, 1988, 1989]), where it serves to provide a stable equivariant Whitehead group for locally linear  $G$ -actions that may not admit equivariant handle decomposition, a topological equivariant  $s$ -cobordism theorem, and a complete obstruction to equivariant handle decomposition for those manifolds with no codimension 2 incidences of fixed point sets of different subgroups and no low-dimensional fixed point sets.

**GA 13** *If  $K$  is a locally compact  $G$ -CW complex, is the diagonal  $G$ -action on  $X = K \times Q_G$  a  $Q_G$ -manifold? What if  $K$  is a locally compact  $G$ -ANR?* **942. ?**

**GA 14** *Given a non-compact  $Q_G$ -manifold  $M$ , what is the obstruction to compactifying it to a  $Q_G$ -manifold by the addition of an equivariant  $Z$ -set?* **943. ?**

*How does this compare with the obstructions of CHAPMAN and SIEBENMANN [1976]? What about compactification by addition of a  $Q_G$ -manifold?*

- ? 944. **GA 15** *Let  $X \subset M$  be a compact  $G$ -ANR in a  $Q_G$ -manifold. Does  $X$  always have a mapping cylinder neighborhood  $N$ ? If not, what is the obstruction, and what about the possibility of the boundary  $\partial(N)$  of  $N$  not being a  $Q_G$ -manifold but some other  $G$ -space?*

A *mapping cylinder neighborhood* is a neighborhood  $N$  that is a compact  $Q_G$ -manifold with equivariantly bicollared (in  $M$ ) boundary  $L$  also a  $Q_G$  manifold and such that  $N$  is the mapping cylinder of a map from  $L$  to  $X$ . The issue of splitting by a non locally linear co-dimension one submanifold has come up in the finite-dimensional case for  $G$  finite, but is likely irrelevant here, as it is expected that a “Toruńczyk Criterion” will be available.

- ? 945. **GA 16** *If  $M$  is a compact  $Q_G$ -manifold, is the space  $\mathcal{H}_G(M)$  of equivariant homeomorphisms of  $M$  a  $G$ -ANR?*

- ? 946. **GA 17** *If the answers to the above are “yes”, what is the effect on the equivariant Whitehead group of stabilization by  $Q_G$ ?*

There is an equivariant Whitehead group  $Wh_G(X)$  in this context defined by ILLMAN [1985, 1985, 1989].

- ? 947. **GA 18** *Describe the subgroup of elements of  $Wh_G(K)$  that may be represented by  $G$ -CW pairs  $(L, K)$  such that upon stabilization by  $Q_G$ , the inclusion  $K \times Q_G \rightarrow L \times Q_G$  is homotopic to an equivariant homeomorphism. In particular, if such an element is represented by a smooth  $G$ - $h$ -cobordism  $(W^{n+1}; M^n, N^n)$  with each non-empty component  $\alpha^H$  of the fixed point set of each closed subgroup  $H$  of  $G$  at least 6-dimensional and none of them of codimension 2 in another, is then  $(W^{n+1}; M^n, N^n)$  equivariantly homeomorphic with  $M \times I; M \times 0, M \times 1$ ?*

- ? 948. **GA 19** *If  $(N, M)$  is a pair of compact  $Q_G$ -manifolds such that  $M$  is an equivariant  $Z$ -set in  $N$  and the inclusion is an equivariant homotopy equivalence, is there a finite-dimensional  $G$ - $h$ -cobordism  $(W, M')$  that stabilizes to  $(N, M)$ , where the  $G$ -action is locally smoothable?*

- ? 949. **GA 20** *Let  $G = \mathbb{Z}_p$  and let  $\mathcal{A}^G(Q)$  denote the space of  $G$ -actions on  $Q$  with*

the uniform convergence topology. Is  $\mathcal{A}^G(Q)$  connected? Path connected? Locally connected? Locally contractible? An ANR? A Hilbert manifold?

**GA 21** What about the subspace  $\mathcal{A}^G(Q, A)$  of  $\mathbf{Z}_p$  actions with prescribed fixed point set  $A$ ? **950. ?**

**GA 22** Same as above for  $\mathcal{A}_A^G(Q)$ ,  $\mathbf{Z}_p$  actions with fixed point set a  $\mathbf{Z}$ -set homeomorphic with  $A$ , connected? Path connected? Locally connected? Locally contractible? An ANR? A Hilbert manifold? **951. ?**

FERRY [1978] showed  $\mathcal{A}_\emptyset^G(M)$  is a Hilbert manifold for all compact Hilbert cube manifolds. These questions are good for  $M$  finite-dimensional, too. EDMONDS showed  $\mathcal{A}_\emptyset^G(M)$  is locally contractible when  $M$  is a finite dimensional manifold (EDMONDS [1976]).

**GA 23** Let  $C$  be a convex Hilbert cube in a locally convex linear space  $X$ . Let  $Gl(C)$  denote the restrictions to  $C$  of those invertible, continuous linear transformations  $T$  of  $X$  such that  $T(C) = C$ . Give necessary and sufficient conditions that two transformations  $S, T \in Gl(C)$  be topologically conjugate on  $C$ . **952. ?**

Examples include unitary operators of separable Hilbert space  $\ell_2$  restricted to the unit ball  $B$  of  $\ell_2$ , equipped with the weak\* topology.

**GA 24** Same as above but in the case that  $S$  and  $T$  are periodic and the fixed point sets of  $T^k$  and  $S^k$  are always infinite dimensional. **953. ?**

Liem has shown that the infinite dimensionality will ensure that the orbit spaces are Hilbert cubes (VO THANH LIEM [1981]).

**GA 25** If  $C$  is as above and  $S, T \in Gl(C)$  are such that on  $C$ , no orbit of  $S$  or  $T$  is finite except for the origin, when are  $S$  and  $T$  conjugate on  $C$ ? **954. ?**

**GA 26** Let  $X$  be a locally compact, convex subset of  $\ell_2$ , and let  $G$  be a discrete group acting freely and properly discontinuously on  $X$ . Under what conditions is the action of  $G$  linearizeable, i.e., topologically conjugate with the restriction to some locally compact, convex subset  $C$  of  $\ell_2$  of a subgroup of  $Gl(C)$ ? **955. ?**

In the above, the orbit space  $X/G$  will be an Eilenberg-MacLane space of type  $K(G, 1)$  and a  $Q$ -manifold. For a given group, they will all be homotopy



equivalent. Thus, they will be homeomorphic if simple homotopy equivalent (infinite simple homotopy equivalent if not compact). A homeomorphism between the orbit spaces will lift to an equivariant homeomorphism, and an equivariant homeomorphism will generate a homeomorphism of orbit spaces. A complication can occur at the ends of  $X$ . Are shape theoretic invariants at the ends sufficient to classify these actions? Do stable homotopy considerations enter? If  $G$  is finitely generated free abelian, and if  $X/G$  is compact, then  $X/G$  is homeomorphic with  $T^k \times Q$ , as all homotopy equivalences are simple in this case and are thus homotopic to homeomorphisms.

Note that the based-free question above is contained in this problem, for one may take a convex Hilbert cube and delete an extreme point, leaving a locally compact convex set.

These questions can obviously be formulated for Lie groups as well.

- ? 956. **GA 27** *For a given finite or discrete group  $G$  classify the free properly discontinuous actions of  $G$  on contractible  $Q$ -manifolds.*

This is much more general than the convex  $Q$ -manifold problem above because of complications at the ends in the manifolds. For example, note the implications of M. Davis' examples of compact  $n$ -manifolds with contractible universal cover not homeomorphic with  $\mathbb{R}^n$  (DAVIS [1983]).

It is quite certain that in this  $Q$ -manifold setting there are questions that are not algebraically obstructed and that can be answered directly using the elementary but powerful techniques, such as controlled engulfing, that are available. In the past decade, there has begun to be a flow of information from the topological theory of manifolds to the algebraic  $K$ -theory achievable, in retrospect, by  $Q$ -manifold methods. Many of these results amount to "vanishing" theorems in the algebraic  $K$ -theory and are of considerable importance. FERRY [1977b], FARRELL and HSIANG [1981, 1983], FARRELL and JONES [1986a, 1986b, 1988a, 1988b, 19 $\infty$ ], QUINN [1979, 1982a, 1982b, 1985].

There are several other questions concerning group actions on  $Q$ -manifolds here and there. See, e.g., the section on infinite dimensional manifolds in nature.

- ? 957. **GA 28** *Let  $G$  be a finitely presented group. Is  $G$  semistable at  $\infty$ ?*

Semistability at infinity of a finitely presented group is a topological property of the universal covers  $\tilde{X}$  of the finite cell complexes  $X$  with  $\pi_1(X) = G$ :  $X$  is semistable at  $\infty$  provided that each two proper maps  $r, s : [0, 1] \rightarrow \tilde{X}$  converging to the same end of  $\tilde{X}$  are properly homotopic. Note that this is a shape-theoretic property of the ends. Mihalik has made an extensive study of this property MIHALIK [1983, 1985, 1986, 1986, 1987, 19 $\infty$ ], which forces  $H^2(G, Z[G])$  to be free abelian (GEOGHEGAN and MIHALIK [1985]). He has shown that 0-ended and 2-ended groups are semistable at  $\infty$ , and has reduced

the general case to 1-ended groups (MIHALIK [19 $\infty$ ]).  $G$  is said to be *simply connected at infinity* provided that for each compact set  $C$  of  $\tilde{X}$  there is a larger one  $K$  such that loops in  $\tilde{X} - K$  are null homotopic in  $\tilde{X} - C$ . This forces  $H^2(G, Z[G])$  to vanish. See GEOGHEGAN and MIHALIK [1985] for this and further explanation of the relation between the shape theory of the ends of  $\tilde{X}$  and the cohomology of  $G$ .

## 8. HS: Spaces of Automorphisms and Mappings

### 8.1. Introduction

There has been a significant amount of work on this topic over the past decade, but the most fundamental problem, determining whether or when the homeomorphism group  $\mathcal{H}(M^n)$  of a compact  $n$ -manifold  $M^n$ ,  $n > 2$ , when equipped with the compact-open topology, is an ANR, and hence a Hilbert manifold, is untouched. Chapter X of GEOGHEGAN [1979] is devoted to this problem and contains an excellent discussion of several reductions to simpler ones. That discussion will not be repeated below. As a point of information, it is known that when  $M$  is compact and  $PL$ , the subgroup  $\mathcal{H}^{PL}(M^n)$  is a  $\sigma$ -manifold (KEESLING and WILSON [1975], GEOGHEGAN and HAVER [1976]). Also, for  $n = 1, 2$ ,  $\mathcal{H}(M^n)$  is known to be a Hilbert manifold (LUKE and MASON [1972]).

There have been interesting developments in at least two directions: measure preserving transformations, by Nguyen To Nhu, J. Oxtoby, and V. Prasad, and uniform and Lipschitz isomorphisms, by K. Sakai and R. Wong.

### 8.2. Progress on Problems of Section HS

**HS 1** (79HS1-3) *Let  $M^n$  be a compact  $n$ -manifold. Is  $\mathcal{H}(M^n)$  a Hilbert manifold?* **958. ?**

Open. The answer is “Yes”, for  $n = 1$  (R. Anderson, R. Bing) and for  $n = 2$ , LUKE and MASON [1972]. The discussion in GEOGHEGAN [1979] shows that this is equivalent to showing that  $\mathcal{H}(M^n)$  is an ANR, or that  $\mathcal{H}_\partial(M^n)$  is an ANR, where  $\mathcal{H}_\partial(M^n)$  is the homeomorphisms that are stationary on the boundary.

**HS 2** (79HS4) *Is every open set in  $\mathcal{H}_\partial(B^n)$  dominated by a CW-complex?* **959. ?**

Open. GEOGHEGAN [1979] indicates how this implies  $\mathcal{H}(M^n)$  is a Hilbert manifold. It is known that if  $M^n$  is a  $PL$  manifold, then in the identity component  $\mathcal{H}_o(M^n)$  of  $\mathcal{H}(M^n)$ , the  $PL$ -homeomorphisms are dense and even form an fd-cap set (GEOGHEGAN and HAVER [1976]). They are also known to be an ANR and even a manifold modeled on the linear span  $\sigma$  in  $\ell_2$  of an

orthonormal basis (KEESLING and WILSON [1975]). The fd-cap property is defined after HS18 and ensures that for each open set  $U$  of  $\mathcal{H}_o(M^n)$ , the inclusion  $U \cap \mathcal{P}\mathcal{L}_o(M^n) \hookrightarrow U$  is an isomorphism on  $\pi_n$  for each  $n$ . It follows that in order to settle (HS1) it is sufficient to demonstrate Kozłowski's *haziness*, which is the strong local homotopy negligibility. (See the discussion around (ANR4).)

- ? 960. **HS 3** (79HS5) *Let  $M$  be a compact  $n$ -manifold. Let  $\overline{\mathcal{H}}(M)$  denote the closure of  $\mathcal{H}(M^n)$  in the space of continuous self-maps of  $M$  in the uniform convergence topology. Is there a (continuous) map  $\overline{\mathcal{H}}(M) \rightarrow \overline{\mathcal{H}}(M)$  arbitrarily close to the identity map of  $\overline{\mathcal{H}}(M)$  with image in  $\mathcal{H}(M^n)$ ?*

Open.

- ? 961. **HS 4** (79HS6) *Is  $\overline{\mathcal{H}}(M)$  an ANR?*

Open. If  $\overline{\mathcal{H}}(M)$  is an ANR then it is a Hilbert manifold, since it is  $\ell_2$ -stable according to Toruńczyk and to GEOGHEGAN [1972, 1973]. Haver has shown that  $\overline{\mathcal{H}}(M) - \mathcal{H}(M^n)$  is a countable union of  $Z$ -sets, so that if  $\overline{\mathcal{H}}(M)$  is an ANR, so is  $\mathcal{H}(M)$ .

### 8.3. More Problems on Mapping Spaces

- ? 962. **HS 5** *Let  $M$  be a compact  $Q$ -manifold and as in (HS3) let  $\overline{\mathcal{H}}(M)$  denote the uniform closure of its homeomorphism group. Is  $\overline{\mathcal{H}}(M)$  a Hilbert manifold?*

This is open even for  $M = Q$ .

- ? 963. **HS 6** *Let  $M^n$  be a compact  $n$ -manifold. Let  $\mathcal{R}(M)$  denote the space of retractions of  $M$  (compact-open topology) and let  $\mathcal{R}_o = \mathcal{R}(M) - id_M$ . Is  $\mathcal{R}(M)$  an ANR? If  $\partial M = \emptyset$ , is  $\mathcal{R}_o(M)$  a Hilbert manifold? If  $\partial M \neq \emptyset$  is  $\mathcal{R}(M)$  a Hilbert manifold?*

The answer is yes for  $n = 1, 2, \infty$ , by work of Basmanov and Savchenko, Cauty, and Chapman and Sakai, respectively (BASMANOV and SAVCHENKO [1987], CAUTY [1986], CHAPMAN [1977b], SAKAI [1981a], cf. NHU ET AL [19 $\infty$ ]).

- ? 964. **HS 7** *Is  $\mathcal{R}(M) - id_M$  locally homeomorphic with  $\mathcal{H}(M)$ ?*

Let  $X$  be a non-discrete metric compactum and  $Y$  a separable metric space without isolated points. Let  $C(X, Y)$  be the space of maps from  $X$  to  $Y$  with the sup metric, and let  $LIP(X, Y) \subset C(X, Y)$  be the Lipschitz maps. Let

$$k\text{-}LIP(X, Y) = \{f \in LIP(X, Y) \mid lip(f) \leq k\},$$

where  $\text{lip}(f)$  is the Lipschitz constant. Let  $PL(X, Y)$  denote the  $PL$  maps, and let  $\mathcal{H}^{LIP}(X)$  be the Lipschitz homeomorphisms (both  $h$  and  $h^{-1}$  being required to be Lipschitz).

**HS 8** *If  $Y$  is not a smooth manifold, can one put a smooth structure on  $C(X, Y)$  in a “natural” way?* **965. ?**

If  $Y$  is a topologically complete ANR, then  $C(X, Y)$  is a Hilbert manifold (GEOGHEGAN [1972, 1973]). HENDERSON [1969] showed each Hilbert manifold embeds in Hilbert space and admits a smooth structure. KUIPER and BURGHELEA [1969] showed that homotopy equivalent smooth Hilbert manifolds are diffeomorphic, so the structure is unique.

Let  $Q^\infty = \lim_{\rightarrow} (Q = Q \times 0 \hookrightarrow Q \times Q = Q \times Q \times 0 \hookrightarrow \dots)$ . Re-equipped with the bounded-weak\* topology (direct limit of radius  $n$  balls, each with the (analysts’) weak\* topology),  $\ell_2$  becomes homeomorphic with  $Q^\infty$ .

**HS 9** *Can a topology be induced “naturally” on  $C(X, Y)$  to get a  $Q^\infty$ -manifold?* **966. ?**

If  $X$  and  $Y$  are polyhedra, then  $PL(X, Y)$  is a  $\sigma$ -manifold (KEESLING and WILSON [1975]). Let  $\mathbb{R}^\infty = \lim_{\rightarrow} (\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \dots)$ . Then  $\mathbb{R}^\infty$  and  $\sigma$  may be regarded as topologies on the same set.

**HS 10** *Is there a natural way to topologize  $PL(X, Y)$  as an  $\mathbb{R}^\infty$ -manifold?* **967. ?**

**HS 11** *Under what conditions is  $LIP(X, Y)$  a  $\Sigma$ -manifold? Does it suffice for  $Y$  to be a locally compact ALNE (= absolute neighborhood extensor for the class of metric spaces and locally Lipschitz maps)?* **968. ?**

Sufficient is that  $Y$  be a locally compact locally convex set in a normed linear space, an Euclidean polyhedron, or a Lipschitz manifold. Cf. SAKAI [1989a], SAKAI and WONG [1989a].

**HS 12** *Under what conditions is  $k\text{-}LIP(X, Y)$  a  $Q$ -manifold?* **969. ?**

Sufficient is that  $Y$  be an open subset of a locally compact convex set in a normed linear space (SAKAI and WONG [1989a]).

**HS 13** *If  $X$  and  $Y$  are Euclidean polyhedra, is* **970. ?**

$$(C(X, Y), LIP(X, Y), PL(X, Y))$$

an  $(\ell_2, \Sigma, \sigma)$ -manifold triple?

Yes, if  $Y$  is open in Euclidean space SAKAI [19∞b]. A *manifold triple* implies charts that preserve the containment and simultaneously provide charts for each object, e.g., if  $N$  is a locally flat codimension 3 submanifold of  $M^n$ , then  $(M, N)$  is a  $(\mathbb{R}^n, \mathbf{R}^{n-3})$ -manifold pair.  $(C(X, Y), LIP(X, Y))$  and  $(C(X, Y), PL(X, Y))$  are known to be  $(\ell_2, \Sigma)$ - and  $(\ell_2, \sigma)$ -manifold pairs (SAKAI [19∞a], GEOGHEGAN [1973]).

? 971. HS 14 If  $X$  is a Lipschitz  $n$ -manifold, is  $\mathcal{H}^{LIP}(X)$  a  $\Sigma$ -manifold?

Yes, if  $X$  is an Euclidean polyhedron of dimension  $\leq 2$ ;  $(\mathcal{H}(X), \mathcal{H}^{LIP}(X))$  is then an  $(\ell_2, \Sigma)$ -manifold pair SAKAI and WONG [1989b].

? 972. HS 15 If  $X$  is a  $Q$ -manifold, what metric conditions on  $X$  guarantee that  $\mathcal{H}^{LIP}(X)$  be a  $\Sigma$ -manifold? A cap set for  $\mathcal{H}(X)$ ?

A dense  $\sigma$ -compact subset  $Z = \cup Z_i$  in  $Y$  has the *compact absorption property* (cap) provided that each  $Z_i$  is a  $Z$ -set in  $Y$  and that any map  $f : (A, B) \rightarrow (Y, Z)$  of a compact pair that embeds  $B$  may be approximated rel.  $B$  by embeddings into  $Z$ .

? 973. HS 16 Is  $\mathcal{H}^{fd}(X \times Q)$  an  $\ell_2 \times \sigma$ -manifold for polyhedral  $X$ ?

$\mathcal{H}^{fd}(X \times Q) = \{h \times id \mid n \in \mathbb{N}, h \in \mathcal{H}(X \times I^n)\}$  is an  $\ell_2 \times \sigma$ -manifold when  $X$  is a  $PL$  manifold (SAKAI and WONG [19∞]). Let

$$\mathcal{H}^{PL}(X \times Q) = \{h \times id \mid n \in \mathbb{N}, h \in \mathcal{H}^{PL}(X \times I^n)\}.$$

? 974. HS 17 Are

$$(\mathcal{H}_o(X), \mathcal{H}_o^{LIP}(X), \mathcal{H}_o^{PL}(X))$$

and

$$(\mathcal{H}_o(X \times Q), \mathcal{H}_o^{LIP}(X \times Q), \mathcal{H}_o^{PL}(X \times Q))$$

$(\ell_2, \Sigma, \sigma)$ -manifold triples when  $X$  is a  $PL$  manifold?

$\mathcal{H}_o$  denotes the identity component. With appropriate dimension restrictions,  $(\mathcal{H}_o(X), \mathcal{H}_o^{PL}(X))$  and  $(\mathcal{H}_o(X), \mathcal{H}_o^{LIP}(X))$  are  $(\ell_2, \sigma)$ - and  $(\ell_2, \Sigma)$ -manifold pairs.  $(\mathcal{H}_o(X \times Q), \mathcal{H}_o^{PL}(X \times Q))$  and  $(\mathcal{H}_o(X \times Q), \mathcal{H}_o^{LIP}(X \times Q))$  are  $(\ell_2, \sigma)$ - and  $(\ell_2, \Sigma)$ -manifold pairs with no dimension hypothesis (SAKAI and WONG [1989b, 19∞]).

Define the following filtered retraction spaces:

$$\mathcal{R}^{fd}(X \times Q) = \{i_n \circ r \circ p_n \mid n \in \mathbb{N}, r \in \mathcal{R}(X \times I^n)\}$$

and

$$\mathcal{R}^{PL}(X \times Q) = \{i_n \circ r \circ p_n \mid f \in \mathcal{R}^{PL}(X \times I^n)\},$$

where  $i_n : X \times I^n \rightarrow X \times Q$  is inclusion and  $p_n : X \times Q \rightarrow X \times I^n$  is projection. Here,  $Q = \prod_{i \in \mathbb{N}} I_i$ .

**HS 18** Let  $X$  be a  $PL$  manifold. Is  $\mathcal{R}_o^{PL}(X)$  (respectively,  $\mathcal{R}^{PL}(X \times Q)$ ) a **975. ?**  
 $\sigma$ -manifold or a  $fd$ -cap set for  $\mathcal{R}_o(X)$  (respectively,  $\mathcal{R}(X \times Q)$ )?

*fd-cap sets* are defined exactly as cap sets except that all compact sets in the definition are additionally required to be finite-dimensional.

**HS 19** Let  $X$  be a  $Q$ -manifold. Is there a metric condition on  $X$  ensuring **976. ?**  
that  $\mathcal{R}^{LIP}(X)$  be a  $\Sigma$ -manifold or cap set for  $\mathcal{R}^{LIP}(X)$ ?

**HS 20** Let  $X$  be a Lipschitz  $n$ -manifold. Is  $\mathcal{R}_o^{LIP}(X)$  a  $\Sigma$ -manifold or a cap **977. ?**  
set for  $\mathcal{R}_o(X)$ ?

**HS 21** Let  $X$  be a  $PL$  manifold. Are **978. ?**  
 $(\mathcal{R}_o(X), \mathcal{R}_o^{LIP}(X), \mathcal{R}_o^{PL}(X))$

and

$(\mathcal{R}(X \times Q), \mathcal{R}^{LIP}(X \times Q), \mathcal{R}^{PL}(X \times Q))$   
 $(\ell_2, \Sigma, \sigma)$ -manifold triples?

**HS 22** Let  $\lambda$  denote Lebesgue product measure on  $Q$ . Let  $\mathcal{H}_\lambda(Q)$  denote **979. ?**  
the subgroup of  $\mathcal{H}(Q)$  of measure preserving homeomorphisms (under the compact-open topology). Is  $\mathcal{H}_\lambda(Q)$  locally contractible? An AR? A Hilbert space? What about  $\mathcal{H}_\lambda(I^n)$ ?

J. Oxtoby and V. Prasad have shown (OXTOPY and PRASAD [1978]) that  $Z$ -set unknotting may be achieved in  $Q$  by members of  $\mathcal{H}_\lambda(Q)$ , but not with control. PRASAD [1979] has shown that  $\mathcal{H}_\lambda(I^n)$  is a dense  $G_\delta$  in  $\mathcal{H}(I^n)$ . Nhu has shown that the space of all measure preserving bijections of Lebesgue measure on a separable complete metric space is an ANR (NGUYEN TO NHU and TA KHAC CU [19∞]).

**HS 23** Let  $F$  be the group of  $PL$  homeomorphisms of the line  $\mathbb{R}^1$  generated by **980. ?**  
 $p(t)$  and  $q(t)$ , where  $p(t) = t$  if  $t \leq 0$ ,  $p(t) = 2t$ , if  $0 \leq t \leq 1$ , and  $p(t) = t + 1$ , if  $t \geq 1$ , while  $q(t)$  is analogously defined using 1 and 2 as the singularities, again with slope 2 between them. Is  $F$  amenable?

A discrete group is *amenable* provided that it admits a translation invariant mean, i.e., averaging functional on the set of bounded measurable real valued functions. (A geometric heuristic definition good for subgroups  $G$  of Lie groups is that the ratio of the number of points of  $G$  near the surfaces of balls to those in the interiors in the Lie group goes to zero “sufficiently rapidly” as diameter goes to infinity to allow an integral to “work”.)  $F$  is the “Thompson-Minc” group. See BRIN and SQUIER [1985] (cf. HASTINGS and HELLER [1982], DYDAK [1977], BROWN and GEOGHEGAN [1984], BROWN [1987, 1990]), where it is shown that the entire group of  $PL$  homeomorphisms of  $\mathbb{R}^1$  with finitely many singularities contains no free subgroup of rank 2 and a complete presentation is given. A negative answer would give a finitely presented counter example to a conjecture of von Neumann.

## 9. LS: Linear Spaces

### 9.1. Introduction

There has been a lot of movement in this branch of infinite dimensional topology. The locally convex, complete, metrizable vector spaces were characterized topologically by TORUŃCZYK [1981], which was available when GEOGHEGAN [1979] was prepared. Since then, the focus has been on the classification of incomplete subspaces of Fréchet spaces and the study of the topology of the non locally convex vector spaces. Despite many advances, the topology of these spaces retains its mystery. The basic reference is still BESSAGA and PEŁCZYŃSKI [1975]. The separate chapter in the present book by Dobrowolski and Mogilski should be consulted on this topic for the authoritative discussion on incomplete spaces.

Some terminology is as follows: a linear space is a topological vector space, a real vector space with a topology under which addition and scalar multiplication is continuous. A linear metric space has topology determined by a metric, and will always have translation invariant metrics. If there is a complete metric, then there are complete translation invariant ones; complete linear metric spaces are called  $F$ -spaces.

### 9.2. Progress on Problems of Section LS

- ? 981. **LS 1 (79LS1)** *Is every linear metric space an AR? Is every  $F$ -space an AR? Is every admissible  $F$ -space an AR?*

Open. Yes for locally convex linear metric spaces (Dugundji) and for  $\sigma$ -compact admissible convex subsets of linear metric spaces Dobrowolski [1985]. (A convex subset  $C$  of a linear metric space is called *admissible* provided that every compact subset of  $C$  admits maps into  $C$  arbitrarily close to the identity

with ranges contained in convex hulls of finitely many vectors. KLEE [1960a, 1960b].)

**LS 2 (79LS2)** *Let  $X$  be an  $F$ -space with invariant metric  $d$ . Let*

**982. ?**

$$\tilde{X} = \{\lambda: X - 0 \rightarrow \mathbb{R} \mid \rho(\lambda, 0) \equiv \sum_{x \in X-0} d(\lambda(x)x, 0) < \infty\}.$$

*Assume without loss of generality that for each  $x$   $d(tx, 0)$  is strictly increasing in  $t$ .  $\tilde{X}$  is an  $F$ -space with invariant metric  $\rho$ . Let  $\mu: \tilde{X} \rightarrow X$  by*

$$\mu(\lambda) = \sum_{x \in X-0} \lambda(x)x.$$

*Does  $\mu$  admit a continuous cross section?*

Open. The interest of this problem stems from the fact that  $\tilde{X}$  is homeomorphic with a Hilbert space, so is an AR; existence of a cross section implies  $X$  and the kernel of  $\mu$  are AR's.

**LS 3 (79LS3)** *Is every infinite dimensional  $F$ -space  $X$  homeomorphic with  $X \times s$ ? with  $X \times Q$ ? with  $X \times \mathbb{R}$ ?* **983. ?**

Open. Van Mill [1987] gave incomplete normed (hence not  $F$ -space) counter examples to all three questions.

**LS 4 (79LS4)** *Let  $X$  be an infinite dimensional  $F$ -space. (a) Are compacta negligible in  $X$ ? (b) Do homeomorphisms between compacta of  $X$  extend to homeomorphisms of  $X$ ?* **984. ?**

Open. Partial solutions: (a) Yes, if  $X$  has a strictly weaker linear Hausdorff topology (Dobrowolski and Riley). (b) Yes, for finite-dimensional compacta (DOBROWOLSKI [1989], cf. BORGES [1987]). A set  $A$  in a space  $X$  is *negligible* provided that  $X - A \approx X$ .

**LS 5 (79LS5)** *Does every infinite dimensional  $F$ -space  $X$  contain an fd-cap set?*

Yes, any Hamel basis. Rephrase by asking for cap sets. (If  $X$  is an AR, then Mazur's Lemma implies "Yes".)

**LS 6 (79LS6)** *Let  $K$  be a convex subset of an  $F$ -space  $X$ . Is  $K$  an AR? If  $K$  is closed is it an AR? A retract of  $X$ ? If  $K$  is compact is it an AR?* **985. ?**

Open. If  $\overline{K}$  is an AR, so is  $K$ . In particular, although  $L_p$  is known to be an AR, it is not known for  $p \in (0, 1)$  whether linear subspaces of  $L_p$  are AR's, much less convex compacta (see comment to LS7).



- ? **986. LS 7** (79LS7) (Schauder) *Has every compact convex subset of a linear metric space the fixed point property?*

Open. Note Roberts' example in  $L_p$ ,  $0 < p < 1$ , which should be dealt with first ROBERTS [1976]. Is it a counter example to any or all of (LS7)? See NGUYEN TO NHU and LE HOANG TRI [19 $\infty$ ].

- ? **987. LS 8** (79LS8) *For each  $\epsilon > 0$  does there exist an open cover  $\alpha$  of  $\ell_1$  such that for each point  $p$  the sum of the diameters of the elements of  $\alpha$  containing  $p$  is less than  $\epsilon$ ?*

Open.

- ? **988. LS 9** (79LS9) *Let  $X$  be an  $F$ -space. Is every convex subset of  $X$  homeomorphic with a convex subset of a Hilbert space? What if  $X$  is a Banach space?*

Open. Bessaga can show that locally compact convex sets in Banach spaces may be affinely embedded in Hilbert spaces.

**LS 10** (79LS10) *Is every closed convex subset of a Banach space either locally compact or homeomorphic with a Hilbert space?*

Yes, for the separable case DOBROWOLSKI and TORUŃCZYK [1979].

- ? **989. LS 11** (79LS11) *Is every infinite dimensional separable normed linear space homeomorphic with some pre-Hilbert space, i.e., a linear subspace of a Hilbert space?*

Open. Yes, for the  $\sigma$ -compact spaces DOBROWOLSKI [1989].

**LS 12** (79LS12) *Let  $X$  be an infinite dimensional separable pre-Hilbert space. Is one of the following true?*

- $X \times R \approx X$
- $X \times X \approx X$
- $X_f^\infty \approx X$
- $X^\infty \approx X$

No. VAN MILL [1987] and Pol (unpublished). Open for the Borelian case, even for the  $\sigma$ -compact spaces.  $X_f^\infty$  is the subspace of  $X^\infty$  comprised of points with at most finitely many non-zero coordinates.

**LS 13** (79LS13) *If a  $\sigma$ -compact separable normed linear space  $E$  contains a homeomorph  $Q'$  of  $Q$ , is  $E \approx \{x \in \ell_2 \mid \sum i^2 \cdot x_i^2 < \infty\}$ ?*

Yes. DOBROWOLSKI and MOGILSKI [1982].

**LS 14** (79LS14) *Let  $E$  be a locally convex linear metric space, and let  $X$  be an incomplete retract of  $E$ . Must  $X \times E^\infty \approx E^\infty$ ?*

No. POL [1984]. Open for the Borelian case.

**LS 15** (79LS15) *Let  $X$  be a Banach space and  $GL(X)$  its general linear group. Let  $\|\cdot\|$  denote the operator norm and “ $w$ ” the pointwise convergence topology on  $GL(X)$ . Is the identity map  $(GL(X), \|\cdot\|) \rightarrow (GL(X), w)$  a homotopy equivalence?* **990. ?**

Open. It is possible that for “infinitely divisible” spaces  $X$ , the technique of Wong’s Thesis (WONG [1967]) can prove  $(GL(X), w)$  contractible.

### 9.3. More Problems on Linear Spaces

**LS 16** *Let  $W$  be a convex subset of a Hilbert space. Under what conditions is  $W$  locally homotopy-negligible in its closure?* **991. ?**

(A set  $A$  is *locally homotopy-negligible* in the space  $X$  provided that the inclusions  $U - A \hookrightarrow U$  are weak homotopy equivalences for all open sets  $U$  of  $X$ .)

**LS 17** *Is every locally contractible closed additive subgroup of a Hilbert space an ANR?* **992. ?**

**LS 18** *Is every locally connected closed additive subgroup of a Hilbert space an ANR?* **993. ?**

**LS 19** *Is every equiconnected space a retract of a convex subset of an  $F$ -space?* **994. ?**

$X$  is equiconnected provided that there is a map  $\kappa : X \times I \times X \rightarrow X$  such that  $\kappa(x, 0, x) = x$ ,  $\kappa(x, 1, y) = y$ , and  $\kappa(x, t, x) = x$ .

? 995. **LS 20** *Is every linear metric space admissible?*

Every locally convex linear metric space is admissible NAGUMO [1951]. There are admissible linear metric spaces that are not locally convex KLEE [1960b]. A linear metric space or a convey subset of it is admissible if and only if it has the CEP KLEE [1960a], DOBROWOLSKI [1985], VAN DER BIJL and VAN MILL [1988]. (A space  $X$  has the *Compact Extension Property (CEP)* provided that for every separable metric space  $Y$  and compact subset  $A \subset Y$ , each map  $f: A \rightarrow X$  extends to  $Y$ .) CEP is (strictly) weaker than ANR (VAN MILL [1986]), so a positive answer to (LS1) implies admissibility.

? 996. **LS 21** *Is every locally convex linear metric space homeomorphic with a normed linear space?*

This is true for complete spaces (combined work of Anderson, Bessaga, Pełczyński, and Kadec (BESSAGA and PEŁCZYŃSKI [1975]) and for  $\sigma$ -compact spaces DOBROWOLSKI [1989].

? 997. **LS 22** *Let  $\mathcal{U}$  denote Urysohn's universal-up-to-isometry separable metric space URYSOHN [1927]. Let  $\mathcal{U}_0$  be an isometric copy of  $\mathcal{U}$  in a Banach space  $X$  containing the zero of  $X$ . Let  $\overline{\mathcal{U}}$  be the closed linear span of  $\mathcal{U}$  in  $X$ . Is  $\overline{\mathcal{U}}$  a universal separable Banach space up to linear isometry? If not, characterize the separable Banach spaces that embed in  $\overline{\mathcal{U}}$  via linear isometries.*

M. R. Holmes in unpublished work has shown that  $\overline{\mathcal{U}}$  is uniquely determined up to linear isometry.

? 998. **LS 23** *Let  $\overline{\mathcal{U}}$  be as above. Does  $\overline{\mathcal{U}}$  have a Schauder basis?*

## 10. NLC: Non Locally Compact Manifolds

### 10.1. Introduction

There has been a great deal of activity in this area over the past decade by Anderson, Bestvina, Bowers, Curtis, Dijkstra, Dobrowolski, Heisey, J. Henderson, Vo Thanh Liem, van Mill, Mogilski, Nguyen To Nhu, Sakai, Toruńczyk, Walsh, and Wong, to mention a few.

Terminology:  $\mathbb{R}^\infty$  is the direct limit of  $\mathbb{R}^n$ 's, and  $Q^\infty$  is the analog for  $Q$ .  $s = \prod_{i \geq 1} \mathbb{R}_i$ ,  $\ell_2$  is the Hilbert space of square-summable sequences of reals, and

$$\sigma = \{x \in \ell_2 \mid x_i = 0 \text{ for almost all } i\}.$$

$\Sigma$  denotes the linear span in  $\ell_2$  of  $Q = \{x \in \ell_2 \mid \|x_i\| \leq \frac{1}{i}\}$ .

### 10.2. Progress on Problems of Section NLC

**NLC 1** (79NLC1) *Let  $M \subset N$  be  $s$ -manifolds and let  $R \subset M$ . Suppose that  $M$  has local codimension 1 at each point of  $M - R$ . Does  $M$  have local codimension 1 at points of  $R$  when  $R$  is (a) a point, (b) compact, or (c) a  $Z$ -set in both  $M$  and  $N$ ?* **999. ?**

Open. Kuiper has a counter example for local codimension 2.

**NLC 2** (79NLC2) *Same as (79NLC1) for codimension  $> 2$ .* **1000. ?**

Open.

**NLC 3** (79NLC3) *Let  $M$  be separable  $C^\infty$   $\ell_2$ -manifold. Can each homeomorphism of  $M$  with itself be approximated by diffeomorphisms?* **1001. ?**

Open.

**NLC 4** (79NLC4) *Let  $M$  and  $K$  be  $s$ -manifolds with  $K \subset M$  and  $K$  a  $Z$ -set in  $M$ . Under what conditions on the pair  $(M, K)$  does there exist an embedding  $h$  of  $M$  in  $s$  such that the topological boundary of  $h(M)$  is  $h(K)$ ?* **1002. ?**

Open. Sakai has addressed this problem, but has not given a definitive solution (SAKAI [1983]).

**NLC 5** (79NLC5) *Let  $\xi: E \rightarrow B$  be a fiber bundle over a paracompact base  $B$  with fiber  $F$  an  $s$ -manifold. Suppose  $K$  is a closed subset of  $E$  such that  $K \cap \xi^{-1}(b)$  is a  $Z$ -set in each  $p^{-1}(b)$ . Is there a fiber preserving homeomorphism of  $E - K$  onto  $E$ ?*

Yes, when  $B$  is a polyhedron CHAPMAN and WONG [1974a, 1974b, 1974c], cf. SAKAI [1982], FERRY [1977b], TORUŃCZYK and WEST [1989] for  $Q$ -manifold bundle theorems.

**NLC 6** (79NLC6) *Does every homeomorphism between  $Z$ -sets in  $\mathbb{R}^\infty$  or  $Q^\infty$  extend to an ambient homeomorphism? If so, is there an appropriate analog for the Anderson-McCharen  $Z$ -set unknotting for these manifolds?*

No, by VO THANH LIEM [1981]. However, if one adds the hypothesis that the  $Z$ -sets are infinitely deficient, then Yes.

**NLC 7** (79NLC7) *Are countable unions of  $Z$ -sets strongly negligible in  $\mathbb{R}^\infty$  and  $Q^\infty$ -manifolds?*

No; each compact set is a  $Z$ -set and the spaces are  $\sigma$ -compact.

**NLC 8** (79NLC8) *Is there an analog of Ferry's  $\alpha$ -approximation theorem for  $\mathbb{R}^\infty$ -manifolds?*

Yes, VO THANH LIEM [1987].

? **1003. NLC 9** (79NLC9) *Are  $(C^p, b^*)$ -manifolds stable? Do they embed as open subsets of the model? Are they classified by homotopy type?*

These are open. (A  $(C^p, b^*)$ -manifold is a Banach manifold with charts of class  $C^p$  that are simultaneously continuous in the bounded-weak\* topology. (See work of HEISEY [1975], R. Graf, and R. Palais' 1970 ICM talk.)

### 10.3. More Problems on Non Locally Compact Manifolds

? **1004. NLC 10** *When does a Hilbert manifold support a topological group structure?*

? **1005. NLC 11** *If  $M$  is a Hilbert manifold that admits the structure of an associative  $H$ -space, does it admit a topological group structure?*

Separable, complete metric ANR topological groups are Hilbert manifolds (DOBROWOLSKI and TORUŃCZYK [1981]).

**NLC 12** *Let  $G$  be a cell-like upper semi-continuous decomposition of  $\sigma$ . If  $\sigma/G$  is an ANR, must it be the countable union of finite-dimensional compacta?*

No. Van Mill announced a counter-example to appear in Proceedings of the Amer. Math. Soc. while this chapter was in proof. It is known that each compactum  $K$  of  $\sigma/G$  is a strong  $Z$ -set, i.e., the identity mapping of  $\sigma/G$  may be approximated by maps  $h$  such that the closure of  $h(\sigma/G)$  misses  $K$ . It is also known that if  $\sigma/G$  is the countable union of finite dimensional compacta, then it is an ANR, and that if it is a  $\sigma$ -manifold, then it is in fact  $\sigma$ .

? **1006. NLC 13** *If  $\sigma/G$  is not  $\sigma$ , but is the union of countably many finite dimension compacta, is  $\sigma/G \times \mathbb{R}^2$  homeomorphic with  $\sigma$ ? If so, is  $\sigma/G \times \mathbb{R}^1$  homeomorphic with  $\sigma$ ?*

Yes, if the closure of the non-degeneracy set of  $G$  is finite-dimensional.

? **1007. NLC 14** *If  $G$  is as in NLC 13 and if  $H$  is the cell-like decomposition of  $\sigma \times \mathbb{R}^1$*

whose nondegenerate elements are  $\{g \times 0 \mid g \in G\}$ . Is  $\sigma \times \mathbb{R}^1 / H$  homeomorphic with  $\sigma$ ?

**NLC 15** Consider (79NLC9): manifolds with two topologies. If we consider  $\mathbb{R}^\infty$  and  $\sigma$ , they have the same underlying set, and the direct limit topology is finer than the metric one. Do we get anything of interest if we consider manifolds with an atlas of charts that are homeomorphisms in both topologies simultaneously? Does a  $\sigma$ -manifold support more than one such structure? **1008. ?**

**NLC 16** Find intrinsic conditions on metrics of  $\sigma$ -manifolds and  $\Sigma$ -manifolds so that their metric completions are  $l_2$ -manifolds. **1009. ?**

Each  $\sigma$ - or  $\Sigma$ -manifold may be embedded in an  $l_2$ -manifold as a fdcap set or a cap set. Specifically, if  $K$  is a simplicial complex, and we take the  $\ell_1$  barycentric metric  $d(x, y) = \sum_{v \in K^0} |x(v) - y(v)|$ , then SAKAI [1987a] showed that the metric completion of the underlying space  $|K|$  is an  $l_2$ -manifold and  $|K|$  is an fd-cap set in it if and only if  $K$  is the combinatorial structure of a  $\sigma$ -manifold. (cf. Sakai [1987], Sakai [1988].)

**NLC 17** Let  $\mathcal{I} = \{A \in 2^Q \mid \dim(A) = \infty\}$  under the Hausdorff metric. Is  $\mathcal{I}$  homeomorphic with  $\sigma^\infty$  (Cartesian power)?

Yes. Solved by Dijkstra, van Mill, and Mogilski (in preparation). The point is to develop techniques to deal with  $\sigma^\infty$  and analogous spaces. Here,  $2^Q$  denotes the hyperspace of non void closed subsets of  $Q$ .

## 11. TC: Topological Characterizations

### 11.1. Introduction

The characterization of  $Q$ -manifolds and  $l_2$ -manifolds by TORÚNCZYK [1980, 1981] has been followed up throughout the decade. Much work has been done on obtaining analogous characterizations of manifolds modeled on the incomplete linear metric spaces. In general, see the article by Dobrowolski and Mogilski in this volume.

The most pressing need is for a simple and useful topological characterization of ANR's (more precisely, Absolute Neighborhood Extensors for the class of metric spaces) among the metric spaces of infinite dimension. In the finite dimensional spaces, we have Kuratowski's theorem that the ANR's coincide with the locally contractible spaces, which was extended to the countable unions of finite dimensional compacta by GEOGHEGAN and HAVER [1976] and to metrizable spaces with Property C by GRESHAM [1980]. This fails even for

a basis of contractible open sets for strongly infinite dimensional spaces (BOR-SUK [1967, Chapter V, section 11], cf. DAVERMAN and WALSH [1983b]).

(A space  $X$  has *Property C* provided that for every sequence  $\{\mathcal{U}_i\}_{i \geq 1}$  of open covers of  $X$  there is a sequence  $\{\mathcal{U}_i\}_{i \geq 1}$  of families of pairwise disjoint open sets with  $\mathcal{U}_i$  refining  $\mathcal{C}_i$  and  $\bigcup \mathcal{U}_i$  covering  $X$ .)

### 11.2. Progress on Problems of Section TC

? **1010. TC 1** (79TC1) *Let  $G$  be a complete metrizable topological group which is an ANR. Is  $G$  a manifold modeled on some Fréchet space? In particular, if  $G$  is not locally compact and separable, is it an  $s$ -manifold?*

Open. The separable case is answered affirmatively in DOBROWOLSKI and TORUŃCZYK [1981].

? **1011. TC 2** (79TC2) *If  $X \times s$  is homeomorphic with  $H$  for a non-separable Hilbert space  $H$ , is  $X \approx H$ ?*

Open. The answer is “No” if  $H$  is separable. This follows from R.D. Anderson’s solution ANDERSON [1964] to the Scottish Book problem of Borsuk whether the product of a triod with  $Q$  is a Hilbert cube.

**TC 3** (79TC3) *Is  $X \approx s$  if  $X$  is a complete separable AR such that each compact subset is a  $Z$ -set?*

No, ANDERSON ET AL [1982], DIJKSTRA [1987].

? **1012. TC 4** (79TC4) *Let  $X$  be a topologically complete separable metric space. If  $X$  is an ANR,  $Y \subset X$ , and  $Y$  is an  $s$ -manifold, when can we conclude that  $X$  is?*

If  $X - Y$  is a countable union of strong  $Z$ -sets of  $X$ , then  $X$  is an  $s$ -manifold TORUŃCZYK [1978, 1981, 1985]. (A is a *strong*  $Z$ -set of ANR  $X$  provided that for every open cover  $\mathcal{U}$  of  $X$  the identity of  $X$  may be approximated  $\mathcal{U}$ -closely by maps of  $X$  onto a set  $X' \subset X - A$  with closure missing  $A$ .)

**TC 5** (79TC5) *Characterize  $\sigma$ - and  $\Sigma$ -manifolds topologically.*

This has been done, DOBROWOLSKI and MOGILSKI [1982], HENDERSON [1985], MOGILSKI [1984], BESTVINA ET AL [1986].

? **1013. TC 6** (79TC6) *If  $G$  is a locally contractible separable metric topological group which is the countable union of compact finite dimensional subsets and not locally compact, then is  $G$  a  $\sigma$ -manifold?*

Open.

**TC 7** (79TC7) Give practical conditions on the inverse sequence  $\{X_n; f_n\}$ , **1014. ?** where each  $X_n$  is an AR and each  $f_n$  is a CE map, is the inverse limit  $X$  homeomorphic with  $Q$ ?

There is a simple characterization: It is not hard to see that  $X$  is an AR. Then by Toruńczyk's Theorem (TORUŃCZYK [1980]) one need only show that each pair  $f, g: Q \rightarrow X$ , of maps may be approximated by maps  $f', g': Q \rightarrow X$  with disjoint images. For a given  $\epsilon > 0$ , there would then have to be an integer  $n(\epsilon)$  such that the projections of  $f$  and  $g$  into  $X_i$  may be  $\epsilon$ -approximated by maps with disjoint images whenever  $i > n(\epsilon)$ . That this will also suffice follows from the fact (LACHER [1977]) that CE maps between ANR's admit " $\epsilon$  cross sections" for all  $\epsilon > 0$ . However, this condition may not be easy to apply. Simpler criteria should be sought. To begin, is it sufficient to require that all point inverses of an projections  $X \rightarrow X_n$  be infinite-dimensional?

**TC 8** (79TC8) Under what conditions is a direct limit of ANR's and embeddings an  $\mathbb{R}^\infty$ - or  $Q^\infty$ -manifold?

Done in HEISEY and TORUŃCZYK [1981].

### 11.3. More Problems on Topological Characterization

**TC 9** Mogilski's characterization (MOGILSKI [1984]) of  $\sigma$ -manifolds goes as **1015. ?** follows:  $X$  is a  $\sigma$ -manifold if and only if it is an ANR and

- (1)  $X$  is a countable union of finite-dimensional compacta,
- (2) each compact subset of  $X$  is a strong  $Z$ -set,
- (3) each  $f: A \rightarrow X$  from a finite-dimensional compactum into  $X$  that restricts to an embedding on a closed subset  $B$  of  $A$  may be approximated rel.  $B$  by embeddings.

HENDERSON [1985] weakened (3) to (3') Each map  $f: R^k \rightarrow X$  may be approximated by injections. Can these conditions be replaced by one more reminiscent of the disjoint disks property?

**TC 10** Show that the following conditions on  $X$  characterize the Nöbeling **1016. ?** spaces:

- $X$  is complete metric,
- $\dim(X) = n$ ,
- $X \in LC^{n-1}$ ,



- $X \in C^{n-1}$ ,
- $X$  satisfies  $n$ -SDAP.

$n$ -SDAP is Toruńczyk's Strong Discrete  $n$ -Cells Approximation Property: for each map  $f: I^n \times \mathbb{Z} \rightarrow X$  and each open cover  $\mathcal{U}$  of  $X$ , there is a map  $g: I^n \times \mathbb{Z} \rightarrow X$  that is  $\mathcal{U}$ -close to  $f$  and embeds  $I^n \times \mathbb{Z}$  onto a discrete set of cells in  $X$ .

? **1017. TC 11** Let  $n$  and  $k$  be fixed integers,  $n > 1, 0 \leq k \leq n$ . Let  $\mathcal{M}_k^n$  denote the tame compacta in  $\mathbb{R}^n$  of dimension at most  $k$ . GEOGHEGAN and SUMMERHILL [1974] proved the existence of a  $\mathcal{M}_k^n$ -absorber and denoted it by  $B_k^n$ , with  $s_k^n$  being the complement in  $\mathbb{R}^n$  of  $B_{n-k-1}^n$ . It is known that  $B_k^n \approx B_k^m$  if and only if  $s_k^n \approx s_k^m$  which occurs when  $n = m$  or  $k$  is less than half of  $m$  and of  $n$  (DIJKSTRA ET AL [19∞]). Give characterizations of these spaces.

? **1018. TC 12** Has every ANR homology  $n$ -manifold a CE resolution by a topological  $n$ -manifold?

QUINN [1983, 1987] has reduced this question to a single controlled surgery index obstruction, for  $n > 4$  ( $n = 4$  if  $\partial X$  is a manifold). If the obstruction always vanishes, then combined results of Cannon, Edwards, and Quinn show that  $X$  is an  $n$ -manifold,  $n > 4$ , provided it is an ANR homology  $n$ -manifold and has the DDP. A space  $X$  is a homology  $n$ -manifold if for each

$$x \in X, H_i(X, X - x; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}).$$

The last three are not infinite-dimensional questions, but they are of interest.

## 12. N: Infinite Dimensional Spaces in Nature

### 12.1. Introduction

It bears repeating that the past vitality of the field and its future strength depend fundamentally on its connections with other branches of mathematics. Historically, infinite dimensional topology of the sort under consideration here was motivated by founders of the field of functional analysis (Fréchet, Banach) and developed strongly by functional analysts (Bessaga, Kadets, Pełczyński and Klee with the aims of classifying vector spaces and convex sets (still a major focus even after the tremendous advances of the past) and by Eells to support Global Analysis.

The field has gained strength from the initially unexpected connection of  $Q$ -manifolds with finite dimensional manifolds unearthed by Chapman: the

Whitehead group and other  $K$ -theoretic invariants fundamental to the analysis of homeomorphisms of  $n$ -manifolds survive intact upon stabilization by Cartesian product with  $Q$ , and in most cases, the topological questions to which they are important also survive.

In essence, the  $Q$ -manifolds form a simultaneous stabilization of the finite-dimensional manifolds, the locally compact simplicial complexes, and the locally compact ANR's, in which local complexities are stabilized away but global aspects of the topology is retained by the local compactness with the result that the first-order global homeomorphism theory of these spaces becomes in the  $Q$ -manifolds virtually the same as the  $K$ -theory. The simplification is at times more subtle than might be expected. For example, CHAPMAN [1980] discovered and HUGHES [1983, 1985] has exploited the fact that engulfing is canonical in Hilbert cube manifolds (in the sense that one may do engulfing continuously parameterized by a parameter in an arbitrary metric space). It is this simplification that has made it possible to prove several theorems of vital import to finite dimensional manifolds and polyhedra in the setting of Hilbert cube manifolds and even to fashion  $Q$ -manifolds into a tool for investigating finite dimensional questions of this sort.

Some examples are the following:

- Chapman's proof (CHAPMAN [1974]) of the topological invariance of Whitehead torsion for compact polyhedra (that homeomorphisms are simple homotopy equivalences) first stabilizes a homeomorphism and then in the  $Q$ -manifold setting deforms it to something that is obviously a stabilization of a simple homotopy equivalence.
- Compact ANR's were shown to be homotopy equivalent with compact polyhedra by demonstrating (WEST [1977]) that they may be embedded in  $Q$ -manifolds with mapping cylinder neighborhoods, which, being compact  $Q$ -manifolds, are homotopy equivalent to compact ANR's by work of Chapman.
- the proof that homeomorphism groups of compact  $Q$ -manifolds are ANR's (independently by FERRY [1977b] and TORUŃCZYK [1977]) provided the first widely applicable topological condition (beyond being a homeomorphism) guaranteeing that certain homotopy equivalences were simple. This was explicitly shown by FERRY [1977b] to imply that homotopy equivalences between finite CW-complexes are simple provided they are controlled homotopy equivalences with sufficiently fine control in the target. This should perhaps be considered as the theorem that really began the burgeoning field now known as "controlled topology". It and the ensuing work by Chapman and Ferry directly motivated QUINN [1979, 1982a, 1982b] to prove the Connell-Hollingsworth Conjectures (CONNELL and HOLLINGSWORTH [1969]). It was also picked up immediately by FARRELL and HSIANG [1981] to prove the vanishing of

the Whitehead group and reduced projective class group of the integral group ring of any discrete torsion-free subgroup of the isometries of Euclidean  $n$ -space that has odd- order holonomy. To my knowledge, this was the first time that manifold techniques were applied to obtain computations in algebraic  $K$ -theory, an enterprise that is now a big business.

The working out of these ideas is currently one of the principal currents in infinite dimensional topology.

Function spaces have received a certain amount of attention, but they should receive much more; it is here that infinite dimensional manifolds naturally arise, unavoidably, and with the most importance in mathematics. However, except for the ANR property of homeomorphism groups of compact manifolds, there has been little contact with other areas of mathematics. The general lack of interaction with other branches of mathematics in this area is no doubt one of the reasons for its relative neglect. It needs an infusion of new ideas and problems.

There was a promising movement in Hilbert manifolds in the late 1960's, but that was abandoned when D. Henderson showed they all embed as open subsets of the model space and that homotopy equivalences are always homotopic to homeomorphisms (HENDERSON [1969]). A problem of P. A. Smith concerning linearization of  $\mathbb{Z}_p$  actions on  $S^3$  is formulated as a fixed point problem on a Hilbert manifold and is included as (N5) to suggest a new direction.

Several papers have pointed out mapping space ideas of promise in the function spaces department that should be followed up:

- GEOGHEGAN [1976] followed by JONES [1976] and COLVIN [1985] have examined spaces of mappings into flat and hyperbolic manifolds and found that several mapping space constructions lead through natural restrictions to spaces of maps that have compact  $Q$ -manifold closures in other spaces. (Examples are maps  $f$  into Riemannian manifolds of nowhere positive sectional curvature that have  $k$ -th derivative of bounded norm or bounded "energy" .)
- SAKAI [1980a], SAKAI and WONG [1989a, 1989b, 1989c] have been developing a theory of Lipschitz homeomorphisms and mappings of manifolds with an eye to proving that they fit in between the piecewise linear maps and the continuous ones and that charts may be had for, say, the homeomorphism spaces that are modelled on various pairs and triples of vector spaces with the sub-elements of the pairs and triples being incomplete subspaces of Hilbert spaces.
- Sakai and Wong in the above series also develop the idea of a stabilized finite dimensional homeomorphism of a  $Q$ -manifold and investigate the

role these play in the above setting. These papers produce new and natural manifolds of maps modelled on various incomplete linear subspaces of Hilbert space.

- J. Oxtoby and V. Prasad (OXTOPY and PRASAD [1978], PRASAD [1979]) have done foundational work on spaces of measures on  $Q$  or  $I^n$ .
- FEDORČUK [1986, 1982], NGUYEN TO NHU [19 $\infty$ ] and NGUYEN TO NHU and TA KHAC CU [19 $\infty$ ] have studied spaces of measures on ANR's and spaces of measure preserving transformations.

Hyperspaces is another point of contact. There has been a certain amount of work done by Curtis, Sakai, Nhu, and Heisey and West, as well as many others. This has been mostly of a follow up nature to theorems of the 1970's. It appears, however, that there are opportunities in new directions generated by work in differential geometry by GROMOV [1979, 1981, 1983, 1986, 1987] GROVE [1987] and currently being exploited by GROVE and PETERSON [1988]. Of particular interest for infinite dimensional topologists is the use of the hyperspace of metric compacta (isometry classes) with a metric that combines the Hausdorff metric and the natural Lipschitz metric when the spaces are Lipschitz equivalent. In this space, the subspaces comprised of Riemannian manifolds with curvature and diameter bounds are of interest to the differential geometers and are shown under suitable hypotheses to form precompact sets. This is used in proofs that within these bounds, the set of homotopy types or even homeomorphism types of manifolds are finite. These spaces and subspaces and their limits, completions, and compactifications should be a fruitful field for investigation.

## 12.2. Progress on Problems of Section N

**N 1 (79N1)** *Let  $M$  be a compact  $n$ -manifold,  $n > 2$ . Is  $\mathcal{H}_\partial(M)$  a Hilbert manifold?* **1019. ?**

Open. This is discussed in the section on homeomorphism spaces.

**N 2 (79N2)** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces with  $X$  compact. Under what conditions is the space  $1\text{-LIP}(X, Y)$  of 1-Lipschitz maps a  $Q$ -manifold?* **1020. ?**

This is generally open, but see COLVIN [1985], SAKAI and WONG [1989a, 1989b, 19 $\infty$ ], and GEOGHEGAN [1976]. ( $f$  is 1-Lipschitz if it is weakly contracting, i.e., if  $\rho(f(x), f(x')) \leq d(x, x')$  for all  $x, x'$  in  $Y$ .) This should be revised to ask about the  $k$ -Lipschitz maps ( $\rho(f(x), f(x')) \leq kd(x, x')$ ) for other  $k$ , too. (See (HS11).)

**N 3 (79N3)** *Is the space of all  $\mathbb{Z}_2$  actions on a compact  $Q$ -manifold a Hilbert manifold?* **1021. ?**

manifold? Is it  $LC^0$ ? What about compact  $n$ -manifolds?

Open. See Section GA.

- ? **1022.** N 4 (79N4) Let  $G$  be a compact Lie group. Let  $2^G$  be its hyperspace of non-void closed subsets with the Hausdorff metric induced from a translation invariant metric on  $G$ . Let  $G$  act on  $2^G$  by, say, left translation. What is the structure of the orbit space  $2^G/G$ ?

Open. Toruńczyk and West examined the case  $G = S^1$  in TORUŃCZYK and WEST [1978] and found that there is a wealth of  $Q$ -manifold Eilenberg-MacLane spaces of type  $K(Z_{(P)}, 2)$  in it for any subset  $P$  of primes, where  $Z_{(P)}$  denotes the integers localized away from the primes in  $P$ . (The structure occurs as a direct system of  $Q$ -manifold  $K(Z, 2) = CP^\infty = BS^1$ 's according to the lattice of inclusions of the closed subgroups of  $S^1$  in such a way that the union of the manifolds in a particular subsystem is in fact a  $Q$ -manifold and a  $K(Z_{(P)}, 2)$ .) For example,  $(2^{S^1} - \{S^1\})/G$  is a  $Q$ -manifold Eilenberg-MacLane space with second homotopy group isomorphic with the rational numbers. Heisey and West have extended this analysis to the case  $G = S^1 \times S^1$  (HEISEY and WEST [1988], HEISEY and WEST [19 $\infty$ ]). In this context, it should be of interest to relate the topology of the hyperspace of closed subgroups of a Lie group to the hyperspace of all closed subsets. The closed subgroups have been found to exhibit interesting topology even in the simple case of  $R^2$  (HUBBARD and POUREZZA [1979]), where knotting phenomena are present.

### 12.3. More Problems on Infinite Dimensional Spaces in Nature

- ? **1023.** N 5 (P. Smith) Let  $\alpha$  be a free action of the group  $\mathbb{Z}_p$ ,  $p$  prime, on the sphere  $S^3$ . Let  $\mathcal{E}$  be the space of all locally flat unknotted simple closed curves in  $S^3$ . Let  $\alpha_*$  denote the induced  $\mathbb{Z}_p$ -action on  $\mathcal{E}$ . Must  $\alpha_*$  have a fixed point?

This is equivalent to a longstanding conjecture of P. Smith. If the answer is “Yes”, then all free  $\mathbb{Z}_p$  actions on  $S^3$  are conjugate to linear ones. This is true if  $p = 2$  (LIVESAY [1960]). Both the action and the simple closed curves may be taken smooth, if desired, in which case a certain amount of smooth machinery is available.  $\mathcal{E}$  is an infinite dimensional manifold homotopy equivalent with the space of great circles on  $S^3$ . The Lefschetz number of  $\alpha_*$  is non-zero, indicating a fixed point, were the Lefschetz fixed point theorem valid in this situation. Is there an invariant compact ANR (e.g.,  $Q$ -manifold of simple closed curves) in  $\mathcal{E}$  homotopy equivalent with  $\mathcal{E}$ ? (The Lefschetz theorem applies to compact ANR's.)

**N 6** Let  $M^n$  be a complete Riemannian manifold of nowhere positive sectional curvature. Let  $P$  be a compact polyhedron and  $k > 0$ . Is the space  $X$  of  $k$ -Lipschitz maps from  $P$  to  $M$  a  $Q$ -manifold? If so, can some of the very interesting dynamics associated with  $M$  be lifted to  $X$  or  $Q$ -manifolds associated with  $X$ ? **1024. ?**

See ANOSOV [1969], BALLMANN ET AL [1985], FARRELL and HSIANG [1981, 1983], FARRELL and JONES [1986a, 1986b, 1988a, 1988b, 19 $\infty$ ]. This is one of the most promising possibilities in the field of  $Q$ -manifolds at present. The work of Farrell and Jones combines a great deal of  $n$ -manifold theory with the dynamics and arrives at a situation where controlled  $h$ -cobordism theorems are needed (and proved) to establish vanishing theorems for the  $K$ -theory. The  $h$ -cobordism theorems are the moral equivalent of homeomorphism theorems in  $Q$ -manifold theory. The connections and structures that exist in the  $Q$ -manifolds of this nature should be understood.

**N 7** Can  $Q$ -manifold function spaces such as the above be used to give a technically simpler proof of the vanishing of the Whitehead group,  $\tilde{K}_0$ , and  $K_{-i}$  groups of  $Z[\pi_1(M)]$  than that of Farrell and Jones? (Where  $M$  is as above.) **1025. ?**

More problems on naturally occurring spaces are in Sections GA, HS, NLC, LS, and TC.

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# PART VII

## TOPOLOGY ARISING FROM ANALYSIS

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Open Problems in Topology

J. van Mill and G.M. Reed (Editors)

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## CHAPTER 31

### **Problems in $C_p$ -theory**

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By a space we mean a Tikhonov topological space. The main role in  $C_p$ -theory is played by the topological ring  $C_p(X)$  formed by all continuous real-valued functions on a space  $X$  in the topology of pointwise convergence. The starting point is J. Nagata's Theorem in NAGATA [1949]: *if the topological rings  $C_p(X)$  and  $C_p(Y)$  are topologically isomorphic then the spaces  $X$  and  $Y$  are homeomorphic*. This result leads to the following general questions. When are  $C_p(X)$  and  $C_p(Y)$  linearly homeomorphic as linear topological spaces? When are  $C_p(X)$  and  $C_p(Y)$  homeomorphic as topological spaces? We say that  $X$  and  $Y$  are  $l$ -equivalent, and write  $X \stackrel{l}{\sim} Y$ , if  $C_p(X)$  and  $C_p(Y)$  are linearly homeomorphic. If  $C_p(X)$  and  $C_p(Y)$  are just homeomorphic, we write  $X \stackrel{t}{\sim} Y$  and say that the spaces  $X$  and  $Y$  are  $t$ -equivalent. It is known that  $l$ -equivalent spaces  $X$  and  $Y$  need not be homeomorphic: for example the spaces  $I$  and  $I \times \{0, 1\}$  (where  $I$  is the unit segment) are  $l$ -equivalent and not homeomorphic. One can show that all uncountable zero-dimensional metrizable compacta are  $l$ -equivalent (BAARS and DE GROOT [1988]).

When are  $X$  and  $Y$   $l$ -equivalent?  $t$ -equivalent? Which topological properties are preserved by  $l$ -equivalence? By  $t$ -equivalence? We call such properties  $l$ -invariants and  $t$ -invariants. The  $C_p$ -theory is formed around questions of this kind. This brings it in contact with many parts of general topology as well as with topological theory of function spaces and with general theory of linear topological spaces.

A major direction in  $C_p$ -theory is represented by duality theorems: here we find topological properties of  $X$  which can be characterized by topological or by linear topological properties of  $C_p(X)$ .

Another important general problem in  $C_p$ -theory is the following one. Given a class  $\mathcal{P}$  of topological spaces, characterize those topological spaces  $Y$  which can be topologically embedded into  $C_p(X)$  for some  $X \in \mathcal{P}$ . Observe that the compact spaces  $Y$  which can be topologically embedded into  $C_p(X)$  where  $X$  is compact are exactly the Eberlein compacta (ARKHANGEL'SKIĬ [1984]). One of the principal advantages of the topology of pointwise convergence is that it provides us with a better supply of compact sets in  $C(X)$  than practically any natural topology on  $C(X)$ —as this topology is the weakest one.

The reader is referred to the rather comprehensive surveys ARKHANGEL'SKIĬ [1988b], [1987] and [1984] for more details and more discussions. See also VAN MILL [1987b] and BAARS, DE GROOT and VAN MILL [1986].

Below we use the following notations. By  $\bigoplus$  we denote the free topological sum;  $M \simeq L$  means that  $M$  and  $L$  are linearly homeomorphic;  $\mathbb{N}^+$  is the set of all positive integers and  $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$ ; by  $X$ ,  $Y$  and  $Z$  we denote Tikhonov spaces. We say that  $X$  and  $Y$  are  $u$ -equivalent, and write  $X \stackrel{u}{\sim} Y$ , if  $C_p(X)$  and  $C_p(Y)$  are uniformly homeomorphic as uniform spaces (with respect to the uniformity of pointwise convergence). By  $\mathbb{R}$  we denote the linear topological space of real numbers. For the definitions of *tightness*, *Fréchet-Urysohn property*, *Souslin number* and of other topological concepts see ENGELKING [1989].

When  $X$  is compact then  $C_B(X)$  is the Banach space  $C(X)$ .

There are many good problems in  $C_p$ -theory; some of them, which I consider very interesting, important, natural and probably difficult to solve are presented here—but of course this list is far from being complete. Good luck to you if you are going to attack them!

- ? **1026. Problem 1.** *Let  $X$  be an infinite space. Is it true that  $C_p(X) \times \mathbb{R}$  is linearly homeomorphic to  $C_p(X)$ ? What if  $X$  is compact?*
- ? **1027. Problem 2.** *Is it true that the topological spaces  $C_p(X) \times \mathbb{R}$  and  $C_p(X)$  are homeomorphic for any infinite space  $X$ ? What if  $X$  is compact?*

Problems 1 and 2 may be reformulated as follows. Let  $X$  be an infinite topological space and let  $X^+$  be the space obtained by adding one new isolated point to  $X$ . Are the spaces  $X$  and  $X^+$   $l$ -equivalent? Is it true that  $X \overset{t}{\sim} X^+$ ?

It is also quite natural to ask whether the spaces  $X$  and  $X^+$  are  $u$ -equivalent for any infinite space  $X$ .

One can show that the answer to problem 2 is “yes” if  $X$  contains a non-trivial convergent sequence (see GUL'KO [1986]). Observe that  $C_p(X)$  can always be represented in the form  $C_p(X) \simeq M \times \mathbb{R}$  where  $M$  is some locally convex linear topological space over  $\mathbb{R}$ .

Observe that for every infinite-dimensional Banach space  $B$  the space  $B \times \mathbb{R}$  is homeomorphic to  $B$  by Kadetz's Theorem and its generalizations. On the other hand, it is still unknown whether  $B \times \mathbb{R}$  is linearly homeomorphic to  $B$  for every infinite-dimensional Banach space  $B$  (this is a well-known and quite important open problem in the theory of Banach spaces).

The last question is related to Problem 1 in the following way. Let  $X$  be compact and assume that  $C_p(X) \times \mathbb{R}$  is linearly homeomorphic to  $C_p(X)$ . Then the Banach spaces  $C_B(X) \times \mathbb{R}$  and  $C_B(X)$  are also linearly homeomorphic (see PAVLOVSKIĬ [1982] and ARKHANGEL'SKIĬ [1980]). It follows that a positive answer to Problem 1 for compact  $X$  would imply a positive answer to the following question on Banach spaces which is a special case of the question on Banach spaces discussed above:

- ? **1028. Problem 3.** *Let  $X$  be an infinite compact space. Is it true that the Banach spaces  $C_B(X)$  and  $C_B(X) \times \mathbb{R}$  are linearly homeomorphic?*

(They are (topologically) homeomorphic.)

Observe that according to VAN MILL [1987a] there exists an infinite-dimensional normable linear topological space  $L$  such that  $L$  is not homeomorphic to  $L \times \mathbb{R}$ .

Other general problems on  $C_p(X)$  of the same type as Problems 1 and 2 are the following ones. Find out when  $C_p(X) \times C_p(X)$  is linearly homeomorphic to  $C_p(X)$ . Find out under what restrictions on a topological space  $X$

the topological space  $C_p(X) \times C_p(X)$  is homeomorphic to the topological space  $C_p(X)$ . In other words, when is  $X \bigoplus X \overset{l}{\sim} X$ ? When is  $X \bigoplus X \overset{t}{\sim} X$ ?

Let  $T(\omega_1 + 1)$  be the space of all ordinal numbers not exceeding the first uncountable ordinal number  $\omega_1$  with the usual topology. It was shown by S. P. Gul'ko that  $C_p(T(\omega_1 + 1)) \times C_p(T(\omega_1 + 1))$  is not homeomorphic to  $C_p(T(\omega_1 + 1))$ . Thus  $T(\omega_1 + 1) \bigoplus T(\omega_1 + 1)$  is not  $t$ -equivalent to  $T(\omega_1 + 1)$ .

**Problem 4.** *Let  $X$  be an infinite metrizable space. Is it true that  $C_p(X)$  is linearly homeomorphic (is homeomorphic) to  $C_p(X) \times C_p(X)$ ? Is this true for every compact metrizable space  $X$ ?* **1029. ?**

**Problem 5.** *Is it true that for every space  $X$  the space  $C_p(X) \times C_p(X)$  can be represented as a continuous image of the space  $C_p(X)$ ? What if  $X$  is compact?* **1030. ?**

It was shown by MARCISZEWSKI [1983] that there exists a metrizable linear topological space  $L$  which cannot be continuously mapped onto its square  $L \times L$ . In [19 $\infty$ ], MARCISZEWSKI has also constructed a compact space  $X$  such that  $C_p(X) \times C_p(X)$  is not homeomorphic to  $C_p(X)$ .

Observe that  $\mathbb{R} = C_p(\{0\})$  can be mapped continuously onto  $\mathbb{R} \times \mathbb{R} = C_p(\{0\}) \times C_p(\{0\}) = C_p(\{0, 1\})$ .

If the answer to Problem 5 is “yes” then the following will be true: if  $C_p(X)$  is Lindelöf then  $C_p(X) \times C_p(X)$  is Lindelöf. The last assertion is not proved so far. Thus we have:

**Problem 6.** *Let  $X$  be a space such that  $C_p(X)$  is Lindelöf. Is it true that the space  $C_p(X) \times C_p(X)$  is Lindelöf? What if  $X$  is compact?* **1031. ?**

This is one of many questions of the following kind: assume that  $C_p(X)$  has topological property  $\mathcal{P}$ , does it follow then that  $C_p(X) \times C_p(X)$  also has property  $\mathcal{P}$ ? It is well known that many topological properties are not productive in general; among such properties are normality, paracompactness, Lindelöfness, countable tightness and many other properties. But the spaces  $C_p(X)$  are always “regular” topological objects—first, they are formed in a standard way, and, second, these spaces are prevented from turning pathological by very strong algebraic barriers—by the natural ring algebraic structure of  $C_p(X)$  which “sticks” to the topology of  $C_p(X)$ . So that we may expect that many topological properties may become productive for the spaces  $C_p(X)$  or at least that they will be preserved by the square operation. For example it was established in this direction that if the tightness of  $C_p(X)$  is countable then the tightness of  $C_p(X) \times C_p(X)$  is countable, that if  $C_p(X)$  is a Fréchet-Urysohn space then  $C_p(X) \times C_p(X)$  is a Fréchet-Urysohn space (see PYTKEEV [1982a]).

? **1032. Problem 7.** *Let  $X$  be a space such that  $C_p(X)$  is normal. Is it true that  $C_p(X) \times C_p(X)$  is normal? What if  $X$  is compact?*

The Souslin number  $c(C_p(X))$  of the space  $C_p(X)$  is always countable (as  $C_p(X)$  is dense in  $\mathbb{R}^X$ —see ARKHANGEL'SKIĬ [1982]). It follows that  $C_p(X)$  is paracompact if and only if it is Lindelöf. Thus we don't have to formulate “the square problem” for paracompact  $C_p(X)$ —it is equivalent to Problem 6.

H. H. CORSON [1959] has shown that if  $Y$  is a dense subspace of the product of any family of separable metrizable spaces and the space  $Y \times Y$  is normal then  $Y$  is collectionwise normal. Thus if the answer to Problem 7 is “yes” this would imply that  $C_p(X)$  is collectionwise normal whenever it is normal. Actually the last equivalence was shown to be true by E. A. Reznichenko (see ARKHANGEL'SKIĬ [1987, 1989b]). But the following question remains open:

? **1033. Problem 8.** *Let  $X$  be a zero-dimensional space,  $D = \{0, 1\}$  and assume that the space  $C_p(X, D)$  is normal. Is it then true that  $C_p(X, D)$  is collectionwise normal? What if  $X$  is compact?*

Some positive results in this direction were obtained under special set-theoretic assumptions (see ARKHANGEL'SKIĬ [1987, 1989b]).

The following problems may provide an approach to Problems 6 and 7.

? **1034. Problem 9.** *Is it possible to represent the Sorgenfrey line (the “arrow space”) as a closed subspace of  $C_p(X)$  where  $C_p(X)$  is Lindelöf? where  $C_p(X)$  is normal?*

If the answer to the last question is positive then Problem 6 would be answered negatively. Observe that it is not possible to embed the Sorgenfrey line into  $C_p(X)$  where  $X$  is compact. This depends on the following fact: if  $X$  is compact then the closure of every countable subset  $A \subseteq C_p(X)$  in  $C_p(X)$  is a space with a countable network (see ARKHANGEL'SKIĬ [1976]).

Problem 9 is a particular case of the following general question.

? **1035. Problem 10.** *Is it possible to represent the Sorgenfrey line as a closed subspace of some linear topological space which is Lindelöf?*

It is also unknown whether one can represent the Sorgenfrey line as a closed subspace of a Lindelöf topological group (see ARKHANGEL'SKIĬ [1981]).

There are several interesting open questions similar to Problems 9 and 10.

? **1036. Problem 11.** *Is it possible to represent every Lindelöf space as a closed subspace of a Lindelöf linear topological space? of a Lindelöf  $C_p(X)$ ?*

? **1037. Problem 12.** *Is it possible to represent every paracompact space as a closed subspace of a paracompact linear topological space?*

**Problem 13.** *Is it possible to represent every normal space as a closed subspace of a normal linear topological space?* **1038. ?**

In Problems 11, 12 and 13 we may restrict ourselves to locally convex linear topological spaces over  $\mathbb{R}$ .

In connection with Problem 13 observe that not every normal space admits a closed embedding into a normal space  $C_p(X)$ . Indeed, according to Reznichenko's Theorem (see ARKHANGEL'SKIĬ [1989b]), if  $C_p(X)$  is normal then  $C_p(X)$  is collectionwise normal. It follows that if  $X$  is a normal space which is not collectionwise normal then  $X$  is not homeomorphic to a closed subspace of  $C_p(X)$  where  $C_p(X)$  is normal. But the following question remains open:

**Problem 14.** *Let  $X$  be a collectionwise normal space. Is  $X$  homeomorphic to a closed subspace of a collectionwise normal linear topological space (not necessarily locally convex)?* **1039. ?**

**Problem 15.** *Is it true that every space  $Y$  of countable tightness is homeomorphic to a subspace (to a closed subspace) of  $C_p(X)$  where the tightness of  $C_p(X)$  is countable?* **1040. ?**

The tightness of  $C_p(X)$  is countable if and only if  $X^n$  is Lindelöf for each  $n \in \mathbb{N}^+$  (ARKHANGEL'SKIĬ and PYTKEEV [1982b]). Thus we can reformulate Problem 15 in the following way. Let  $Y$  be a space of countable tightness. Is it possible to find a space  $X$  such that  $X^n$  is Lindelöf for every  $n \in \mathbb{N}^+$  and  $Y$  is homeomorphic to a subspace of  $C_p(X)$ ? This reformulation suggests the following version of Problem 15:

**Problem 16.** *Is it true that every space  $Y$  of countable tightness is homeomorphic to a subspace (to a closed subspace) of  $C_p(X)$  where  $X$  is Lindelöf?* **1041. ?**

**Problem 17.** *Is it true that every space  $Y$  of countable tightness is homeomorphic to a subspace (to a closed subspace) of a linear topological space of countable tightness?* **1042. ?**

Observe that every countable space  $Y$  is homeomorphic to a closed subspace of  $C_p(X)$ , where  $C_p(X)$  has a countable network and hence is a space of countable tightness. Indeed, one can take  $X$  to be  $C_p(Y)$ —then  $C_p(Y)$  has a countable base and  $C_p(C_p(Y))$  has a countable network (ARKHANGEL'SKIĬ [1976]).

Not every countable space can be realized as a subspace of  $C_p(X)$  where  $X$  is compact (V. V. Uspenskii, see USPENSKIĬ [1978] and ARKHANGEL'SKIĬ [1989b]). The countable Fréchet-Urysohn fan can serve as a counterexample (USPENSKIĬ [1978]).

In connection with Problem 15 it should be mentioned that not every countable Fréchet-Urysohn space can be topologically embedded into a Fréchet-Urysohn linear topological space. This follows from NYIKOS [1981]: if  $G$  is a Fréchet-Urysohn topological group then  $G$  is a  $\langle 4 - FU \rangle$ -space (i.e.,  $G$  is a strongly Fréchet-Urysohn space, see MICHAEL [1971] and NYIKOS [1981]).

? **1043. Problem 18.** (N. V. Velichko) *Let  $C_p(X)$  be a hereditarily Lindelöf space. Is it true that  $(C_p(X))^n$  is hereditarily Lindelöf for all  $n \in \mathbb{N}^+$ ?*

P. ZENOR [1980] and VELICHKO [1981] have independently shown that  $(C_p(X))^n$  is hereditarily Lindelöf for all  $n \in \mathbb{N}^+$  if and only if  $X^n$  is hereditarily separable for all  $n \in \mathbb{N}^+$ . Thus we can reformulate Problem 18 in the following way: is it true that if  $C_p(X)$  is hereditarily Lindelöf then  $X^n$  is hereditarily separable for all  $n \in \mathbb{N}^+$ .

Let us recall that the *spread* of  $Y$ , denoted  $s(Y)$ , is the supremum of cardinalities of discrete subspaces of  $Y$ . Very close to Problem 18 is the following question:

? **1044. Problem 19.** *Let the spread of  $C_p(X)$  be countable. Is it true that the spread of  $(C_p(X))^n$  is countable for all  $n \in \mathbb{N}^+$ ?*

It is known that  $s(C_p(X))^n \leq \aleph_0$  for all  $n \in \mathbb{N}^+$  if and only if  $s(X^n) \leq \aleph_0$  for all  $n \in \mathbb{N}^+$  (see ARKHANGEL'SKIĬ [1989b]). Thus Problem 19 can be stated in this form: let  $s(C_p(X)) \leq \aleph_0$ ; is it true that  $s(X^n) \leq \aleph_0$  for all  $n \in \mathbb{N}^+$ ?

If  $X$  is a zero-dimensional space then the answer to Problems 18 and 19 is positive (ARKHANGEL'SKIĬ [1989b]). Another case when Problems 18 and 19 get positive solutions is described by M. Asanov (see ARKHANGEL'SKIĬ [1989b]). It is known that if  $hl(C_p(X) \times C_p(X)) \leq \aleph_0$  then all  $X^n$  are hereditarily separable and that if  $s(C_p(X) \times C_p(X)) \leq \aleph_0$  then  $s(X^n) \leq \aleph_0$  for all  $n \in \mathbb{N}^+$  (see ARKHANGEL'SKIĬ [1989b, 1984]). Observe that Velichko has shown that if  $C_p(X)$  is hereditarily separable then  $(C_p(X))^n$  is hereditarily separable for all  $n \in \mathbb{N}^+$  and  $X^n$  is hereditarily Lindelöf for all  $n \in \mathbb{N}^+$  (VELICHKO [1981]).

Let  $SA$  denote the following assertion: every hereditarily separable space is hereditarily Lindelöf. S. Todorčević has shown in TODORČEVIĆ [1983] that  $SA$  is consistent with the usual system **ZFC** of axioms of set theory. Arkhangel'skiĭ has proved that under  $SA$  the answer to Problems 18 and 19 is positive (ARKHANGEL'SKIĬ [1989a]) so that it is not possible to construct counterexamples to Problems 18 and 19 in **ZFC**.

Problems 18 and 19 can also be formulated for higher cardinal numbers.

? **1045. Problem 20.** *Let  $X$  and  $Y$  be  $t$ -equivalent spaces. Is it true that  $\dim X = \dim Y$ ? What if  $X$  and  $Y$  are compact?*

For a compact space  $X$ ,  $\dim X$  is the Lebesgue covering dimension of  $X$ . If  $X$  is not compact we put  $\dim X = \dim \beta X$ , where  $\beta X$  is the Čech-Stone

compactification of  $X$ . V. G. Pestov has shown that if  $X$  is  $l$ -equivalent to  $Y$  then  $\dim X = \dim Y$  (PESTOV [1982]). Another problem inspired by the theorem of Pestov is the following one:

**Problem 21.** Assume that  $C_p(X)$  can be mapped by a linear continuous mapping onto  $C_p(Y)$ . Is it true that  $\dim Y \leq \dim X$ ? What if  $X$  and  $Y$  are compact? **1046. ?**

**Problem 22.** Assume that  $C_p(X)$  can be mapped by an open linear continuous mapping onto  $C_p(Y)$ . Is it true that  $\dim Y \leq \dim X$ ? What if  $X$  and  $Y$  are compact? **1047. ?**

It can be shown that if  $X$  is a compact space which does not contain a topological copy of the Tikhonov cube  $I^{\aleph_1}$  then  $C_p(X)$  cannot be mapped by a linear continuous mapping onto  $C_p(I^{\aleph_1})$  (A. V. Arkhangel'skiĭ).

The following question is obviously related to Problems 21 and 22.

**Problem 23.** Let  $C_p(Y)$  be a linear topological factor (shortly: an  $l$ -factor) of  $C_p(X)$ —i.e.,  $C_p(X) \simeq C_p(Y) \times M$  for some linear topological space  $M$ . Is it true that  $\dim Y \leq \dim X$ ? **1048. ?**

For compact metrizable spaces this was shown to be true by A. N. DRANISHNIKOV [1986]. For all compact spaces this was proved by Arkhangel'skiĭ and Choban. For arbitrary Tikhonov spaces the question remains open.

Let us say that  $X$  and  $Y$  are *weakly topologically equivalent* (notation:  $X \stackrel{w}{\sim} Y$ ) if  $X$  is homeomorphic to a closed subspace of  $Y$  and  $Y$  is homeomorphic to a closed subspace of  $X$ .

**Problem 24.** Is it true that weakly topologically equivalent metrizable spaces are always  $l$ -equivalent? Is this true at least for compact metrizable spaces? **1049. ?**

For non-metrizable compact spaces the answer to the last question is “no”. Indeed let  $X = I^{\aleph_1}$  and  $Y = I^{\aleph_1} \oplus A$  where  $A$  is any compact subspace of  $I^{\aleph_1}$  the Souslin number of which is uncountable. Then  $X$  and  $Y$  are not  $t$ -equivalent as the Souslin number is preserved by  $t$ -equivalence in the class of compact spaces (ARKHANGEL'SKIĬ [1987]).

**Problem 25.** Is it true that every infinite compact space is  $t$ -equivalent ( $l$ -equivalent) to a compact space containing a non-trivial convergent sequence? **1050. ?**

If the answer is “yes” then  $X^+ \stackrel{t}{\sim} X$  ( $X^+ \stackrel{l}{\sim} X$ ) for every infinite compact space  $X$ . A related question is:

? **1051. Problem 26.** *Is it true that every non-empty compact space is  $l$ -equivalent ( $t$ -equivalent) to a compact space containing a point of countable character?*

For a space  $X$  we put  $C_{p,1}(X) = C_p(X)$  and  $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ .

? **1052. Problem 27.** (S. P. Gul'ko) *Let  $X$  be a compact space such that all spaces  $C_{p,n}(X)$ , where  $n \in \mathbb{N}^+$ , are Lindelöf. Is it true that  $X$  is a Corson compactum?*

Recall that Corson compacta are defined to be compact subspaces of  $\Sigma$ -products of unit segments (see ARKHANGEL'SKIĬ [1989b] and GUL'KO [1979]). It is known (SOKOLOV [1986]) that if  $X$  is a Corson compactum, then  $C_{p,n}(X)$  is Lindelöf for all  $n \in \mathbb{N}^+$ . Gul'ko has shown that if  $X$  is a compact space such that  $C_p(X)$  is a Lindelöf  $\Sigma$ -space then  $X$  is a Corson compactum and O. G. Okunev under the same restrictions on  $X$  and  $C_p(X)$  has shown that all  $C_{p,n}(X)$  are Lindelöf  $\Sigma$ -spaces (see ARKHANGEL'SKIĬ [1989b]).

? **1053. Problem 28.** *Let  $X$  be a Lindelöf space and let  $Y$  be a compact subspace of  $C_p(X)$ . Is it true that the tightness of  $Y$  is countable?*

One can show that under the assumptions in Problem 28 the space  $Y$  cannot be homeomorphic to the compactum  $T(\omega_1 + 1)$  (see ARKHANGEL'SKIĬ and USPENSKIĬ [1986]). It was shown by ARKHANGEL'SKIĬ [1988a] that under additional set-theoretic assumptions (consistent with **ZFC**) the answer to the last question is positive. Of course if  $X^n$  is Lindelöf for all  $n \in \mathbb{N}^+$  then the answer to Problem 28 is “yes”—in this case the tightness of the whole space  $C_p(X)$  is countable (ARKHANGEL'SKIĬ [1976]).

It is not clear at all what happens if we formulate Problem 28 for higher cardinals.

? **1054. Problem 29.** (O. G. Okunev) *Let  $Y$  be a compact subspace of an infinite space  $C_p(X)$ . Is it true that the tightness of  $Y$  does not exceed the Lindelöf degree of  $C_p(X)$ ?*

Problem 28 can be reformulated in the following way. Let  $X$  be a compact space such that there exists a Lindelöf subspace  $Y$  of  $C_p(X)$  separating the points of  $X$ —i.e., such that for every two different points  $x_1, x_2 \in X$  one can find  $f \in Y$  such that  $f(x_1) \neq f(x_2)$ . Is it true that the tightness of  $X$  is countable? To show that this question is equivalent to Problem 28 one only has to apply the evaluation mapping  $\psi: X \rightarrow C_p(Y)$ .

Let us recall that a space  $Y$  is *co-Lindelöf* if there exists a Lindelöf space  $X$  such that  $Y$  is homeomorphic to a subspace of  $C_p(X)$ .

? **1055. Problem 30.** *Is it true that every continuous image of a compact co-Lindelöf*



space is co-Lindelöf?

The answer “yes” to the last question will imply a positive answer to Problem 28. Indeed, every subspace of a co-Lindelöf space is co-Lindelöf, and if  $X$  is a compact space of uncountable tightness then some closed subspace of  $X$  can be mapped continuously onto  $T(\omega_1 + 1)$  (see ARKHANGEL'SKIĬ [1978]). One can state Problem 30 for perfect mappings as well.

**Problem 31.** *Let  $X$  be a compact space such that  $\aleph_1$  is a caliber of  $C_p(X)$ . 1056. ? Is it true that  $X$  is metrizable?*

Our assumptions imply that  $X$  has an  $\aleph_1$ -inaccessible diagonal in the sense of M. Hušek (see HUŠEK [1977] and ARKHANGEL'SKIĬ [1987]). It was shown by ZHOU in [1982] that it is consistent with **ZFC** to assert every such compact space is metrizable. Thus one cannot expect to construct a counterexample in **ZFC**.

**Problem 32.** *Let  $X$  and  $Y$  be  $t$ -equivalent compact spaces. Is it true that 1057. ? the tightness of  $X$  and  $Y$  are equal, i.e., is it true that  $t(X) = t(Y)$ ?*

For non-compact spaces O. G. Okunev has shown that even  $l$ -equivalence does not preserve tightness (see ARKHANGEL'SKIĬ [1989b, 1985]). On the other hand, it was shown by V. V. Tkachuk that in the class of compact spaces tightness is preserved by  $l$ -equivalence (see ARKHANGEL'SKIĬ [1989b]).

**Problem 33.** *Let  $X \overset{l}{\sim} Y$ , where  $X$  is a compact Fréchet-Urysohn space. 1058. ? Must  $Y$  be a Fréchet-Urysohn space as well? What if  $X \overset{t}{\sim} Y$  and  $Y$  is compact?*

For compact sequential spaces the answer is “yes”.

**Problem 34.** *Let  $X$  be a Hewitt-Nachbin space such that  $C_p(X)$  is Lindelöf. 1059. ? Must then  $X$  be Lindelöf?*

If  $X$  is the  $\Sigma$ -product of uncountably many unit segments then  $X$  is normal countably compact but not compact and hence not Lindelöf while  $C_p(X)$  is Lindelöf (see ARKHANGEL'SKIĬ [1987]). I do not have any idea how difficult Problem 34 will prove to be.

**Problem 35.** *Let  $X$  and  $Y$  be  $l$ -equivalent separable metrizable spaces and 1060. ? let one of them be Čech-complete. Is it true that the other space is also Čech-complete<sup>1</sup>?*

---

<sup>1</sup>Remark by the editors: this question was answered recently in the affirmative by Pol, Baars, de Groot and Pelant.

This question is quite natural as Okunev has shown that in the class of separable metrizable spaces local compactness is preserved by  $l$ -equivalence (see ARKHANGEL'SKIĬ [1987]).

Let  $Y$  be a (closed) subspace of  $X$ . A  $t$ -extender ( $l$ -extender) from  $Y$  to  $X$  is a continuous (linear continuous) mapping  $\psi: C_p(Y) \rightarrow C_p(X)$  such that  $\psi(g)|Y = g$  for every  $g \in C_p(Y)$ . We say that  $Y$  is  $t$ -embedded ( $l$ -embedded) in  $X$  if there exists a  $t$ -extender ( $l$ -extender) from  $Y$  to  $X$ . Every closed subspace of any metrizable space is  $l$ -embedded in it. Each compact metrizable space is  $l$ -embedded in every space containing it (see DUGUNDJI [1951]).

? 1061. **Problem 36.** Let  $\tau > \aleph_0$ . Is it true that  $D^\tau$  is  $t$ -embedded in  $I^\tau$ ?

Here  $D^\tau = \{0, 1\}^\tau$ . Observe that  $D^\tau$  is not  $l$ -embedded in  $I^\tau$  for  $\tau > \aleph_0$ .

? 1062. **Problem 37.** Is it true that every compact space is  $l$ -embedded (is  $t$ -embedded) into some topologically homogeneous compact space?

It was shown by D. Motorov that there exists a compact metrizable space which is not a retract of any topologically homogeneous compact space. This was generalized by ARKHANGEL'SKIĬ in [1985]. If the answer to Problem 37 is positive (at least in the case of  $t$ -embeddings) then there exists a topologically homogeneous compact space  $X$  such that the cellularity  $c(X)$  of  $X$  is as large as we want—in particular we can choose such  $X$  so that  $c(X) > 2^{\aleph_0}$ . This would answer a well-known question of E. K. van Douwen.

? 1063. **Problem 38.** Let  $S$  be the convergent sequence together with its limit point, i.e.,  $S = \{0, \frac{1}{n}: n \in \mathbb{N}^+\}$ . Is it true that for every compact metrizable space  $X$  the spaces  $X$  and  $X \times S$  are  $l$ -equivalent? or  $t$ -equivalent?

The answer is “yes” for all infinite compact polyhedra.

? 1064. **Problem 39.** Let  $X$  be a compact space such that  $C_p(C_p(X)) (= C_{p,2}(X))$  is Lindelöf. Is it true that  $C_p(X)$  is Lindelöf?

More problems on  $C_p(X)$  are formulated in the surveys ARKHANGEL'SKIĬ [1988b, 1987] and in the book ARKHANGEL'SKIĬ [1989b].

I think that most of the problems formulated in this article are interesting and difficult enough—I venture to speculate that at least half of them will remain unsolved in 1996 and at that at least five of them will remain open at the beginning of the third millennium.

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## CHAPTER 32

### **Problems in Topology arising from Analysis**

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## 1. Topologically Equivalent Measures on the Cantor Space

Two measures  $\mu$  and  $\nu$  defined on the family of all Borel subsets of a topological space  $X$  are said to be *homeomorphic* or *topologically equivalent* provided there exists a homeomorphism  $h: X \rightarrow X$  such that  $\mu = \nu h^{-1}$ . This means that for each Borel set  $E$ ,  $\mu(E) = \nu(h^{-1}(E))$ . The measure  $\mu$  is said to be a *continuous image* of  $\nu$  if  $h$  is only required to be continuous. OXToby and ULAM [1941] characterized those probability measures,  $\mu$ , on the finite dimensional cubes  $[0, 1]^n$ , which are homeomorphic to Lebesgue measure— $\mu$  must give each point measure zero, each nonempty open set positive measure and the boundary of the cube must have  $\mu$  measure zero. Later OXToby and PRASAD [1978] extended this theorem to the Hilbert cube. The situation regarding the Cantor set remains unsolved—even for product measures.

Let  $X = \{0, 1\}^{\mathbb{N}}$  and for each  $r$ ,  $0 \leq r \leq 1$ , let  $\mu(r)$  be the infinite product probability measure on  $X$  determined by  $r$ :  $\mu(r) = \prod_{n=1}^{\infty} \mu_n$ , where  $\mu_n(0) = 1 - r$  and  $\mu_n(1) = r$ , for all  $n$ . For each  $r$ , let  $E(r) = \{s: \mu(r) \text{ is homeomorphic to } \mu(s)\}$ .

First, let us note when one of these product measures is a continuous image of another.

**1.1. THEOREM.** *The measure  $\mu(r)$  is a continuous image of  $\mu(s)$  if and only if there is positive integer  $n$  and integers  $a_i$ ,  $0 \leq i \leq n$ , such that*

$$0 \leq a_i \leq \binom{n}{i}, \quad (1)$$

and

$$r = \sum_{i=0}^n a_i s^i (1-s)^{n-i}. \quad (2)$$

PROOF. Suppose  $f: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$  is continuous and for each Borel set  $E$ ,

$$\mu(r)(E) = \mu(s)(f^{-1}(E)). \quad (3)$$

Let  $E = \langle 1 \rangle$ . Then  $f^{-1}(E)$  is a clopen subset of  $\{0, 1\}^{\mathbb{N}}$ . Therefore, there is a positive integer  $n$  and a subset  $\mathcal{E}$  of  $\{0, 1\}^n$  such that

$$f^{-1}(\langle 1 \rangle) = \bigcup \{ \langle e \rangle : e \in \mathcal{E} \}. \quad (4)$$

For each  $i$ ,  $0 \leq i \leq n$ , let  $a_i$  be the number of sequences  $e = (q_1, \dots, q_n)$  of  $\mathcal{E}$  such that  $\#(e) = \sum_{p=1}^n q_p = i$ . Thus,  $0 \leq a_i \leq \binom{n}{i}$  and if  $\#(e) = i$ , then  $\mu(s)(\langle e \rangle) = s^i (1-s)^{n-i}$ . Thus

$$r = \mu(r)(\langle 1 \rangle) = \sum_{i=0}^n a_i s^i (1-s)^{n-i}. \quad (5)$$

Conversely, let us assume that (1) and (2) hold. Let  $\mathcal{E}$  be a subset of  $\{0, 1\}^n$  such that  $\mathcal{E}$  has exactly  $a_i$  members  $e$  with  $\#(e) = i$ . Notice that if  $\sigma \in \{0, 1\}^{\mathbb{N}}$ , then  $\sigma$  has a unique representation as

$$\sigma = t_1 * t_2 * t_2 * t_3 \cdots, \quad (6)$$

where for each  $i$ ,  $t_i$  is in  $\{0, 1\}^n$ . Define  $f: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$  by  $f(\sigma)(i) = 1$ , if and only if  $t_i \in \mathcal{E}$ . Clearly,  $f$  is a continuous map of  $\{0, 1\}^{\mathbb{N}}$  into  $\{0, 1\}^{\mathbb{N}}$  and for all  $k$ ,

$$\begin{aligned} 1 - r &= \mu(r)(\{\sigma: \sigma(k) = 0\}) = \sum_{i=0}^n \left[ \binom{n}{i} - a_i \right] s^i (1-s)^{n-i} \\ &= \mu(s)(f^{-1}(\{\sigma: \sigma(k) = 0\})). \end{aligned} \quad (7)$$

From this it follows that  $\mu(r)$  is the image of  $\mu(s)$  under  $f$ .  $\square$

**1.2. EXAMPLE.**  $\mu(1/2)$  is the image of  $\mu(1/\sqrt{2})$ .

Let us note that there are many maps which take  $\mu(s)$  to  $\mu(r)$ . For if  $f$  is such map, then since  $\mu(s) = \mu(s) \circ h$ , where  $h$  is a homeomorphism induced by a permutation,  $\mu(r) = \mu(s) \circ f \circ h$ . Theorem 1.1 characterizes those shift invariant product measures  $\mu(s)$  and  $\mu(r)$  such that each is a continuous image of the other.

**1.3. THEOREM.** *Each of  $\mu(r)$  and  $\mu(s)$  is the continuous image of the other if and only if there are positive integers  $n$  and  $m$ , integers  $a_i$ ,  $0 \leq i \leq n$ , integers  $b_j$ ,  $0 \leq j \leq m$  such that*

$$0 \leq a_i \leq \binom{n}{i}, 0 \leq b_j \leq \binom{m}{j}, \quad (8)$$

$$r = \sum_{i=0}^n a_i s^i (1-s)^{n-i}, \quad (9)$$

and

$$s = \sum_{j=0}^m b_j r^j (1-r)^{m-j}. \quad (10)$$

$\square$

**1065. Problem 1.4.** *Is it true that  $\mu(r)$  and  $\mu(s)$  are homeomorphic if and only if equations (8), (9) and (10) hold?*

Let us note that for integers  $a_i$  and  $b_j$  satisfying the given constraints, there is always a solution of equations (9) and (10). This may be seen by applying



Brouwer's fixed point theorem to the map given by:

$$(r, s) \rightarrow \left( \sum_{i=0}^n a_i s^i (1-s)^{n-i}, \sum_{j=0}^m r^j (1-r)^{m-j} \right).$$

A number of references can be drawn from Theorem 1.1. For each  $r$ , let  $F(r) = \{s: \text{each of } \mu(r) \text{ and } \mu(s) \text{ is a continuous image of the other}\}$ . NAVARRO-BERMUDEZ [1979, 1984] showed:

**1.5. THEOREM.** *For each  $r$ ,  $F(r)$  is countable and  $F(r) \supseteq E(r)$ . If  $r$  is rational or transcendental, then  $E(r) = F(r)$  and consists of exactly its obvious members:  $E(r) = \{r, 1-r\}$ .  $\square$*

Huang extended this theorem by proving the same result in case  $r$  is an algebraic integer of degree two. The situation is more complicated for the other algebraic numbers. For example, HUANG [1986] proved:

**1.6. THEOREM.** *For each  $n > 2$ , there is an algebraic integer  $r \in (0, 1)$  of degree  $n$  and a number  $s \in (0, 1)$  such that  $r$  and  $s$  satisfy relations of the form (9) and (10) and  $s \neq r$  and  $s \neq 1-r$ .*

Let us examine Huang's algebraic integer of degree three. It is the unique real solution of

$$r^3 + r^2 - 1 = 0 \tag{A}$$

(It is perhaps worth noting that  $1/r$  is the smallest Pisot-Vijayaraghavan number.) Now, set

$$s = r^2. \tag{B}$$

Clearly,  $s \neq r$  and  $s \neq 1-r$ . OXToby and NAVARRO-BERMUDEZ [1988] showed that for this  $r$  and  $s$ , the measures  $\mu(r)$ ,  $\mu(1-r)$ ,  $\mu(s)$ , and  $\mu(1-s)$  are topologically equivalent.

**Problem 1.7.** *Let  $r$  be the root of eq. (A) between 0 and 1. Does  $E(r)$  or  $F(r)$  consist of exactly the four numbers  $r$ ,  $1-r$ ,  $r^2$  and  $1-r^2$ ? **1066. ?***

**Problem 1.8.** *For each  $r$ , is it true that there are only finitely many numbers  $s$  such that  $\mu(s)$  and  $\mu(r)$  are homeomorphic? **1067. ?***

## 2. Two-Point Sets

MAZURKIEWICZ [1914] showed that there is a "two-point" subset  $M$  of  $\mathbb{R}^2$ , i.e.,  $M$  meets each line in exactly 2 points. Direct generalizations of this result were given by ERDÖS and BAGEMIHL [1957]. The axiom of choice plays a central role in the construction of  $M$ . The problem naturally arises as to how effective such a construction can be.

? **1068. Problem 2.1.** *Is there a Borel set  $M$  in  $\mathbb{R}^2$  which meets each straight line in exactly two points? Can  $M$  be a  $G_\delta$  set?*

LARMAN [1968] has shown that  $M$  cannot be an  $F_\sigma$  set. But, even whether  $M$  can be a  $G_\delta$  set is unknown. It is known that if  $M$  is an analytic set then  $M$  is a Borel set. This follows for example from the fact that every analytic subset  $A$  of  $\mathbb{R}^2$  such that each vertical fiber  $A_x$  has cardinality  $\leq 2$  lies in a Borel set  $B$  such that each vertical fiber has cardinality  $\leq 2$ . MILLER has shown that  $V = L$  implies that  $M$  can be taken to be a coanalytic set (MILLER [1989]).

I have proven the following.

**2.2. THEOREM.** *A two point set  $M$  must always be totally disconnected, i.e., every connected subset of  $M$  consists of a single point.*  $\square$

Larman's Theorem follows from this since each  $\sigma$ -compact subset of  $\mathbb{R}^2$  which meets each vertical line in two points contains the graph of a continuous function defined on some interval.

? **1069. Problem 2.3.** *Must a two-point set  $M$  always be zero-dimensional?*

Note that if  $E$  is a subset of the plane which meets each line in  $2^\omega$  points then there is a two-point set  $M$  lying in  $E$ . Since there is such a subset  $E$  of the plane which is both zero-dimensional and of planar Lebesgue measure 0,  $M$  can be both zero-dimensional and of Lebesgue measure 0. On the other hand, one can construct  $M$  such that  $M$  meets each closed subset of  $\mathbb{R}^2$  which has positive Lebesgue measure. Thus,  $M$  can also be taken to be non-Lebesgue measurable. It should be noted that the property of being a partial two-point set cannot necessarily be extended. For example, the unit circle meets each line in no more than two points but of course we cannot even add a single point to this set and retain this property.

? **1070. Problem 2.4.** *Can a zero-dimensional partial two-point set always be extended to a two-point set?*

(van Mill and I note that this is true assuming **CH** holds).

### 3. Pisot-Vijayaraghavan Numbers

Let  $S$  be the set of all Pisot-Vijayaraghavan numbers. Thus,  $x \in S$  if and only if  $x$  is an algebraic number,  $x > 1$  and all its conjugates have moduli less than 1. SALEM [1983] proved that the countable set  $S$  is also a closed subset of  $\mathbb{R}$ . SIEGEL [1944] showed that the smallest element of  $S$  is the root of  $x^3 - x^2 - 1$ . PISOT and DUFRENOY [1953] showed that the smallest number in the Cantor-Bendixson derived set of  $S$  is the root of  $x^2 - x - 1$ .

**Problem 3.1.** *What is the order type of the set  $S$  of all Pisot-Vijayaraghavan numbers?* **1071. ?**

**Problem 3.2.** *What is the Cantor-Bendixson derived set order of  $S$ ?* **1072. ?**

#### 4. Finite Shift Maximal Sequences Arising in Dynamical Systems

A particular countable linear order type arises in one-dimensional dynamics. A simple case occurs in the iteration of the critical point in a scaled family of unimodal maps of the unit interval one-dimensional dynamics. For example, consider the quadratic map  $q(x) = 4x(1 - x)$  on the unit interval,  $[0, 1]$ . For each  $\lambda$ ,  $0 \leq \lambda \leq 1$ , consider the itinerary,  $I_{\lambda q}(1/2)$ , of the critical point of the scaled map,  $\lambda q$ . Thus

$$I_{\lambda q}(1/2)(i) = \begin{cases} R, & \text{if } (\lambda q)^i(1/2) > 1/2, \\ C, & \text{if } (\lambda q)^i(1/2) = 1/2, \\ L, & \text{if } (\lambda q)^i(1/2) < 1/2. \end{cases}$$

We make the convention that the sequence stops at the first  $C$  if there is a  $C$  in the sequence. Thus, a finite itinerary arises from a value of  $\lambda$  such that  $1/2$  is periodic under  $\lambda q$ . The set of all possible itineraries has been abstractly characterized as follows. First, consider the parity-lexicographic order on the space  $S$  of all finite and infinite sequences of  $R$ ,  $L$  and  $C$  such that if the sequence has a  $C$  there is only one  $C$  and it is the last term of the sequence. Thus, if  $A = (A_1, A_2, \dots)$  and  $B = (B_1, B_2, \dots)$  are elements of  $S$ , then  $A \leq B$  provided  $A_i < B_i$ , where  $i$  is the first place where  $A$  and  $B$  disagree and we use the order  $L < C < R$  if there are an even number of  $R$ 's preceding  $A_i$  in  $A$  and we use the reverse order if there are an odd number. An element  $A$  of  $S$  is said to be *shift maximal* provided  $A$  is not less than any of its shifts,  $\sigma^i(A) = (A_{i+1}, A_{i+2}, \dots)$  in the parity-lexicographic order.

**4.1. THEOREM.** *An element  $A = (A_1, A_2, A_3, \dots)$  is the itinerary of  $1/2$  under the quadratic map,  $q$ , for some value of  $\lambda$  if and only if  $A$  is shift maximal.  $\square$*

This theorem is true not only for the quadratic map but for a general wide class of maps of  $[0, 1]$  onto  $[0, 1]$  (See COLLET and ECKMAN [1980] and BEYER, MAULDIN and STEIN [1986].)

**Problem 4.2.** *What is the order type of the countable set of finite shift-maximal sequences in the parity-lexicographic order?* **1073. ?**

#### 5. Borel Selectors and Matchings

Consider the hyperspace of all compact subsets of the unit interval,  $\mathcal{K}(I)$ . There are exactly 2 continuous selectors. If  $f: \mathcal{K}(I) \rightarrow I$  is continuous and

for each compact set  $K$ ,  $f(K) \in K$ , then either  $f(K) = \max(K)$  for all  $K$  or else  $f(K) = \min(K)$  for all  $K$ . In MAULDIN [1980], I showed that there are  $\omega_1$  Borel measurable selectors  $f_\alpha: \mathcal{K}(I) \rightarrow I$  such that if  $K$  is an uncountable compact set, then the values  $f_\alpha(K)$  are distinct.

? **1074. Problem 5.1.** *Can one prove in **ZFC** that there are continuum many Borel measurable selectors on  $\mathcal{K}(I)$  such that for each uncountable compact set  $K$ , the selected points of  $K$  are all distinct?*

There does exist such a family of Borel selectors if instead of the uncountable compact sets, one considers the family of compact perfect sets (MAULDIN [1979]).

? **1075. Problem 5.2.** *Let  $B$  be a Borel subset of  $[0, 1] \times [0, 1]$  such that each horizontal and each vertical fiber of  $B$  is co-meager. Can  $B$  be filled up by a collection of pairwise disjoint graphs of Borel isomorphisms of  $[0, 1]$  onto  $[0, 1]$ ?*

DEBS and SAINT-RAYMOND [1989] have shown that  $B$  does contain a Borel matching—the graph of some Borel isomorphism. This result is false if co-meager is replaced by Lebesgue measure one. An example of such a set is given in GRAF and MAULDIN [1985] and in more detail in MAULDIN and SCHLEE [1989]. More problems on this theme are given in MAULDIN [1989].

## 6. Dynamical Systems on $S^1 \times \mathbb{R}$ —Invariant Continua

Fix  $a > 0$  and  $B > 0$  and define a map  $T: S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$  by

$$T(e^{i2\pi x}, y) = (e^{i2\pi ax}, B(y - A(x))).$$

In order for the map to be well-defined and continuous, we assume  $A: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, has period 1 and that  $a$  is a positive integer. For convenience, we assume  $\|A\| = 1$ . Note that  $T$  maps the fiber  $\{e^{i2\pi x}\} \times \mathbb{R}$  one-to-one and onto  $\{e^{i2\pi ax}\} \times \mathbb{R}$ . Also,  $T$  restricted to the fiber is an orientation preserving similarity map with similarity ratio

$$B: \|T(e^{2\pi ix}, y) - T(e^{2\pi ix}, z)\| = B|y - z|.$$

This map or close relatives have been studied by KAPLAN, MALLET-PARET and YORKE [1984], MOSER [1969] and FREDRICKSON ET AL [1983]. In order to examine the dynamics of  $T$ , note that

$$T^n(e^{i2\pi x}, y) = (e^{2\pi i a^n x}, B^n y - \sum_{p=0}^{n-1} B^{n-p} A(a^p x)).$$

If  $a = 1$ , then the dynamics are quite simple. If  $B = 1$ , then  $T^n(e^{2\pi ix}, y) = (e^{2\pi ix}, y - nA(x))$  and the asymptotic behaviour is clear. If  $B \neq 1$ , then the graph  $\mathcal{G}$ , of

$$f(x) = \left(\frac{B}{B-1}\right)A(x)$$

lifted to the cylinder is invariant. If  $0 < B < 1$ , this graph is a universal attractor. In fact, for each  $x$  and  $y$ ,  $T^n(e^{2\pi ix}, y) \rightarrow (e^{2\pi ix}, \mathcal{G}(x))$ . If  $B > 1$ , this graph is a repeller. The points of the cylinder above the graph iterate to  $+\infty$  and those below iterate to  $-\infty$ .

From this point on, we assume  $a \geq 2$ . Now the map  $T$  is  $a$ -to-1:

$$T^{-1}(e^{2\pi ix}, y) = \left\{ (e^{2\pi i((x+k)/a)}, B^{-1}y + A((x+k)/a)) : k = 0, \dots, a-1 \right\}.$$

If  $B > 1$ , then the graph of the continuous, period 1 function  $f$  which satisfies the functional equation:

$$f(ax) = B(f(x) - A(x))$$

is invariant. Or, setting  $b = 1/B$ ,

$$f(x) = A(x) + bf(x).$$

The unique solution of this equation is the Weierstrass function:

$$f(x) = \sum_{p=0}^{\infty} b^p A(a^p x).$$

The graph of  $f$  on the cylinder is a nowhere differentiable invariant 1-torus. It is also a universal repeller. The points of the cylinder above the graph iterate to  $+\infty$  and those below iterate to  $-\infty$ . The capacity dimension of this graph is  $2 + \log b / \log a$ , in some cases (KAPLAN, MALLET-PARET and YORKE [1984]). The Hausdorff dimension of this set is a long standing unsolved problem. It is widely believed that the capacity dimension is the Hausdorff dimension. The best estimates in the general case are given in MAULDIN and WILLIAMS [1986].

**Problem 6.1.** Find the Hausdorff dimension,  $\gamma$ , of this graph. Moreover, find **1076. ?** the exact Hausdorff dimension function—if there is one. In other words, find a slowly varying function  $L(t)$  such that  $0 < \mathcal{H}^h(f) < \infty$ , where  $h(t) = t^\gamma L(t)$ .

If  $0 < B < 1$ , then  $T$  has an attracting continuum  $M$ . This is seen by noticing that if  $|y| \leq \frac{B}{1-B}$ , then

$$|B(y - A(x))| \leq B(|y| + |A(x)|) \leq B\left(\frac{B}{1-B} + 1\right) = \frac{B}{1-B}.$$

Thus the “can”,

$$K = S^1 \times \left[ \frac{-B}{1-B}, \frac{B}{1-B} \right],$$

is mapped into itself,  $T(K) \subseteq K$ . Set

$$M = \bigcap_{n=0}^{\infty} T^n(K).$$

Then  $M$  is an invariant continuum which separates  $S^1 \times \mathbb{R}$  and  $M$  attracts the orbit of all points. Pat Carter and I have shown that  $T$  acts chaotically on the continuum  $M$ . The case  $0 < B < 1$  is very different from the case  $1 < B$ , in fact I conjecture:

**? 1077. Problem 6.2.** *Is it true that  $M$  is a Sierpiński curve? In particular, is this true if  $A$  is the tent map on  $[0, 1]$ ?*

Let us remark that in general  $M$  is not a graph in this case. Let us assume  $M$  is the graph of a function from  $S^1$  into  $\mathbb{R}$ . Since the graph is compact, there is a continuous period one map  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $M$  is the graph of the lift of  $f$  to the cylinder. Since

$$T(e^{i2\pi x}, f(x)) = (e^{i2\pi ax}, B(f(x) - A(x))),$$

the function  $f$  must satisfy the functional equation

$$f(ax) = B(f(x) - A(x)),$$

for all  $x$ . Or,

$$f(x) = A(x) + \frac{1}{B}f(ax).$$

However, Pat Carter and I have shown that for some functions, the unique solution of this equation which is continuous at zero does not have period one. This class includes the case when  $A$  is nonnegative. In particular, if  $A$  is the tent map,  $M$  is not a graph.

**? 1078. Problem 6.3.** *Let  $A$  be a non-constant, continuous, period one map of  $\mathbb{R}$  into  $\mathbb{R}$  with  $\|A\| = 1$ ,  $a$  is an integer,  $a \geq 2$  and  $0 < B < 1$ . Is it true that the unique continuous solution of*

$$f(x) = A(x) + \frac{1}{B}f(ax)$$

*does not have period one, or more generally, is not periodic?*

## 7. Borel Cross-Sections

Let  $X$  be an indecomposable continuum and consider the decomposition of  $X$  into its composants and let  $R$  be the corresponding equivalence relation:  $R$  is a Borel subset of  $X \times X$  and each equivalence class is a meager, dense,  $F_\sigma$  subset of  $X$ . I have raised the following question over the past fifteen years, but it probably has been known much earlier.

**Problem 7.1.** *Is there a Borel subset  $B$  of  $X$  which meets each equivalence class in exactly one point?* **1079. ?**

While this question remains unsolved, there is one case for which the answer is no. The continuum  $X$  is said to be *strictly transitive in the sense of category* provided that for each subset  $E$  of  $X$  which has the Baire property and which can be expressed as the union of some composants either  $E$  or  $X \setminus E$  is meager (KURATOWSKI [1968]).

**7.2. THEOREM.** *Let  $X$  be an indecomposable continuum which is strictly transitive in the sense of category. There is no Borel cross-section for the composants of  $X$ .*

PROOF. Assume that there is a Borel cross-section  $B$ . For each subset  $E$  of  $X$ , let  $\text{sat}(E)$  be the union of all composants which meet  $E$ . Notice that if  $E$  is a Borel set, then  $\text{sat}(E)$  of  $E$  is analytic, since  $\text{sat}(E) = \text{proj}_2(R \cap (E \times X))$  and, therefore,  $\text{sat}(E)$  has the Baire property. Define a probability measure,  $\mu$ , on the Borel subsets of  $B$  as follows:  $\mu(E) = 1$ , if  $\text{sat}(E)$  is co-meager, and  $\mu(E) = 0$ , otherwise. Then  $\mu$  gives each singleton measure 0, and each Borel subset of  $B$  has measure 0 or 1. This is impossible.  $\square$

There are a number of indecomposable continua which are strictly transitive: Knaster continua (KURATOWSKI [1968]) and those admitting a Polish group action for which the orbit decomposition consists of the composants (ROGERS [1986]).

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# PART VIII

## DYNAMICS

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CHAPTER 33

**Continuum Theory and Topological Dynamics**

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One can argue quite convincingly that continuum theory first arose from problems in dynamics even before there was a definition of a topological space. It is rather difficult (and, we suppose, pointless) to give a definition of a dynamical system that is general enough to cover all of the situations in which the term is used. There is a common underlying goal in the study of dynamical systems though, and that is to gain an understanding in some qualitative sense (topological or statistical) of the orbit structure of iterative processes. Here we will restrict our consideration to dynamical systems consisting of self maps of metric spaces. We make no claim to being encyclopedic, even with this restriction. Our intention is to show the reader that strange topology has come up in dynamics from the beginning, to impart some of the flavor of the field of dynamical systems, and to introduce the reader both to some of the applications of continuum theory to the study of dynamical systems and to the problems coming into continuum theory from dynamical systems. We begin with some definitions and notation.

Let  $X$  be a metric space and let  $f: X \rightarrow X$  be a continuous map. Given  $x$  in  $X$ , the *orbit* of  $x$  is the set  $O(x) = \{f^n(x) : n \geq 0\}$ , where by  $f^n(x)$  we mean  $f(f^{n-1}(x))$  for  $n \geq 1$  and  $f^0(x) = x$ . The point  $x$  is a *fixed point* of  $f$  if  $f(x) = x$ . More generally,  $x$  is a *periodic point* if  $f^n(x) = x$  for some  $n \geq 1$ ; in this case  $O(x)$  is finite and the *period* of  $x$  (or of  $O(x)$ ) is the cardinality of  $O(x)$ . If  $x$  is in  $X$  and there is an open set  $O$  in  $X$  containing  $x$  such that  $\{f^{-n}(O) : n \geq 0\}$  is a disjoint collection, then  $x$  is a *wandering point* for  $f$ . If there is a point  $x$  in  $X$  such that  $O(x)$  is dense in  $X$ , then  $f$  is said to be *transitive*. A closed set  $A$  in  $X$  is an *attracting set* for  $f$ , or an *attractor*, if there is an open set  $U$  containing  $A$  such that  $\bigcap_{n=0}^{\infty} f^n(U) = A$ , and  $f(U) \subseteq U$ . (Please note that this is not the only definition of attracting set being used.) A closed set  $D$  of the space  $X$  is *minimal* for  $f$  if  $f(D) = D$  and for  $x \in D$ ,  $\{f^n(x) : n \geq 0\}$  is dense in  $D$ . If  $D = X$ , then  $f$  is *minimal*.

A continuum  $X$  is *indecomposable* if every proper subcontinuum of  $X$  is nowhere dense in  $X$ . If it is not indecomposable, then it is *decomposable*. A continuum is *hereditarily indecomposable* (*decomposable*) if every nondegenerate subcontinuum is indecomposable (decomposable). An *open chain*  $C$  in the space  $X$  is a finite collection  $C = \{C_0, C_1, \dots, C_n\}$  of open sets such that  $C_i \cap C_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ . An *open circular chain*  $C$  is a finite collection  $C = \{C_0, \dots, C_n\}$  of open sets such that  $C_i \cap C_j \neq \emptyset$  if and only if  $|i - j| \leq 1$  or  $i = 0, j = n$ . A continuum is *chainable* (*circularly chainable*) if for each  $\epsilon > 0$ , it has an open chain (circular chain) cover of mesh less than  $\epsilon$ .

In 1913, C. Carathéodory, needing to deal with the fact that although an open, connected, simply connected subset  $U$  of  $S^2$  is homeomorphic to the interior of a disk, its boundary need not be a simple closed curve, or even anything close to a simple closed curve, developed in a pair of papers his theory of prime ends. (For a modern, topological treatment and the original references, see MATHER [1982].) Central to Carathéodory's theory is the

following theorem:

**1. THEOREM (Carathéodory).** *Let  $W$  be a connected, simply connected open set in  $S^2$  with  $\overline{W} - W$  containing more than one point. There is a compactification  $W^*$  of  $W$  such that  $W^*$  is homeomorphic to a closed disk  $D$  where points in  $W$  are associated with points in the interior of  $D$ . ( $W^* - W$  is the collection of Carathéodory's prime ends.) Further, if  $F: S^2 \rightarrow S^2$  is a homeomorphism with  $F(W) = W$ , then  $F|_W$  extends to a map  $F^*: W^* \rightarrow W^*$ .*

What exactly does this do for us? It means that one way we can study the dynamics of  $F$  is by studying the corresponding dynamics on  $S^1$  and  $D$ , which have been extensively studied. For what Carathéodory has found is a *conjugacy*: there is a homeomorphism  $\beta$  from  $D$  to  $W^*$  and if  $\hat{F} = \beta^{-1}F^*\beta$ , then  $\beta\hat{F} = F^*\beta$ . The maps  $\hat{F}$  and  $F^*$  will have the same dynamical properties. Associated with the prime ends are continua in  $\overline{W} - W$  which Carathéodory calls impressions and principal sets. (If the map  $\beta$  above were only a continuous surjection (but still  $\beta\hat{F} = F^*\beta$ ), then we would say that  $\hat{F}$  and  $F^*$  were *semi-conjugate*. In this case  $\hat{F}$  and  $F^*$  share some, but not all, dynamical properties.)

An important concept due to Poincaré is that of rotation number. If  $g$  is an orientation preserving homeomorphism of  $S^1$ ,  $\pi$  is the standard covering map from  $\mathbb{R}$  to  $S^1$  (i.e.,  $\pi(x) = \exp(2\pi ix)$ ) and  $G: \mathbb{R} \rightarrow \mathbb{R}$  is a map such that  $\pi G = g\pi$ , define  $p_G(x) = \lim_{n \rightarrow \infty} G^n(x)/n$  for  $x$  in  $S^1$ . The *rotation number*  $r(g)$  is then the unique number  $r$  in  $[0, 1)$  such that  $p_G(x) - r$  is an integer. This number is independent of the choice of  $x$  and  $G$ . Loosely, what this number measures is the average rotation under iteration of  $g$ , of a point on the circle. A homeomorphism  $g$  of  $S^1$  has a rational rotation number if and only if it has periodic points. It has a fixed point if and only if it has rotation number 0.

By means of prime end theory, one can talk about the rotation numbers of those points of  $\overline{W} - W$  accessible from  $W$  (i.e., those points  $p$  in  $\overline{W} - W$  such that there is an arc  $A$  in  $\overline{W}$  such that  $A \cap (\overline{W} - W) = \{p\}$ .)

In [1932], G. D. BIRKHOFF used the notion of rotation number of accessible points to study the dynamics of an annulus map having an unusual invariant set  $G$ . This set  $G$  is the boundary set for an open set  $G_1$ , which contains one boundary circle, and for another disjoint open set  $G_2$ , which contains the other boundary circle. The set  $G$  has the property that it contains a dense set of points accessible from  $G_1$  with one rotation number, and another dense set of points accessible from  $G_2$  with, surprisingly, a different rotation number. CHARPENTIER, in [1934], later proved that this continuum is indecomposable. (Our source for this information and some of what follows is ALLIGOOD and YORKE [1989].)

CARTWRIGHT and LITTLEWOOD in [1945] and [1951] further developed the study of the relationship between the dynamics of prime ends and the dynamics of the boundary of an invariant region in the course of studying second

order differential equations in the plane. In their investigations, they found that, at certain parameter values, an associated Poincaré homeomorphism admits a certain invariant plane separating continuum and they conjectured that this continuum contains an indecomposable continuum. It has recently been proven that this continuum of Cartwright and Littlewood *is* indecomposable (see BARGE and GILLETTE [1988]). Also, J. Mather has used prime ends to study the invariant sets of area-preserving homeomorphism of the annulus in MATHER [1979] and [1981], and K. Alligood and J. Yorke have used them to investigate the dynamics of accessible points on basin boundaries in ALLIGOOD and YORKE [1989].

In early “dynamical systems” as practiced by Poincaré, the system was typically the solution to a differential equation modeling some physical process. Periodic orbits correspond to periodic physical phenomena and it was just such phenomena that were most commonly observed in nature. This at least partially explains why much of the theory at present has to do with periodic points and their distribution.

A beautiful theorem regarding periodic points due to A. N. Sarkovskii is the following. Let the integers be ordered by  $3 \triangleleft 5 \triangleleft 7 \triangleleft 9 \triangleleft \dots \triangleleft 2(3) \triangleleft 2(5) \triangleleft 2(7) \triangleleft \dots \triangleleft 2^2(3) \triangleleft 2^2(5) \triangleleft \dots \triangleleft 2^{n+1} \triangleleft 2^n \triangleleft \dots \triangleleft 2 \triangleleft 1$ .

**2. THEOREM** (Sarkovskii’s Theorem, SARKOVSKIĬ [1964]). *If  $f: I \rightarrow I$  is a continuous map of the compact interval  $I$  and  $f$  has a periodic point of period  $n$ , then  $f$  has a periodic point of period  $m$  for all  $m$  such that  $n \triangleleft m$ .*

Spaces other than  $I$  on which this theorem remains valid have been found (e.g., hereditarily decomposable chainable continua (MINC and TRANSUE [1989a]), and certain ordered spaces (SCHIRMER [1985]), and modifications of the theorem work on the circle (BLOCK, GUCKENHEIMER, MISIUREWICZ and YOUNG [1979]) and on certain trees (ALSEDA, LLIBRE and MISIUREWICZ [1989], BALDWIN [1988], and IMRICH and KALINOWSKI [1985]). A nice proof of the Sarkovskii Theorem can be found in BLOCK, GUCKENHEIMER, MISIUREWICZ and YOUNG [1979].

One sees from Sarkovskii’s Theorem that if a map of a compact interval has a periodic orbit of period not a power of two then the map has infinitely many periodic orbits of infinitely many different periods. Such a map also has orbits exhibiting various types of complicated behavior. (See LI and YORKE [1975].)

Maps with the above dynamical properties are sometimes referred to as being chaotic. In physical applications, the space  $X$  may be a collection of possible states of some system and the map  $f: X \rightarrow X$  the law by which states evolve. Chaotic dynamical properties of  $f$  then correspond to complicated and computationally unpredictable evolution in the physical system.

The most widely used measure of complexity of a dynamical system is topological entropy. There are a number of equivalent definitions of topological entropy, one of which we give now. Let  $f: X \rightarrow X$  be a map of the compact

topological space  $X$  with metric  $d$ . Let  $\epsilon > 0$  and  $n$  in  $\mathbf{N}$  be given. The set  $E \subseteq X$  is said to be  $(n, \epsilon)$ -separated (under  $f$ ) provided that for each  $x, y$  in  $E$ ,  $x \neq y$ , there is a  $k$  in  $\{0, 1, \dots, n-1\}$  such that  $d(f^k(x), f^k(y)) \geq \epsilon$ . Let  $S(n, \epsilon, f) = \max\{|E| : E \text{ is } (n, \epsilon)\text{-separated}\}$ . Thus  $S(n, \epsilon, f)$  is the greatest number of orbit segments  $\{x, f(x), \dots, f^{n-1}(x)\}$  of length  $n$  that can be distinguished one from another provided we can only distinguish between points of  $X$  that are at least  $\epsilon$  apart. Now let  $h(f, \epsilon) = \limsup_{n \rightarrow \infty} \ln S(n, \epsilon, f)/n$  and let  $h(f) = \lim_{\epsilon \rightarrow 0} h(f, \epsilon)$ . The number  $h(f)$  is called the *topological entropy* of  $f$ . It is an easy exercise to show that  $h = h(f)$  is independent of the (equivalent) metric used and that  $h$  is an invariant of topological conjugacy.

If  $h(f) > 0$  then, for some  $\epsilon > 0$ , the number  $S(n, \epsilon, f)$  of distinguishable orbit segments of length  $n$  grows exponentially with  $n$ . This behavior is consistent with complicated dynamics and, in fact, is sometimes given as the definition of chaotic dynamics. For maps  $f: I \rightarrow I$  of the compact interval  $I$  there is the pleasing (in view of Sarkovskii's Theorem) result:  $h(f) > 0$  if and only if  $f$  has a periodic orbit of period not a power of 2 (see BOWEN and FRANKS [1976] and MISIUREWICZ [1979]).

Although a connection between periodicity and entropy persists for maps of continua other than the interval, the correspondence is generally less precise. For example, in [1980] KATOK has proven that if  $f: M^2 \rightarrow M^2$  is a  $C^{1+\alpha}$  diffeomorphism of the compact surface  $M^2$  and  $h(f) > 0$ , then some power of  $f$  has periodic orbits of all periods. On the other hand, REES [1981] has constructed an example of a positive entropy homeomorphism of the two-dimensional torus that is minimal. (Every orbit is dense so, in particular, there are no periodic orbits.)

As topologically simple as the compact interval  $I$  is, it is remarkable that maps of  $I$  can have such complicated dynamical properties. Even the much studied quadratic family  $f_\lambda: [0, 1] \rightarrow [0, 1]$  defined by  $f_\lambda(x) = \lambda x(1-x)$ ,  $\lambda$  in  $[0, 4]$ , is not completely understood. (See MAY [1976] and COLLET and ECKMANN [1980] for an introduction to this family.) Homeomorphisms of  $I$ , on the other hand, are dynamically quite trivial. (For the interested reader, R. Devaney's book, *An Introduction to Chaotic Dynamical Systems* (DEVANEY [1976]), contains a good elementary discussion of the dynamics of the interval. Another text we might recommend is P. Walters' *An Introduction to Ergodic Theory* (WALTERS [1982]), the second half of which is about topological dynamics.)

In some sense, possible dynamical properties of homeomorphisms of a space are dictated by the topology. In order to take advantage of this for non-homeomorphisms (endomorphisms) one can pass to inverse limits. By this we mean the following. Let  $f: X \rightarrow X$  be a map of the compact metric space  $X$  and let  $(X, f)$  be the inverse limit space  $(X, f) = \{(x_0, x_1, \dots) : x_i \in X, f(x_{i+1}) = x_i \text{ for } i = 0, 1, \dots\}$ . Then  $(X, f)$  is also a continuum with metric  $d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i$  where  $||$  denotes the

metric in  $X$ . The map  $f$  then induces the shift homeomorphism

$$\hat{f}: (X, f) \rightarrow (X, f) \text{ defined by } \hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots).$$

The dynamical properties of  $f$  and  $\hat{f}$  are nearly identical. For example,  $h(f) = h(\hat{f})$ ,  $n$  is a period of a periodic orbit of  $f$  if and only if  $n$  is a period of a periodic orbit of  $\hat{f}$ , etc. Topologically,  $(X, f)$  may be much more complicated than  $X$ , but at least  $\hat{f}$  is a homeomorphism. Generally speaking, the increased topological complexity of  $(X, f)$  is a reflection of dynamical properties possessed by  $f$  that are not possible for homeomorphisms of  $X$ .

For example, if  $f: I \rightarrow I$  is a map of the compact interval  $I$  possessing a periodic orbit of period not a power of 2, then  $(I, f)$  contains an indecomposable subcontinuum. Conversely, if  $f$  is piecewise monotone and  $(I, f)$  contains an indecomposable subcontinuum then  $f$  has a periodic orbit of period not a power of 2. This and related results can be found in BARGE and MARTIN [1985a, 1987, 1985b], and INGRAM [1987].

Inverse limits have also proven useful in describing attractors in dynamical systems. (For a general discussion of the notion of an attractor see MILNOR [1985].) In [1967], R. F. WILLIAMS shows that if  $A$  is a sufficiently nice one-dimensional attractor of a diffeomorphism  $F$  of the manifold  $M$ , then  $F|A$  is topologically conjugate to the shift map induced on an inverse limit space  $(K, f)$ , where  $f: K \rightarrow K$  is an endomorphism of the branched one-dimensional manifold  $K$ . Conversely, given a differentiable endomorphism  $f$  of a branched one-manifold  $K$ , the inverse limit  $(K, f)$  can be embedded in the 4-sphere  $S^4$  and the shift map extended to a diffeomorphism of  $S^4$  possessing  $(K, f)$  as an attractor. The result is that sufficiently nice, but still very complicated, attractors can be understood, topologically and dynamically, in terms of a one-dimensional map.

In a similar vein, it is not difficult to show (BARGE and MARTIN [1990]) that, given any map  $f: I \rightarrow I$  of the compact interval  $I$ , the inverse limit space  $(I, f)$  can be embedded in the plane  $\mathbb{R}^2$  and the shift map  $\hat{f}$  extended to a homeomorphism  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  in such a way that  $(I, f)$  is an attractor for  $F$ . For some (all?)  $f$  this can be done so that  $F$  is a diffeomorphism (MISIUREWICZ [1985], BARGE [1988]). Deciding which planar attractors or invariant sets can be modeled using inverse limits on relatively simple spaces is a more difficult problem.

In [1982] MICHAEL HANDEL constructed a remarkable example, an area-preserving  $C^\infty$  diffeomorphism  $f$  of the plane with the pseudocircle  $P_C$  as a minimal set. The pseudocircle  $P_C$  can be characterized as a plane separating, hereditarily indecomposable circularly chainable continuum. This extraordinary continuum contains no nontrivial continuous images of arcs, is nearly homogeneous but not homogeneous (i.e., if  $x$  is in  $P_C$ ,  $\{h(x) : h: P_C \rightarrow P_C \text{ is a homeomorphism}\}$  is dense in  $P_C$ , but not equal to  $P_C$ ), and imitates



somewhat a compact abelian topological group with its nice group of “rotations” (KENNEDY and ROGERS [1986]). If the requirement that  $f$  be area preserving is dropped,  $P_C$  can be made an attracting set for  $f$ . There is a well defined irrational rotation number for  $f|_{P_C}$ , but  $f|_{P_C}$  is not semi-conjugate to a rotation on  $S^1$ .

Every nondegenerate proper subcontinuum of a pseudocircle is a continuum known as a pseudoarc. Pseudoarcs can be characterized as chainable hereditarily indecomposable continua. They are homogeneous, have the fixed point property, and don't separate the plane. R. H. Bing showed that if one puts the Hausdorff metric on the collection of all plane continua, then the collection of all pseudoarcs is a dense  $G_\delta$ -subset of this space. GEORGE HENDERSON [1964] has expressed the pseudoarc  $P$  as an inverse limit system on the interval with one bonding map. (His bonding map is surprisingly simple. Its graph looks much like the graph of  $g(x) = x^2$ ,  $x \in [0, 1]$ , with little notches in it.) P. MINC and W. TRANSUE [1989b] have constructed a map  $f$  on  $I$  such that the inverse limit space  $(I, f)$  is a pseudoarc and  $\hat{f}$  is transitive. It then follows from this and the work of BARGE and MARTIN in [1990] that the pseudoarc can be embedded in  $\mathbb{R}^2$  in such a way that the shift map  $\hat{f}$  on  $P$  can be extended to a homeomorphism  $F$  on  $\mathbb{R}^2$  so that  $(I, f) = P$  is a chaotic attractor for  $F$ . Using different techniques, J. KENNEDY, in [1989a, 1989b] and [1990], has constructed a chaotic homeomorphism on the pseudoarc and one with positive entropy.

As we have glimpsed then, the complexity of a dynamical system is reflected in the complexity of its invariant sets, and this is where continuum theory can and has come into the picture. Even in relatively well behaved dynamical systems these invariant sets can be complicated.

At the beginning of the chapter on topological methods in his book on partial differential equations J. SMOLLER [1983] makes the following remarks: “The invention of modern topology goes back to Poincaré, who was led to it in his study of the differential equations of celestial mechanics. Its development was taken over, for quite a while, by people who interestingly enough, seemed to have completely forgotten its origins. Perhaps this really was necessary in order that the subject develop rapidly.” Although topologists have not been looking at these continua from a dynamical perspective, they have been looking at them ever since the early 1920's and the papers and questions of Knaster and Kuratowski in the first volumes of *Fundamenta Mathematica*. A large body of knowledge exists about these continua and many potentially useful tools have been developed. Also, the knowledge gained from considering these dynamical problems will almost surely add to our knowledge of the continua themselves.

We end with a list of some unsolved problems and, for the reader who wishes more details on the topics just discussed, a list of references.

## Unsolved Problems - Continuum Theory and Topological Dynamics

**Problem 1.** Define the Henon Map  $H: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

1080. ?

$$H(x, y) = (y + 1 - ax^2, bx).$$

Are there values of the parameters  $a$  and  $b$  ( $b \neq 0$ ) and a nondegenerate continuum  $\Lambda$  such that  $H(\Lambda) = \Lambda$  and  $H|_\Lambda$  is transitive (HÉNON [1976])?

**Problem 2.** Let  $f: M \rightarrow M$  be a diffeomorphism of the two dimensional manifold  $M$  with  $p \in M$  a hyperbolic fixed saddle ( $Df(p)$  has eigenvalues  $\lambda_1$  and  $\lambda_2$  with  $0 < |\lambda_1| < 1 < |\lambda_2|$ ). Suppose that there is a point  $q$  in the intersection of one branch of the unstable manifold  $W_+^u(p)$  and the stable manifold  $W^s(p)$  and that the intersection of  $W_+^u(p)$  with  $W^s(p)$  at  $q$  is not topologically transverse. Is  $cl(W_+^u(p))$  not chainable? (See BARGE [1987].) 1081. ?

**Problem 3.** Under what conditions (if any) on the continuous map  $f: I \rightarrow I$  of the compact interval  $I$  is there an embedding of the inverse limit space  $(I, f)$  into the plane so that the shift map  $\hat{f}: (I, f) \rightarrow (I, f)$  extends to a diffeomorphism of the plane? 1082. ?

**Problem 4.** If  $M$  is a nonseparating plane continuum and  $f$  is a mapping of  $M$  into  $M$ , does  $f$  have a periodic point? 1083. ?

**Problem 5.** Let  $\{p_1, p_2, \dots, p_n\}$  be a set of  $n \geq 2$  distinct points in the sphere  $S^2$ . Is there a homeomorphism of  $S^2 - \{p_1, p_2, \dots, p_n\}$  such that every orbit of the homeomorphism is dense? 1084. ?

**Problem 6.** Is there a homeomorphism of  $\mathbb{R}^n, n \geq 3$ , such that every orbit of the homeomorphism is dense? 1085. ?

**Problem 7.** The homeomorphism  $f: X \rightarrow X$  of the compact metric space  $(X, d)$  is expansive provided there is a  $\delta > 0$  such that for each  $x, y \in X, x \neq y$ , there is an integer  $n$  for which  $d(f^n(x), f^n(y)) \geq \delta$ . Characterize the planar continua admitting expansive homeomorphisms. 1086. ?

**Problem 8.** Is the Mandelbrot set locally connected?

1087. ?

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## CHAPTER 34

### One-dimensional versus two-dimensional dynamics

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In this note I want to pose some questions about the difference between one-dimensional and two-dimensional dynamical systems. The theory about one-dimensional maps is rather complete. Let us mention some of these results and analyze to what extent they can be generalised to the two-dimensional case.

## 1. The existence of periodic points

One of the best known results on the dynamics of one-dimensional maps is the one by Sarkovskiĭ. Take the following ordering on  $\mathbb{N}$ :

$$\begin{aligned} 3 < 5 < 7 < \dots < 2n+1 < \dots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < \dots \\ < 2^m \cdot 3 < 2^m \cdot 5 < 2^m \cdot 7 < \dots < \dots < 2^n < \dots < 2 < 1. \end{aligned}$$

**1.1. THEOREM** (Sarkovskiĭ, see SARKOVSKIĖ [1964] and BLOCK, GUCKENHEIMER, MISIUREWICZ and YOUNG [1979]). *Let  $I$  be an interval and  $f: I \rightarrow I$  a continuous map. If  $f$  has a periodic orbit of period  $n$  then  $f$  has periodic orbits of period  $m$  for all integers  $n$  such that  $n < m$ .*

In fact one can prove much more than is stated here. For example, if  $f: I \rightarrow I$  has a periodic point  $p$  of period 4 such that  $p < f(p) < f^2(p) < f^3(p)$  then  $f$  has also periodic orbits of periods  $2^n$  for each  $n$ . Furthermore infinite orbits (with certain ordering) also imply the existence of other orbits. Finally, from the theory of MILNOR and THURSTON [1978] it follows that the dynamics of a map  $f: I \rightarrow I$  with a finite number of turning points is essentially determined by the orbits of its turning points.

The idea of the proof of this theorem can be sketched as follows. Let  $p$  be a periodic point of  $f$  of period  $n$  and  $O = \{p, f(p), \dots, f^{n-1}(p)\}$ . Let  $I_1, \dots, I_{n-1}$  be the intervals connecting consecutive points of  $O$ . Because  $f$  is continuous,  $f(I_i)$  covers at least one of the intervals  $I_1, \dots, I_{n-1}$ . In particular, for each infinite sequence of intervals  $J_0, J_1, J_2, \dots$  such that  $J_i$  is equal to one of the intervals  $I_1, \dots, I_{n-1}$  and such that  $f(J_{i+1}) \supset J_i$ , there exists a point  $x \in J_0$  such that  $f^i(x) \in J_i$  for all  $i \geq 0$ . Choosing appropriate sequences  $J_i$  one can construct the required periodic points.

Clearly this theorem is not valid in the two-dimensional case: a rotation on a disc over degree  $2\pi/3$  has only periodic points of period 3 and 1. However, this example is misleading: using W. Thurston's classification of isotopy classes of homeomorphisms on surfaces, see THURSTON [1988] and BLEILER and CASSON [1988], one can prove the following

**1.2. THEOREM** (GAMBAUDO, VAN STRIEN and TRESSER [1990]).

*Let  $D$  be a disc in the Euclidean plane and let  $f: D \rightarrow D$  be a homeomorphism from  $D$  onto its image. Assume that  $f$  has a periodic point  $p$  of period three and let  $O = \{p, f(p), f^2(p)\}$ . Furthermore assume that  $O$  is knotted in the following sense: there exists an arc  $\gamma$  connecting  $p$  and  $f(p)$  in  $D \setminus O$  such*

that  $f^3(\gamma)$  is not homotopic to  $\gamma$  in  $D \setminus O$ . Under these conditions  $f$  has periodic orbits of each period.

That results of this type were to be expected already follows from PH. BOYLAND [1988].

The idea of the proof of this theorem goes roughly as follows. Let  $O$  be a ‘knotted’ periodic orbit of  $f$  of period 3. Using THURSTON [1988], one can prove that  $f: D \setminus O \rightarrow D \setminus O$  is isotopic to a so-called pseudo-Anosov map  $g$  (for this one uses that  $O$  is knotted). Using an index argument due to Nielsen, see ASIMOV and FRANKS [1983], one can show that each periodic orbit of  $g$  persists when one takes maps which are isotopic to  $g$ . In particular it is sufficient to prove that  $g$  has periodic orbits of each period. Since the dynamics of this pseudo-Anosov is ‘supported’ on a branched-manifold (called a train-track), a collapsing procedure associates to  $g$  a continuous interval map with a periodic orbit of period 3. By Sarkovskii’s theorem the interval map and therefore  $g$  has periodic orbits of each period.

In the proof of this theorem it is essential that one ends up with a train-track that collapses to an interval. In general this is not the case: sometimes the train-track collapses to a circle or to some branches manifold.

? 1088. **Question 1.** Describe the periodic orbits  $O$  of  $f: D \rightarrow D$  for which Sarkovskii’s theorem remains valid (in terms of the action of  $f_*: \pi_1(D \setminus O) \rightarrow \pi_1(D \setminus O)$ ).

? 1089. **Question 2.** Is it possible to extend these results to infinite orbits?

Furthermore, if an interval  $f: I \rightarrow I$  is analytic then it has at most a finite number of turning points and therefore, as we remarked above, its dynamics is essentially determined by the orbits of a finite number of points (its turning points).

? 1090. **Question 3.** Is there an analogue of this statement if  $f: D \rightarrow D$  is analytic?

## 2. The boundary of ‘chaos’

In this section we shall say that a map is *chaotic* if its topological entropy is positive. Non-chaotic interval maps can be easily characterised:

**2.1. THEOREM (BOWEN and FRANKS [1976]).** Let  $f: I \rightarrow I$  be continuous. Then  $f$  is non-chaotic if and only if every periodic orbit of  $f$  has period  $2^n$  for some  $n \geq 0$ . More precisely, suppose  $p$  is a periodic point of period  $2^n$  and denote the orbit of  $p$  by  $O$ . Then there exists a periodic point  $q$  of period  $2^{n-1}$  and a component  $J$  of  $I \setminus O$  containing  $q$  such that  $f^i$  maps  $J$  onto a component of  $I \setminus O$  for  $i = 1, \dots, 2^{n-1}$ .



This result shows that there is a clear tree structure on the set of periodic orbits. The transition from non-chaotic to chaotic is also clear:

**2.2. THEOREM.** *Take any one-parameter family  $f_\mu$  of continuous interval maps depending continuously on the parameter. If  $f_\mu$  is chaotic for  $\mu > \mu_0$  and non-chaotic for  $\mu < \mu_0$  then  $f_{\mu_0}$  is non-chaotic, has periodic orbit of periods  $2^n$  for every  $n \in \mathbb{N}$  and no other periods.*

Of course this theorem is again false for homeomorphisms  $f: D \rightarrow D$  (consider the rotation of the disc from above). However, there is an exact analogue of this theorem:

**2.3. THEOREM** (GAMBAUDO, VAN STRIEN and TRESSER [1989a]).

*Let  $f: D \rightarrow D$  be a homeomorphism onto its image. Then  $f$  is non-chaotic if and only if every periodic orbit is rotation compatible. Furthermore, any two periodic orbits are either disjoint, or they lie nested, or one is the parent of the other.*

Here a periodic orbit is rotation compatible, if it is also the periodic orbit of a homeomorphism which is conjugate to a map which is built by successive surgeries of rotations, see GAMBAUDO, VAN STRIEN and TRESSER [1989a]. Partial results in this direction were already obtained in BOYLAND [1987].

Given a periodic orbit structure, it is not hard to construct homeomorphisms which have precisely these periodic orbits. However, if one gives an infinite number of periodic orbits it is not clear whether one can construct  $C^\infty$  diffeomorphisms with these periodic orbits.

**Question 4.** *Which periodic structures can be represented by  $C^\infty$  diffeomorphisms? Is it for example possible to find a  $C^\infty$  diffeomorphism with zero topological entropy which has periodic orbits of periods  $3^n$  for each  $n \in \mathbb{N}$  where each periodic orbit of period  $3^{n+1}$  circles around the periodic orbit of period  $3^n$ ?* **1091. ?**

It is not too difficult to show that there exist such  $C^\infty$  diffeomorphisms with periodic orbits of periods  $p_n$  if  $p_n$  grows sufficiently fast. For this one can use the techniques of FRANKS and YOUNG[1981]. However,

**Question 5.** *Is it possible to construct such periodic orbits if  $f: D \rightarrow D$  is analytic?* **1092. ?**

Related to this is Smale’s question. In 1962, S. Smale asked whether there exists a diffeomorphism  $f: D^2 \rightarrow D^2$  which has no periodic attractors, but such that each of its periodic orbits is hyperbolic. R. BOWEN, J. FRANKS and L. S. YOUNG proved that there exist  $C^2$  diffeomorphisms of the disc with these properties. These example can not be easily made smoother. Indeed:

**2.4. THEOREM** (VAN STRIEN [1990]). *The diffeomorphisms of R. Bowen, J. Franks and L. S. Young are not conjugate to  $C^3$  diffeomorphisms.*

The impossibility of smoothening these diffeomorphisms is caused by the amount of twisting and the invariant curves which these diffeomorphisms have. However, going about it much more carefully, one can prove the following

**2.5. THEOREM** (GAMBAUDO, VAN STRIEN and TRESSER [1989b]). *There exist analytic diffeomorphisms on  $D^2$  such that all of its periodic orbits are hyperbolic and none of them are sinks or sources.*

These diffeomorphisms were not constructed by hand but the existence was deduced from the theory of renormalizations. More specifically, there is a (small) codimension-one surface in the space of analytic diffeomorphism of the disc such that each diffeomorphism which belongs to this surface has the required properties. This surface is the stable-manifold of the so-called renormalization operator. This renormalization operator is first defined using the corresponding renormalization operator acting on interval maps  $f: [0, 1] \rightarrow [0, 1]$ : to  $f$  one associates the restriction of  $f^2$  to some interval (up to rescaling). One can show that this operator has a hyperbolic fixed point (in the space of interval maps). It turns out that one can extend this to diffeomorphisms of the disc which are ‘almost’ one-dimensional (in the sense that they are close to a non-invertible map which sends the disc to a curve). The non-invertible map is a hyperbolic fixed point of this extended renormalization operator and the stable manifold of this operator is the required surface. Since the renormalization operator is only defined for almost one-dimensional maps, we can ask

? **1093. Question 6.** *Can this surface be more globally defined, using a topological two-dimensional analysis of the renormalization operator?*

In GAMBAUDO, VAN STRIEN and TRESSER [1989b] it is shown that this small codimension-one surface separates chaotic and non-chaotic maps. Therefore we ask:

? **1094. Question 7.** *Is it possible to characterize the boundary of the chaotic diffeomorphisms for diffeomorphisms of  $D$  as was done in the one-dimensional case in Theorem 2.2. Has this boundary the structure of a stratified manifold?*

### 3. Finitely many sinks

Of course a  $C^\infty$  map on an interval can have an infinite number of attracting fixed points. However, it turns out that analytic one-dimensional maps can have at most a finite number of periodic attractors:

**3.1. THEOREM** (MARTENS, DE MELO and VAN STRIEN [1990]). *Let  $f: I \rightarrow I$  be analytic. Then  $f$  can have at most a finite number of periodic attractors.*

In this theorem is shown that all periodic attractors of high periods must necessarily attract a critical point of  $f$ . The corresponding statement is false in the two-dimensional case: NEWHOUSE [1979] has shown that many (analytic) diffeomorphisms on  $D^2$  have an infinite number of periodic attractors.

**Question 8.** *Give some geometric properties of the basins of these periodic attractors.* **1095. ?**

**Question 9.** *Does there exist an open set of such diffeomorphisms?* **1096. ?**

The proof of the existence of an infinite number of attractors strongly uses  $C^2$  estimates.

**Question 10.** *Does there exist a one-parameter family of homeomorphism such that for each nearby family there exist parameters for which the corresponding homeomorphism has an infinite number of periodic attractors?* **1097. ?**

#### 4. Homeomorphisms of the plane

Clearly if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism with periodic point then  $f$  has also fixed points. Brouwer proved that the same result also holds on the plane:

**4.1. THEOREM** (Brouwer). *Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an orientation preserving homeomorphism. If  $f$  has a periodic point then  $f$  has a fixed point.*

Using Thurston's classification theorem and Nielsen's condition for unremovable fixed points, J. M. Gambaudo proved the following strengthened version of this result.

**4.2. THEOREM** (GAMBAUDO [1989]). *Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^1$  orientation diffeomorphism. If  $f$  has a periodic point  $O$  then  $f$  has a fixed point  $P$  which is linked with  $O$ .*

Here  $O$  and  $P$  are linked if the corresponding closed curves of the suspension-flow of  $f$  on  $\mathbb{R}^2 \times S^1$  are linked as knots in  $\mathbb{R}^3$ . (Another way to define this would be to say that they are linked if one cannot find an isotopy  $f_\mu$ ,  $\mu \in [0, 1)$  of  $f$  and disjoint curves  $O_\mu$  and  $P_\mu$  through the periodic points such that  $O_\mu$  and  $P_\mu$  are periodic points of  $f_\mu$  and such that  $P_\mu$  goes to infinity as  $\mu \rightarrow 1$  and such that  $O_\mu$  stays bounded.)

**Question 11.** (Gambaudo) *Can one choose  $P$  so that the linking number between  $O$  and  $P$  is non-zero?* **1098. ?**

## 5. Maps of the annulus

An old conjecture of Birkhoff states that any area preserving diffeomorphism  $f: A \rightarrow A$  without periodic orbits is conjugate to a rotation of the annulus. This conjecture is false for smooth maps. Indeed,

**5.1. THEOREM** (HANDEL [1982], HERMAN [1986] and also FOKKINK and OVERSTEEGEN [1990]). *There exists a  $C^\infty$  diffeomorphism of  $A$  which is area-preserving, has no periodic points and which is not conjugate to a rotation of the circle.*

? **1099. Question 12.** *Does there exist an analytic diffeomorphism with these properties?*

? **1100. Question 13.** *Does there exist an analytic diffeomorphism  $f: A \rightarrow A$  without periodic points such that for some  $x$  in the interior  $A$ , the omega-limit of  $x$  contains points of both boundary components of  $A$ ?*

These last two questions may well be related to the existence of  $C^\infty$  Denjoy counter-examples and the non-existence of analytic examples with these properties, see HALL [1981] and YOCCOZ [1984]. More precisely, in the one-dimensional case one shows these counter-examples cannot arise using the distortion of cross-ratio's, see YOCCOZ [1984] and MARTENS, DE MELO and VAN STRIEN [1990]. Is it possible to find a two-dimensional analogue of these distortion results?

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## Index of general terms

The index is organized as follows: First come some terms that are not readily alphabetized. After that everything is in alphabetical order, with the understanding that, for example,  $\alpha$  is alphabetized as ‘alpha’, 2 as ‘two’ etc.

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